Actuarial Mathematics II
MTH5125

Problem Set 1 Solutions

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Exercise 6.1

Consider a fully discrete whole life insurance with sum insured \$200,000 issued to a select life aged 30. The premium payment term is 20 years. Assume the mortality follows the Standard Select Life Table with $i = 5\%$.

a) Write down an expression for the net loss at issue random variable

b) Calculate the annual premium.

c) Calculate the probability that the contract makes a profit

Benefits outgo: whole life insurance(annual, discrete) - \$200,000 to be paid at the end of the year of death: $v^{K_{[30]}+1}$

Income: premiums paid over 20 years - term annuity due (if not mentioned clearly in the question). If the life dies (obviously) premiums are not paid anymore.
Net future loss at issue:

$$L_0^n = 200,000v^{K_{[30]}+1} - P\ddot{a}_{\overline{\min(K_{30}+1,20)}|}$$

Equivalence principle:

$$E\left(L_0^n\right) = 200,000A_{[30]} - P\ddot{a}_{[30]:\overline{20}|} = 0$$

Exercise 6.1

Reminder:

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - \ v^n \ {}_np_x \ddot{a}_{x+n}$$

$$\ddot{a}_{[30]:\overline{20}|} = \ddot{a}_{[30]} - {}_{20}E_{[30]}\ddot{a}_{[50]}$$

Exercise 6.1

(a) Let P be the annual premium. Then

$$L_0 = 200\,000\, v^{K_{[30]}+1} - P\, \ddot{a}_{\overline{\min(K_{[30]+1},20)}|}\,.$$

(b) The equation of value is

$$P\, \ddot{a}_{[30]:\overline{20}|} = 200\,000\, A_{[30]}\,,$$

giving $P = \$1179.74$ since

$$\ddot{a}_{[30]:\overline{20}|} = \ddot{a}_{[30]} - {}_{20}E_{[30]}\, \ddot{a}_{[50]} = 13.0419$$

and $A_{[30]} = 0.07693$.

Exercise 6.1

Assume the life survives more than $n > 20$ years:
Did the policy made a profit at death?
At death:

- **Benefit outgo:** 200,000
- **Income**: premiums for 20 years only - bring them to the time of death?

$$\underbrace{\ddot{a}_{[30]:\overline{20}|} \times P\,(1.05)^{20}}_{FV \text{ at the end of 20 years}} \times (1.05)^{n-20} > 200,000$$

We want the smallest integer such that the accumulations of premiums to time n exceed the sum insured

$$n = 53$$

Exercise 6.1

The probability of profit is ${}_{52}p_{30} = 0.70704$.
Note that the benefit will be paid at time 53 or layter if the life survives for 52 years from issue.

Exercise 6.2

Consider a five-year term insurance issued to a select life aged 40 by a single premium, with sum insured \$1 million payable immediately on death. Assume mortality follows the Standard Select Life Table with UDD between integer ages, $i = 5\%$.

a) Write down an expression for the net loss at issue random variable

b) Calculate the annual premium.

c) Calculate the probability that the contract makes a profit

Exercise 6.2

Net random loss:

$$L_0^n = 1{,}000{,}000 v^{T_{[40]}} \times 1\left(T_{40} \leq 5\right) - P$$

Equivalence principle:

$$E\left(L_0^n\right) = 1{,}000{,}000 \ \bar{A}^1_{[40]:\overline{5}|} - P = 0$$

$$P = 1{,}000{,}000 \ \bar{A}^1_{[40]:\overline{5}|}$$

Exercise 6.2

Note (using UDD) that

$$\bar{A}^{1}_{[x]:\overline{n}|} = \frac{i}{\delta} A^{1}_{[x]:\overline{n}|}$$

$$A^{1}_{x:\overline{n}|} = A_x - \ {}_nE_x A_{x+n}$$

Important: when calculating the term insurance benefit keep in mind that at 45 the life that started the contract at 40 is not anymore select! We use table D!

