Actuarial Mathematics II
MTH5125

Multiple State Models

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Plan

- Based on Chapter 8 (DHW)
- Introduction on health insurance
- Multiple state framework

Health Insurance

So far in this module we have only considered life assurance products and lives exposed to the risk of death
In many other actuarial fields, lives may transition to a number of different states

- in a pension fund we need to consider retirement, death, leaving employment
- health insurance: benefits are contingent on state of health
  - the set of possible transitions as competing risks

Income protection policies

Policy pays income to the policyholder whilst that person is sick

- "sick" needs to be carefully defined in the policy documentation
- if the policyholder recovers the insurance remains in force and the policyholder is then entitled to further income payments in the future if the illness returns or if they become sick again for another reason

These policies usually have a deferred period

- 3 months is quite common
  - this is the period of continuous sickness which must pass before benefits start to be paid
- premiums are generally monthly and are waived whilst benefits are being paid

Critical illness insurance

- Pays a lump sum benefit on diagnosis of critical illness
- 'Critical illness' is some life threatening condition
- Policies will usually have a defined list of conditions which are deemed critical
- These policies are often bought alongside term assurance so the benefit is paid on death or diagnosis with a critical condition within n years

Long term care insurance products

- The cost of long term care for the elderly is becoming an important public policy issue in many countries
- Long term care refers to a variety of services which help meet both the medical and non medical needs of people with a chronic illness or disability who cannot care for themselves for long periods.
- Long term care insurance pays an income when the policyholder requires long term care
  - this income is designed to support the costs of providing the care

Private medical care insurance

- Private or personal healthcare insurance pays the costs of eligible medical treatments including the costs of specialist practitioners and hospitals
- Actuaries working on income protection, critical illness and long termcare insurance use techniques similar to those found for life assurance (and considered in this module)
- Actuaries working on private medical care insurance use general insurance type products.

Introduction

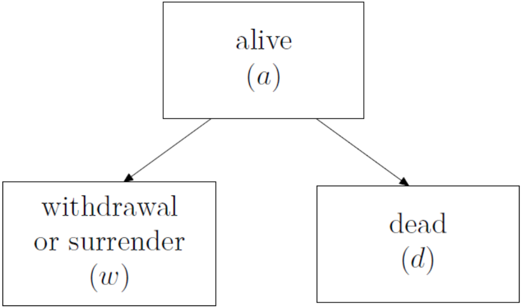
- Multiple state models (also called transition models)
  - what are they?
  - actuarial applications - some examples
- State space
- Transition probabilities
- Premiums and Reserves

Introduction

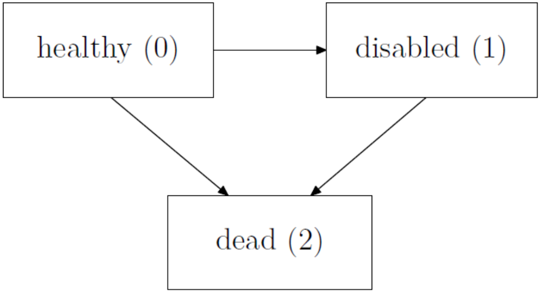
- Multiple state models are probability models that describe the random movements of a subject (often a person, but could be a machinery, organism, etc.) among various states
  - Discrete time or continuous time and discrete state space
- Examples include:
  - basic survival model
  - multiple decrement models
  - health-sickness model
  - disability model
  - pension models
  - multiple life models
  - long term care models

The basic survival model

The withdrawal-death model



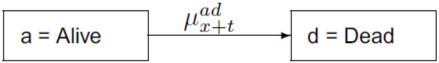
The permanent disability model



Two state: alive dead model

There are two equivalent formulation for **single life**

1. T_x is the current lifetime random variable for (x) with cdf $F_x(t)$, pdf $f_x(t)$ and force of mortality $\mu_{x+t} = \frac{f_x(t)}{1-F_x(t)}$
2. Alive - dead model



