Actuarial Mathematics II
MTH5125

Practice Set: Multiple States

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8.1 You are given the following transition intensities for the permanent disability model for $0 \leq t \leq 5$
$\mu^{01}_{x+t} = 0.02, \mu^{02}_{x+t} = 0.03, \mu^{12}_{x+t} = 0.04$

a) Calculate the probability that a healthy life aged x is still health at age $x + 5$

$$
\begin{aligned}
{}_t p_x^{\overline{00}} &= \exp\left(-\int_0^t \left(\mu^{01}_{x+r} + \mu^{02}_{x+r}\right) dr\right) \\
&= \exp\left(-\int_0^t (0.02 + 0.03)\, dr\right) \\
&= \exp(-0.05t)
\end{aligned}
$$

$${}_5 p_x^{\overline{00}} = \exp(-0.25) = 0.7788$$

b) Calculate the probability that a healthy life aged x is still healthy a age $x+5$ given that (x) survives to age $x+5$

We need to calculate the probability that (x) is in State 0 at time 5, given that (x) is either in state 0 or State 1 at time 5, hence a conditional probability:

$$\frac{{}_5p_x^{\overline{00}}}{{}_5p_x^{\overline{00}} + {}_5p_x^{01}}$$

Now,

$$ {}_{5}p_{x}^{01} = \int_{0}^{5} {}_{r}p_{x}^{\overline{00}} \mu_{x+r}^{01} \; {}_{5-r}p_{x+r}^{\overline{11}} dr $$

$$ \begin{aligned} {}_{5-r}p_{x+r}^{\overline{11}} &= \exp\left(-\int_{0}^{5-r} \mu_{x+r+s}^{12} ds \right) \\ &= \exp\left(-0.04\left(5-r\right)\right) \end{aligned} $$

Hence,

$$
\begin{aligned}
{}_5p_x^{01} &= \int_0^5 e^{-0.05r} \times 0.02 \times e^{-0.04(5-r)} dr \\
&= 0.02 \times e^{-0.2} \int_0^5 e^{-0.01r} dr \\
&= 0.02 \times e^{-0.2} \times \frac{\left(1 - e^{-0.05}\right)}{0.01} \\
&= 0.07986
\end{aligned}
$$

And

$$
\frac{{}_5p_x^{\overline{00}}}{{}_5p_x^{\overline{00}} + {}_5p_x^{01}} = \frac{0.7788}{0.7788 + 0.07986} = 0.90699
$$

8.2 Use the permanent disability model with:
$\mu^{01}_{50+t} = 0.02, \mu^{02}_{50+t} = 0.03, \mu^{12}_{50+t} = 0.11$, $0 \leq t \leq 5$.
An insurance policy will pay a benefit only if the life currently 50 and healthy, has been disable for one full year before age 65. Calculate the probability that the benefit is paid.

In order for the benefit to be paid the life must mve from State 0 to State 1 at age $x + r$ where $r < 14$ and then the life must remain in state 1 continuously for at least 1 year. Hence the probability that the benefit is paid is:

$$\int_0^{14} {}_r p_x^{\overline{00}} \mu^{01}_{x+r} \ {}_1 p_{x+r}^{\overline{11}} dr$$

Now, ${}_{r}p_{x}^{\overline{00}} = \exp\left(-0.05r\right)$ and ${}_{1}p_{x+r}^{\overline{11}} = \exp\left(-0.11\right)$ and thus:

$$
\begin{aligned}
\int_{0}^{14} {}_{r}p_{x}^{\overline{00}} \mu_{x+r}^{01} \; {}_{1}p_{x+r}^{\overline{11}} dr &= \int_{0}^{14} e^{-0.05r} \times 0.02 \times e^{-0.11} dr \\
&= 0.02 \times e^{-0.11} \frac{\left(1 - e^{-0.7}\right)}{0.05} \\
&= 0.18039
\end{aligned}
$$

8.3 Use the sickness-death model (health-sickness) model with
$\mu^{01}_{x+t} = 0.08, \mu^{02}_{x+t} = 0.04, \mu^{10}_{x+t} = 0.1, \mu^{12}_{x+t} = 0.05$, $0 \leq t \leq 5$
a) Calculate the probability that a life aged x who is currently in State 1 remains in State 1 for the next 15 years.

$$
\begin{aligned}
{}_{15}p^{\overline{11}}_{x} &= \exp\left(-\int_0^{15}\left(\mu^{10}_{x+r} + \mu^{12}_{x+r}\right) dr\right) \\
&= \exp\left(-\int_0^{15}(0.1 + 0.05)\, dr\right) \\
&= \exp\left(-0.15 \times 15\right) = 0.10540
\end{aligned}
$$

b) Calculate the probability that a life aged x who is currently in State 1 makes exactly one transition to State 0 and then remains in State 0 over the next 15 years.