Exercise 6.2

(b) The equation of value is

$$P = 10^6 \bar{A}^{\;1}_{[40]:\overline{5}|}$$

and, under UDD,

$$\bar{A}^{\;1}_{[40]:\overline{5}|} = \frac{i}{\delta} A^{\;1}_{[40]:\overline{5}|} = \frac{i}{\delta}\left(A_{[40]} - {}_5E_{[40]}\, A_{45}\right) = 0.002594,$$

leading to $P = \$2594$.

(c) Note that $P(1.05^5) < 10^6$, meaning that there is a loss if the death benefit is payable at time five years (or earlier). Hence the probability of a profit is ${}_5p_{[40]} = 0.99704$.

Exercise 6.3

You are given the following extract from a select life table with a four-year select period. A select individual aged 41 purchased a three-year term insurance with a net premium of \$350 payable annually. The sum insured is paid at the end of the year of death.

$[x]$	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	l_{x+4}	$x+4$
[40]	100 000	99 899	99 724	99 520	99 288	44
[41]	99 802	99 689	99 502	99 283	99 033	45
[42]	99 597	99 471	99 628	99 030	98 752	46

Use an effective rate of interest of 6% per year to calculate@
(a) the sum insured, assuming the equivalence principle,
(b) the standard deviation of L_0, and
(c) $\Pr[L_0 > 0]$.

Exercise 6.3

Let B be the benefit of a 3-year term insurance
Income: Premium: 350 paid annualy (if alive)!
Net random loss:

$$L_0^n = Bv^{K[41]+1} \times 1\left(K_{[41]+1} \leq 3\right) - P\ddot{a}_{\overline{\min(K_{41}+1,3)}|}$$

$$E\left(L_0^n\right) = BA^{1}_{[41]:\overline{3}|} - P\ddot{a}_{[41]:\overline{3}|}$$

Exercise 6.3

Alternative way of writing: let $K = K_{[41]}$ with $K = \{0,1,2\}$

$$L_0^n = \begin{cases} Bv^{K+1} - P\ddot{a}_{\overline{K+1}|} \text{ for } K = \{0,1,2\} \\ -P\ddot{a}_{\overline{2}|} \text{ for } K \geq 3 \end{cases}$$

Exercise 6.3

Equivalence principle:

$$E\left(L_0^n\right) = BA^1_{[41]:\overline{3}|} - P\ddot{a}_{[41]:\overline{3}|} = 0$$

with:

$$BA^1_{[41]:\overline{3}|} = B \sum_{k=\{0,1,2\}} v^{k+1} \ {}_{k|}q_{[41]}$$

$$P\ddot{a}_{[41]:\overline{3}|} = 350 \sum_{k=\{0,1,2,\geq 3\}} \ddot{a}_{\overline{min(k+1,2)}|} \ {}_{k|}q_{[41]}$$

Exercise 6.3

${}_{k|}q_{[41]}$ is the death rate: $P\left[K = k\right]$

Remember also: $E\left(\ddot{a}_{\overline{K+1}|}\right) = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|}\ {}_{k|}q_x \equiv \sum_{k=0}^{\infty} v^{k+1}\ {}_{k}p_x$

Exercise 6.3

The death rate

$$\begin{aligned}
\Pr[K=0] &= \frac{d_{[41]}}{\ell_{[41]}} = \frac{113}{99802} \\
\Pr[K=1] &= \frac{d_{[41]+1}}{\ell_{[41]}} = \frac{187}{99802} \\
\Pr[K=2] &= \frac{d_{[41]+2}}{\ell_{[41]}} = \frac{219}{99802} \\
\Pr[K\geq 3] &= 1-\frac{113+187+219}{99802}
\end{aligned}$$