We define ${}_tp_x^{ij}$ to be the probability of being in state j at age $x+t$, conditional on being in state i at age x.
The time of death is governed by the transition intensity μ_{x+t}^{ad}

Two state: alive dead model

Assumptions on transition intensity:

The intensity μ^{ad}_{x+t} depends only on the current age $x + t$ and not on any other aspect of the life's past history

The superscript '*ad*' on the transition intensity indicates that it refers to the transition from the state labelled '*a*' to the state labelled '*d*

The probability of dying before age $x + t + dt$, conditional on being alive at age $x + t$, is:

$$_{dt}p^{ad}_{x+t} = \mu^{ad}_{x+t}dt + o\left(dt\right)$$

Two state: alive dead model

A function $f(t)$ is said to be '$o(dt)$' if $\lim_{dt \to 0} \frac{f(dt)}{dt} = 0$ - f converges faster to 0 than dt

This is the behaviour of the model over infinitesimal time intervals dt

The key question then is: how does this determine the model's behaviour over extended time intervals, e.g. years?

Two state: alive dead model

Kolmogorov Equation:

$$\frac{d}{dt}\ {}_tp_x^{aa} = -\ {}_tp_x^{aa}\mu_{x+t}^{ad}$$

equivalent to $\frac{d}{dt}\ {}_tp_x = -\ {}_tp_x\mu_{x+t}$ or $\mu_{x+t} = -\frac{1}{{}_tp_x}\frac{d}{dt}\ {}_tp_x$ (please see revision notes 1, eq. 2.18)

The general Markov multiple-state model

- Our aim in any given case is to model the life history of a person initially age (x), of which 'alive or dead' is merely the simplest possible example.
- Assume a finite state space (total of $m+1$ states): $\{0, 1, ..., m\}$
- In most actuarial applications, we need a reference age
  - let x the age at which the multiple state process begins
  - x is the age at time $t = 0$
- Let $Y(t)$ the state of the process at time t (random variable):
  - can take one possible values in the state space
  - $Y(t) = i$ means that the individual is in state i at t
- The process can be denoted by $\{Y(t); t \geq 0\}$

For each pair of distinct states i and j , the probability of making a transition from state i to state j at age $x + t$ (conditional on then being in state i) is governed by a transition intensity μ^{ij}_{x+t}
Transition probabilities:

- ${}_tp^{ij}_x = \Pr\left(Y(x+t) = j; Y(x) = i\right)$
- This is the probability that a life currently aged x in state i and will be in state j after t periods.

Force of transition (also called transition intensity) - for any positive interval of time h :

- $\mu^{ij}_x = \lim_{h\to+0} \frac{1}{h}\ {}_hp^{ij}_x$ for $i \neq j$
- $\mu^{ij}_x = 0$ if it is not possible to transition from state i to state j at any time.

Some assumptions

Assumption 1: The Markov property holds. Any state i and j and times t and $t+s$:

$$\Pr\left[Y(t+s)=j|Y(t)=i\right]$$

is well defined in the sense that its value does not depend on any information about the process before time t.
Assumption 2: For any positive interval of time length (generally very small) h:
$\Pr[2 \text{ or more transitions within a time period of length } h] = o(h)$
Assumption 3 (technical assumption): For a state i and j and any age $x \geq 0$ ${}_tp_x^{ij}$ is differentiable function of t
We need this for $\mu_x^{ij} = \lim_{h\to+0} \frac{1}{h}\ {}_hp_x^{ij}$ to exist.