$$\int_0^{15} {}_r p_x^{\overline{11}} \mu_{x+r}^{10} \; {}_{15-r} p_x^{\overline{00}} \, dr$$

Now,

$$\begin{aligned} {}_{15-r} p_x^{\overline{00}} &= \exp\left(- \int_0^{15-r} \left(\mu_{x+s}^{01} + \mu_{x+s}^{02} \right) ds \right) \\ &= \exp\left(-0.12 \left(15 - r \right) \right) \end{aligned}$$

Hence

$$
\begin{aligned}
& \int_0^{15} {}_rp_x^{\overline{11}} \mu_{x+r}^{10} \; {}_{15-r}p_x^{\overline{00}} dr \\
= & \int_0^{15} e^{-0.15r} \times 0.1 \times e^{-0.12(15-r)} \\
= & \; 0.1e^{-1.8} \frac{\left(1 - e^{-0.45}\right)}{0.03} = 0.19967
\end{aligned}
$$

8.6 Calculate $\bar{A}^{02}_{79:\overline{1}|}$ using the Standard Sickness -Death model tables in Appendix D, with an interest rate of 5% per year.

$$\begin{aligned}\bar{A}^{02}_{79:\overline{1}|} &= \bar{A}^{02}_{79} - v\ {}_1p^{01}_{79}\bar{A}^{12}_{80} - v\ {}_1p^{00}_{79}\bar{A}^{02}_{80} \\ &= 0.0749\end{aligned}$$

8.8 An insurer calculates premiums for permanent disability insurance. A life aged 60 purchases a policy with a five-year term which provides a benefit of $100,000$ on exit from a healthy state.
a) Write down an expression in terms of transition intensities, probabilities and δ for the EPV of this benefit at force of interest δ per year.

The EPV of the benefit is::

$$100,000 \int_0^5 e^{-\delta t} \; {}_tp_{60}^{00} \left(\mu_{60+t}^{01} + \mu_{60+t}^{02} \right) dt$$

b) Calculate the EPV of the benefit when $\mu_x^{01} = 0.01$ and $\mu_x^{02} = 0.015$ for $60 \leq x \leq 65$ and $\delta = 0.05$.

$$
\begin{aligned}
{}_tp_{60}^{00} &= \exp\left(-\int_0^t \left(\mu_{60+s}^{01} + \mu_{60+s}^{02}\right) ds\right) \\
&= \exp\left(-\int_0^t 0.025 ds\right) \\
&= \exp\left(-0.025t\right)
\end{aligned}
$$

Hence the EPV of the benefit is: $100,000 \int_0^5 e^{-0.05t} \times e^{-0.025t} \times 0.025 dt = 10,423.69$

Example 8.7 An insurer issues a 10-year disability income insurance policy to a healthy life aged 60. Premiums are payable continuously while in the healthy state. A benefit of \$20, 000 per year is payable continuously while in the disabled state. A death benefit of \$50, 000 is payable immediately on death. Use the Standard Sickness model in Appendix D3, assume interest is 5% per year and there are no expenses. Calculate the premium.

The equation of value is:

$$P\bar{a}^{00}_{60:\overline{10}|} = 20,000\bar{a}^{01}_{60:\overline{10}|} + 50,000\bar{A}^{02}_{60:\overline{10}|}$$

$$\bar{a}^{00}_{60:\overline{10}|} = \bar{a}^{00}_{60} - v^{10}\ {}_{10}p^{00}_{60}\,\bar{a}^{00}_{70} - v^{10}\ {}_{10}p^{01}_{60}\,\bar{a}^{10}_{70} = 6.5885$$
$$\bar{a}^{01}_{60:\overline{10}|} = \bar{a}^{01}_{60} - v^{10}\ {}_{10}p^{00}_{60}\,\bar{a}^{01}_{70} - v^{10}\ {}_{10}p^{01}_{60}\,\bar{a}^{11}_{70} = 0.6476$$
$$\bar{A}^{02}_{60:\overline{10}|} = \bar{A}^{02}_{60} - v^{10}\ {}_{10}p^{00}_{60}\,\bar{A}^{02}_{70} - v^{10}\ {}_{10}p^{01}_{60}\,\bar{A}^{12}_{70} = 0.16382$$

Hence, as

$$
\begin{aligned}
P &= \frac{20,000\,\bar{a}^{01}_{60:\overline{10}|} + 50,000\,\bar{A}^{02}_{60:\overline{10}|}}{\bar{a}^{00}_{60:\overline{10}|}} \\
&= 3,209
\end{aligned}
$$