Exercise 6.3

k	(1)	(2)	(3)	(1) × (2)	350 × (1) × (3)
0	0.00113224	0.943396	1.00000	0.00107	0.39628
1	0.00187371	0.889996	1.94340	0.00167	1.27448
2	0.00219434	0.839619	2.83339	0.00184	2.17610
≥ 3	0.99479970	0.000000	2.83339	0.00000	986.53036
sum	1.00000			0.004578162	990.3772

where

$$
\begin{aligned}
(1) &= \Pr[K = k] \text{ for } k = 0, 1, 2 \text{ and } \Pr[K \geq 3] \text{ otherwise} \\
(2) &= v^{k+1} \text{ for } k = 0, 1, 2 \text{ and } 0 \text{ otherwise} \\
(3) &= \ddot{a}_{\overline{k+1}|} \text{ for } k = 0, 1, 2 \text{ and } \ddot{a}_{\overline{2}|} \text{ otherwise}
\end{aligned}
$$

Exercise 6.3

By the equivalence principle with $\mathrm{E}[L_0] = 0$, we have

$$B = \frac{990.3772}{0.004578162} = 216,326.4.$$

One can also verify that you can compute the death benefit based on

$$B = \frac{350\, A^{\;\;1}_{[41]:\overline{3}|}}{\ddot{a}_{[41]:\overline{3}|}} = \frac{(350)(2.829649)}{0.004578162} = 216,326.4.$$

(b) The loss-at-issue random variable now can be expressed as

$$L_0 = 216326.4 v^{K+1} - 350\, \ddot{a}_{\overline{K+1}|} \quad \text{for } K = 0, 1, 2$$

and $L_0 = -350\, \ddot{a}_{\overline{2}|}$ for $K \geq 3$. Details for calculating standard deviation of L_0 are summarized in the table below:

k	$\Pr[K=k]$	loss	loss $\times \Pr[K=k]$	loss^2 $\times \Pr[K=k]$
0	0.00113224	203731.4910	230.6733	46995419.0
1	0.00187371	191849.5198	359.4703	68964214.7
2	0.00219434	180640.1130	396.3867	71603337.1
≥ 3	0.99479970	-991.6874	-986.5304	978329.8
sum	1.00000		0.0000	188541300.0

We indeed do not need the calculation of $\mathrm{E}[L_0]$ because according to the equivalence principle, this is zero, as also confirmed in the table above. Hence, the standard deviation of L_0 is

$$\mathrm{SD}[L_0] = \sqrt{\mathrm{Var}[L_0]} = \sqrt{188541300.0} = 13{,}731.03.$$

Exercise 6.3

(c) Note from the table that the loss-at-issue is positive for $k = 0, 1, 2$; always a positive loss if death is prior to the end of the term of the contract. Thus, the required probability is

$$\Pr[L_0 > 0] = \Pr[K \leq 2] = \frac{(113 + 187 + 219)}{99802} = 0.005200297.$$

Exercise 6.5

Consider a 10-year annual premium term insurance issued to a select life aged 50, with sum insured \$100 000 payable at the end of the year of death.

(a) Write down an expression for the net future loss random variable.

(b) Calculate the net annual premium.

(a) Let P be the annual premium. Then

$$L_0 = \begin{cases} 100\,000\, v^{K_{[50]}+1} - P\, \ddot{a}_{\overline{\min(K_{[50]}+1,10)}|} & \text{if } K_{[50]} < 10, \\ -P\, \ddot{a}_{\overline{\min(K_{[50]}+1,10)}|} & \text{if } K_{[50]} \geq 10, \end{cases}$$

or, using an indicator random variable,

$$L_0 = 100\,000\, v^{K_{[50]}+1}\, I(K_{[50]} < 10) - P\, \ddot{a}_{\overline{\min(K_{[50]}+1,10)}|}\,.$$

Exercise 6.5

(b) The equation of value is

$$P\,\ddot{a}_{[50]:\overline{10}|} = 100\,000\,A^{\;\;1}_{[50]:\overline{10}|}$$

and as

$$\ddot{a}_{[50]:\overline{10}|} = \ddot{a}_{[50]} - {}_{10}E_{[50]}\,\ddot{a}_{60} = 8.05643$$

Reminder:

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - \; v^n \;{}_n p_x \, \ddot{a}_{x+n}$$

and

$$A^{\;1}_{[50]:\overline{10|}} = A_{[50]:\overline{10|}} - {}_{10}E_{[50]} = 1 - d\,\ddot{a}_{[50]:\overline{10|}} - {}_{10}E_{[50]} = 0.01438,$$

we find $P = \$178.50$.