Some useful approximations

We can express the transition probabilities in terms of the forces of transition as:

$$ {}_h p_x^{ij} = h\mu_x^{ij} + o\left(h\right) $$

so that for very small values of h, we have the approximation

$$ {}_h p_x^{ij} \approx h\mu_x^{ij} $$

The occupancy probability

- When a person currently age x and is currently in state i, the probability that the person continuously remains in the same state for a length of t periods is called an occupancy probability.
- ${}_tp_x^{\overline{ii}} = \Pr\left(Y(x+s) = i;\ \text{for all } s \in [0,t] \,|\, Y(x) = i\right)$
- ${}_tp_x^{ii} = {}_tp_x^{\overline{ii}} + o(t)$
- ${}_tp_x^{ii}$ includes the possibility that the process leaves state i between x and $x+t$ provided that it is back in state i at age $x+t$

The occupancy probability

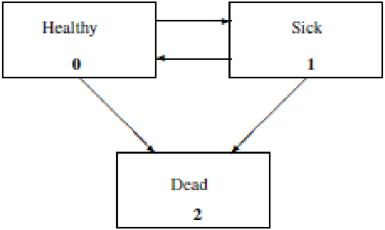
- ${}_{h}p_{x}^{\overline{ii}} = 1 - h \sum_{j=0, j\neq i}^{m} \mu_{x}^{ij} + o(h)$ or equivalently

  $1 - {}_{t}p_{x}^{\overline{ii}} = h \sum_{j=0, j\neq i}^{m} \mu_{x}^{ij} + o(h)$
- Note that $1 - {}_{h}p_{x}^{\overline{ii}}$ probability that the process leaves state i in between x and $x+h$ (possibly returning to i before $x+h$)
- this means that the process must be in a state $j \neq i$ or in state i after making at least two transitions in the interval h:

  $h \sum_{j=0, j\neq i}^{m} \mu_{x}^{ij} + o(h)$ (QED)

The sickness-death model - example

A sickness-death (health-sickness)model can be represented as:



We know:

x	${}_{10}p_x^{00}$	${}_{10}p_x^{01}$	${}_{10}p_x^{11}$	${}_{10}p_x^{10}$
40	0.93705	0.01953	0.70349	0.23942
50	0.83930	0.06557	0.81211	0.06057

Calculate: ${}_{20}p_{40}^{00}$, ${}_{20}p_{40}^{10}$, ${}_{20}p_{40}^{02}$

The sickness death model - example

$$ {}_{20}p_{40}^{00} = {}_{10}p_{40}^{00}\ {}_{10}p_{50}^{00} + {}_{10}p_{40}^{01}\ {}_{10}p_{50}^{10} = 0.78765 $$

$$ {}_{20}p_{40}^{10} = {}_{10}p_{40}^{11}\ {}_{10}p_{50}^{10} + {}_{10}p_{40}^{10}\ {}_{10}p_{50}^{00} = 0.24356 $$

${}_{20}p_{40}^{02}$ many paths to consider:

1. (40) dies before time 10 given that she is healthy now: ${}_{10}p_{40}^{02} = 1 - {}_{10}p_{40}^{00} - {}_{10}p_{40}^{01} = 0.04342$
2. (40) is healthy at time 10 but dies between 10 and 20: ${}_{10}p_{40}^{00}\ {}_{10}p_{50}^{02} = {}_{10}p_{40}^{00}\left(1 - {}_{10}p_{50}^{00} - {}_{10}p_{50}^{01}\right) = 0.08914$
3. (40) is sick at time 10 and dies between 10 and 20: ${}_{10}p_{40}^{01}\ {}_{10}p_{50}^{12} = {}_{10}p_{40}^{01}\left(1 - {}_{10}p_{50}^{10} - {}_{10}p_{50}^{11}\right) = 0.00249$

Hence

$$ {}_{20}p_{40}^{02} = 0.04342 + 0.08914 + 0.00249 = 0.13505 $$

The sickness-death model - example

Given that a healthy life age (40) dies before age 60 what is the probability that she was healthy at 50?

$$\Pr\left[(40) \text{ is healthy at } 50 \middle| \text{Dead before } 60\right] =$$

$$\frac{\Pr[(40) \text{ is healthy at 50 AND Dead before 60}]}{\Pr[(40) \text{ is Dead before 60}]} = \frac{{}_{10}p_{40}^{00}\ {}_{10}p_{50}^{02}}{{}_{20}p_{40}^{02}} = 0.66007$$

The occupancy probability

- For any state i in a multiple state model, the probability that (x) now in state i will remain in state i for t years can be computed using:

$$
{}_tp_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j=0, j\neq i}^{m} \mu_{x+s}^{ij} ds\right) \tag{1}
$$

Proof

For any $h > 0$, consider the probability ${}_{t+h}p_x^{\overline{ii}}$ is the probability that the individual/process stays in state i throughout the time period $[0, t + h]$, given that the process was in state i at age x. We can split this event into two sub-events:

• the process stays in state i from age x until (at least) age $x + t$, given that it was in state i at age x, and

• the process stays in state i from age $x + t$ until (at least) age $x + t + h$, given that it was in state i at age $x + t$

The occupancy probability

The probabilities of these two sub-events are ${}_tp_x^{\overline{ii}}$ and ${}_hp_{x+t}^{\overline{ii}}$, respectively, and using the rules for conditional probabilities, we have

$$
\begin{aligned}
{}_{t+h}p_x^{\overline{ii}} &= {}_tp_x^{\overline{ii}}\ {}_hp_{x+t}^{\overline{ii}} \Leftrightarrow \\
{}_{t+h}p_x^{\overline{ii}} &= {}_tp_x^{\overline{ii}} \left(1 - h \sum_{j=0, j\neq i}^{m} \mu_{x+t}^{ij} + o(h)\right) \Leftrightarrow \\
\frac{{}_{t+h}p_x^{\overline{ii}} - {}_tp_x^{\overline{ii}}}{h} &= -\ {}_tp_x^{\overline{ii}} \sum_{j=0, j\neq i}^{m} \mu_{x+t}^{ij} + \frac{o(h)}{h}
\end{aligned}
$$

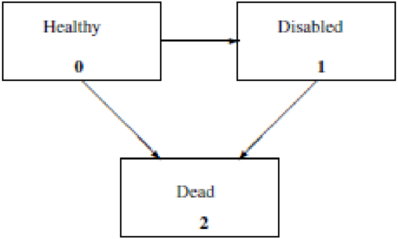
Letting $h \to 0$

$$\frac{1}{{}_t p_x^{\overline{ii}}}\left(\frac{d}{dt}\ {}_t p_x^{\overline{ii}}\right) = -\sum_{j=0, j\neq i}^{m} \mu_{x+t}^{ij} \Leftrightarrow \frac{d}{dt}\ \log_t p_x^{\overline{ii}} = -\sum_{j=0, j\neq i}^{m} \mu_{x+t}^{ij}$$

Integrating over $(0, t)$: $\log_t p_x^{\overline{ii}} - \log \underbrace{{}_0 p_x^{\overline{ii}}}_{=1} = -\int_0^t \sum_{j=0, j\neq i}^{m} \mu_{x+s}^{ij} ds \Leftrightarrow$

$${}_t p_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j=0, j\neq i}^{m} \mu_{x+s}^{ij} ds\right)$$

The permanent disability model



Find ${}_{t+h}p_x^{01}$, for $x \geq 0$ and $t, h > 0$

- the probability that this life is alive and disabled at age $x + t + h$.