A select life aged 45 purchases a fully discrete 20-year endowment insurance with sum insured \$100 000. Calculate the annual premium using the following assumptions:
i) Commission is 10% of the first premium and 2% of each subsequent premium.
ii) Other expenses are \$50 at issue and \$8 at each subsequent date.
Mortality follows Standard Select Table and $i = 5\%$.

Let P be the annual premium. We equate the EPV of premiums with the EPV of the sum insured plus expenses. When the cash flows are complicated, it is often convenient to value each element separately.

EPV of premiums:

$$P\,\ddot{a}_{[45]:\overline{20|}} = 12.9409\,P.$$

EPV of endowment insurance benefit:

$$100\,000\,A_{[45]:\overline{20|}} = 38\,376.55.$$

Exercise 6.6

Reminder:

$$\text{Term insurance: } A^1_{x:\overline{n|}} = A_x - {}_nE_x A_{x+n}$$

$$\text{Endowment insurance} A_{x:\overline{n|}} = A^1_{x:\overline{n|}} + {}_nE_x$$

$$\text{Endowment insurance: } A_{x:\overline{n|}} = A_x -_n E_x A_{x+n} +_n E_x$$

Endowment insurance (continuos case: benefit paid immediately on death:

$$\bar{A}_{x:\overline{n|}} = \frac{i}{\delta}\left(A_x -_n E_x A_{x+n}\right) +_n E_x$$

<u>EPV of initial and renewal expenses:</u>

$$0.08P + 42 + (0.02P + 8)\,\ddot{a}_{[45]:\overline{20}|} = 0.3388\,P + 145.53,$$

where we have split the initial expenses as $0.1P = 0.08P + 0.02P$ and $50 = 42 + 8$. The equation of value is

$$12.9409\,P = 38\,376.55 + 0.3388\,P + 145.53,$$

giving $P = \$3056.80$.

Exercise 6.7

Determine the annual premium for a 20-year term insurance with sum insured \$100 000 payable at the end of the year of death, issued to a select life aged 40 with premiums payable for at most 10 years, with expenses, which are incurred at the beginning of each policy year, as follows:

	Year 1		Years 2+	
	% of premium	Constant	% of premium	Constant
Taxes	4%	0	4%	0
Sales commission	25%	0	5%	0
Policy maintenance	0%	10	0%	5

Assume that mortality follows the Standard Select Life Table and $i = 5\%$

Exercise 6.7

Let P be the annual premium.

EPV of premiums:

$$P\,\ddot{a}_{[40]:\overline{10|}} = 8.08705\,P.$$

EPV of death benefit:

$$100\,000\,A^{\;1}_{[40]:\overline{20|}} = 1453.58.$$

EPV of taxes and commission:

$$0.09\,P\,\ddot{a}_{[40]:\overline{10|}} + 0.2P = 0.92783\,P.$$

EPV of policy maintenance costs:

$$5\,\ddot{a}_{[40]:\overline{20|}} + 5 = 69.97.$$

Equate the EPVs of income and outgo to give $P = \$212.81$.

Exercise 6.8

A fully discrete whole life insurance with unit sum insured is issued to (x). Let L_0 denote the net future loss random variable with the premium determined by the equivalence principle. You are given that $V\left[L_0\right] = 0.75$. Let L_0^* denote the net future loss random variable with the premium determined such that $E\left[L_0^*\right] = -0.5$. Calculate $V\left[L_0^*\right]$.