The permanent disability model

We can write down a formula for this probability by conditioning on which state the life was in at age $x + t$.
Either:

- the life was disabled at age $x + t$ (probability ${}_tp_x^{01}$) and remained disabled between ages $x + t$ and $x + t + h$ (probability ${}_hp_{x+t}^{\overline{11}}$) or,
- the life was healthy at age $x + t$ (probability ${}_hp_x^{\overline{00}}$) and then became disabled between ages $x + t$ and $x + t + h$ $(h\mu_{x+t}^{01} + o(h))$

Note that the probability of the life being healthy at age $x + t$, becoming disabled before age $x + t + h$ and then dying before age $x + t + h$ is $o(h)$ since this involves two transitions in a time interval of length h)

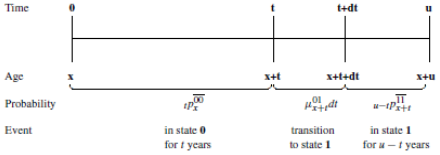
$$ {}_{t+h}p_x^{01} = {}_tp_x^{01}\ {}_hp_{x+t}^{\overline{11}} + {}_tp_x^{\overline{00}}\ h\mu_{x+t}^{01} + o(h) $$

It can be shown that (please email me if you want to see the proof):

$$ {}_{u}p_{x}^{01} = \int_{0}^{u} {}_{t}p_{x}^{\overline{00}} \mu_{x+t}^{01} \; {}_{u-t}p_{x+t}^{\overline{11}} dt \tag{2} $$

The permanent disability model

For the life to move from state 0 to state 1 between ages x and $x + u$, the life must stay in state 0 until some age $x + t$, transfer to state 1 between ages $x + t$ and $x + t + dt$, where dt is small, and then stay in state 1 from age $x + t + dt$ to age $x + u$.



The permanent disability model

The infinitesimal probability of this path is ${}_{t}p_{x}^{\overline{00}}\mu_{x+t}^{01}dt\ {}_{u-t}p_{x+t}^{\overline{11}}$
Note ${}_{u-t}p_{x+t}^{\overline{11}} \simeq {}_{u-t-dt}p_{x+t}^{\overline{11}}$ if dt is small.
Since the age at transfer, $x+t$, can be anywhere between x and $x+u$, the total probability, ${}_{u}p_{x}^{01}$ is the 'sum'(i.e. integral) of these probabilities from $t=0$ to $t=u$.

Move to more general multiple states?

Kolmogorov's forward equations

Idea: we consider the probability of being in the required state, j, at age $x+t+h$, and condition on the state of the process at age $x+t$:

- either it is already in state j,
- or it is in some other state, say k, and a transition to j is required before age $x+t+h$.

$$
{}_{t+h}p_x^{ij} = {}_tp_x^{ij}\ {}_hp_{x+t}^{jj} + \sum_{k=0,k\neq j}^{m} {}_tp_x^{ik}\ {}_hp_{x+t}^{kj}
$$

With:

$$
{}_hp_{x+t}^{jj} = 1 - h \sum_{k=0,k\neq j}^{m} \mu_{x+t}^{jk} + o\left(h\right)
$$

$$
{}_hp_{x+t}^{kj} = h\mu_{x+t}^{kj} + o\left(h\right)
$$

Kolmogorov's forward equations

After substituting and collecting terms, by letting $h \rightarrow \infty$ we get the Kolmogorov's forward equations:

$$\frac{d}{dt}\ {}_t p_x^{ij} = \sum_{k=0, k\neq j}^{m} \left({}_t p_x^{ik} \mu_{x+t}^{kj} - {}_t p_x^{ij} \mu_{x+t}^{jk} \right) \qquad (8.16)$$

Kolmogorov's forward equations

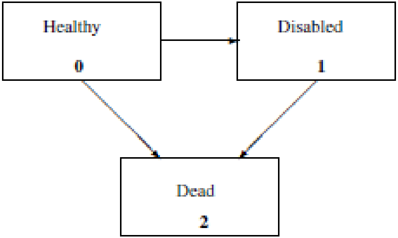
We can solve the Kolmogorov forward equations numerically but we need boundary conditions:
For any model, in any state i :

$${}_0p_x^{ii} = 1$$

and for any two states i and j where $i \neq j$:

$${}_0p_x^{ij} = 0$$

Consider the permanent disability model.



Suppose the transition intensities for this model are all constants, as follows:
$\mu_x^{01} = 0.0279$, $\mu_x^{02} = 0.0229$, $\mu_x^{12} = \mu_x^{02}$.Calculate ${}_{10}p_{60}^{00}$ and ${}_{10}p_{60}^{01}$.

Numerical evaluations of probabilities

For this model, neither state 0 nor state 1 can be re-entered once it has been left, so that ${}_t p_x^{ii} = {}_t p_x^{\overline{ii}}$ for $i = 0, 1$ and any $x, t \geq 0$. We are using:

$$ {}_t p_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j=0, j\neq i}^{m} \mu_{x+s}^{ij} ds \right) $$

$$ {}_t p_{60}^{00} = {}_t p_{60}^{\overline{00}} = \exp\left(-\int_0^t \left(\mu_x^{01} + \mu_x^{02} \right) ds \right) = $$

$$ \exp\left(-\int_0^t (0.0279 + 0.0229)\, ds \right) = \exp(-0.0508t) $$

$$ {}_{10} p_{60}^{00} = \exp(-0.508) = 0.60170 $$

Numerical evaluations of probabilities

Now, $_{10}p_{60}^{01} = \int_0^{10} \left({}_tp_{60}^{00}\mu_{60+t}^{01} \times {}_{10-t}p_{60+t}^{11} \right) dt$

$$\begin{aligned} {}_{10-t}p_{60+t}^{11} &= \exp\left(-\int_0^{10-t} \mu_x^{12} ds \right) = \exp\left(-\mu_x^{12} (10-t) \right) \\ &= \exp\left(-0.0229 (10-t) \right) \end{aligned}$$

Hence:

$$\begin{aligned} {}_{10}p_{60}^{01} &= \int_0^{10} \exp\left(-0.0508t \right) \times 0.0279 \times \exp\left(-0.0229 (10-t) \right) dt \\ &= 0.0279 \exp\left(-0.0229 \right) \int_0^{10} \exp\left(-0.0279t \right) dt = 0.19363 \end{aligned}$$

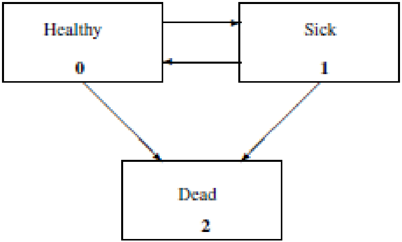
Numerical evaluations of probabilities

Probabilities of the form ${}_tp_x^{\overline{ii}}$ can be evaluated analytically provided the sum of the relevant intensities can be integrated analytically.
In other cases numerical integration can be used.
However deriving an integral formula for the probability which can then be integrated numerically – is tractable only in the simplest cases.

- few states and
- if none of these states can be re-entered once it has been left.

Numerical evaluations of probabilities

Disability income insurance model (sickness-death model):



States 0 and 1 can both be re-entered.
This means, for example, that ${}_tp_x^{\overline{01}}$ is the sum of the probabilities of exactly one transition (0 to 1), plus three transitions (0 to 1, then 1 to 0, then 0 to 1 again), plus five transitions, and so on.
A probability involving k transitions involves multiple integration with k nested integrals.

Numerical evaluations of probabilities

We can use however Kolmogorov's forward equations: $\frac{d}{dt}\ {}_tp_x^{ij} = \sum_{k=0,k\neq j}^{m} \left({}_tp_x^{ik}\mu_{x+t}^{kj} - {}_tp_x^{ij}\mu_{x+t}^{jk} \right)$

We have system of equations to solve simultaneously rather than a single one

In general, the number of equations increases with the number of states in the model.

State dependent annuities

Generalize our definitions of insurance and annuity functions to a multiple state framework

Suppose we have a life aged x currently in state i of a multiple state model.

What is the value of an annuity of 1 per year payable continuously while the life is in some state j (which may be equal to i).