Exercise 6.8

We know that

$$L_0 = v^{K_x+1} - P\,\ddot{a}_{\overline{K_x+1|}} = v^{K_x+1}(1 + P/d) - P/d$$

and

$$L_0^* = v^{K_x+1} - P^*\,\ddot{a}_{\overline{K_x+1|}} = v^{K_x+1}(1 + P^*/d) - P^*/d.$$

Hence

$$\mathrm{V}[L_0] = (1 + P/d)^2\,\mathrm{V}\left[v^{K_x+1}\right] \quad \text{and} \quad \mathrm{V}[L_0^*] = (1 + P^*/d)^2\,\mathrm{V}\left[v^{K_x+1}\right]$$

so that

$$\mathrm{V}[L_0^*] = \frac{(1 + P^*/d)^2}{(1 + P/d)^2}\mathrm{V}[L_0] = \left(\frac{d + P^*}{d + P}\right)^2 \mathrm{V}[L_0].$$

Also

$$\mathrm{E}[L_0] = 0 = A_x - P\,\ddot{a}_x = 1 - (P + d)\,\ddot{a}_x$$

and

$$\mathrm{E}[L_0^*] = -0.5 = A_x - P^*\,\ddot{a}_x = 1 - (P^* + d)\,\ddot{a}_x$$

so that

$$\frac{(P + d)\,\ddot{a}_x}{(P^* + d)\,\ddot{a}_x} = \frac{1}{1.5} = \frac{P + d}{P^* + d}.$$

Thus

$$\mathrm{V}[L_0^*] = 1.5^2\,\mathrm{V}[L_0] = 1.6875.$$

Exercise 6.9

A life insurance company issues a 10-year term insurance policy to a life aged 50, with sum insured $\$100,000$. Level premiums are paid monthly in advance throughout the term. You are given the following premium assumptions.
i) Commission is initial 20% of each premium payment in the first year, and 5% of all premiums after the first year.
ii) Additional initial expenses $\$250$.
iii) Claim expenses are $\$250$
iv) The sum insured and claim expenses are payable one month after the date of death.
Mortality follows the Standard Select Life Table with UDD between integer ages. and $i = 5\%$.
Calculate the gross monthly premium.

Exercise 6.9

Let P denote the total annual premium.

EPV of premiums less premium expenses:

$$P\left(0.95\,\ddot{a}^{(12)}_{[50]:\overline{10|}} - 0.15\,\ddot{a}^{(12)}_{[50]:\overline{1|}}\right) = 7.3321\,P.$$

Note that the premium related expenses are 20% of all the premiums in the first year, not just the first premium.

Exercise 6.9

Reminder from MTH5124:

$$\boxed{\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)\,.}$$

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_x^{(m)} - v^n{}_np_x\,\ddot{a}_{x+n}^{(m)}$$

$$\begin{aligned}
\ddot{a}_{x:\overline{n}|}^{(m)} &\approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x) \\
&\quad - v^n{}_np_x\left(\ddot{a}_{x+n} - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_{x+n})\right) \\
&= \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}\left(1 - v^n{}_np_x\right) \\
&\quad - \frac{m^2-1}{12m^2}\left(\delta + \mu_x - v^n{}_np_x(\delta + \mu_{x+n})\right).
\end{aligned}$$

Exercise 6.9

EPV of death benefit + claim expenses + other expenses:

$$(100\,000 + 250)\, \bar{A}^{\;1}_{[50]:\overline{10}|}\, v^{\frac{1}{12}} + 100.$$

Exercise 6.9

The $v^{\frac{1}{12}}$ term allows for the 1-month delay in paying claims.

Using UDD, we have

$$\bar{A}^{\ 1}_{[50]:\overline{10|}} = \frac{i}{\delta} A^{\ 1}_{[50]:\overline{10|}} = 0.01474$$

which gives the EPV of the benefits and non-premium expenses of 1572.09, leading to $P = \$214.41$. The monthly premium is thus \$17.87.