EPV of continuous state dependent time annuity:

$$\bar{a}_x^{ij} = \int_0^\infty e^{-\delta t} \; {}_t p_x^{ij} \, dt$$

If the annuity is payable at the start of each year, from the current time, conditional on the life being in state j, given that the life is currently in state i.

The EPV of a discrete annuity due is:

$$\ddot{a}_x^{ij} = \sum_{k=0}^\infty v^k \; {}_k p_x^{ij}$$

If $i \neq j \; {}_0 p_x^{ij} = 0$

State dependent annuities

The EPV of a term annuity due is:

$$\ddot{a}^{ij}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k \;\, {}_k p_x^{ij}$$

$$\bar{a}^{ij}_{x:\overline{n}|} = \bar{a}^{ij}_x - e^{-\delta n} \sum_{k=0}^{m} \;{}_n p_x^{ik} \;\, \bar{a}^{kj}_{x+n}$$

For example: alive-dead model: $\bar{a}^{00}_{x:\overline{n}|} = \bar{a}^{00}_x - e^{-\delta n} \;{}_n p_x^{00} \;\, \bar{a}^{00}_{x+n}$

Sickness-death model: $\bar{a}^{00}_{x:\overline{n}|} = \bar{a}^{00}_x - e^{-\delta n} \;{}_n p_x^{00} \;\, \bar{a}^{00}_{x+n} - e^{-\delta n} \;{}_n p_x^{01} \;\, \bar{a}^{10}_{x+n}$

State dependent insurance benefits

For insurance benefits, the payment is usually conditional on making a transition.

- A death benefit is payable on transition into the dead state;
- A critical illness insurance policy might pay a sum insured on death or earlier diagnosis of one of a specified group of illnesses.

Suppose a unit benefit is payable immediately on each future transfer into state j, given that the life is currently in state i (which may be equal to j). Then the expected present value of the benefit is:

$$\bar{A}_x^{ij} = \int_0^{\infty} \sum_{k=0, k\neq j}^{\infty} e^{-\delta t} \ {}_t p_x^{ik} \ \mu_{x+t}^{kj} dt$$

State dependent insurance benefits

Note that this benefit does not require the transition to be directly from state i to state j, and if there is a possibility of transitions into state j than it values a benefit of 1 paid each time the life transitions into state j.

$$
\begin{aligned}
\bar{A}^{ij}_{x:\overline{n}|} &= \int_0^n \sum_{k=0,k\neq j}^{m} e^{-\delta t} \; {}_tp^{ik}_x \; \mu^{kj}_{x+t} dt \\
&= \bar{A}^{ij}_x - e^{-\delta n} \sum_{k=0}^{m} \; {}_np^{ik}_x \bar{A}^{kj}_{x+n}
\end{aligned}
$$

State dependent insurance benefits

Alive - dead model:

$$\bar{A}^{00}_{x:\overline{n}|} = \bar{A}^{00}_{x} - e^{-\delta n}\ {}_{n}p^{00}_{x}\bar{A}^{00}_{x+n}$$

Health -sickness model for $i = 0$ and $j = 1$:

$$\bar{A}^{01}_{x:\overline{n}|} = \bar{A}^{01}_{x} - e^{-\delta n}\ {}_{n}p^{00}_{x}\bar{A}^{01}_{x+n} - e^{-\delta n}\ {}_{n}p^{01}_{x}\bar{A}^{11}_{x+n}$$

Premiums

- We assume that premiums are calculated using the equivalence principle and that lives are in state 0 at the policy inception date.
- Premiums are calculated by solving the equation of value using the appropriate annuity and insurance functions

Policy values

When we consider the reserves that need to be held for an insurance policy more general than life insurance, e.g. disability insurance, it is clear that a different reserve needs to be held, depending on the state currently occupied.
For example, under disability insurance:

- if the life is currently healthy, it is certain that they are currently paying premiums and possible that they might, in future, receive benefits.
- if the life is currently sick, it is certain that they are currently receiving benefits and possible that they might, in future, resume paying premiums.