Actuarial Mathematics II
MTH5125

Revision: Annuities
Chapter 5 (MH)

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Spring Term

Annuities

- **Life annuity**: series of payments as long as a given person is alive on the payment dates.
- Payments:
  - at regular intervals,
  - (usually) of the same amount.
- Used for calculating:
  - pension benefits,
  - premiums,
  - policy values.

Cash flow notations

- The series of cash flows (c_k, k), $k = m, \ldots, n$, is denoted by:

$$\boxed{\sum_{k=m}^{n} (c_k, k)}$$

- The continuous stream of payments $c_\tau \, d\tau$ in any infinitesimal subinterval $(\tau, \tau + d\tau)$ of (s, t), is denoted by:

$$\boxed{\int_s^t (c_\tau d\tau, \ \tau)}$$

- Convention:

$$\sum_{k=m}^{n} (c_k, k) = (0,0) \text{ if } m > n \text{ and } \int_s^t (c_\tau d\tau, \ \tau) = (0,0) \text{ if } s > t$$

- In previous notations, m, n, c_k, c_τ, s and t may be deterministic or random.
- Similar notations and conventions for series of cash flows with $1/m$-thly payments.

Annuities-certain

- Annuity-due:

$$\ddot{a}_{\overline{n}|} = 1 + v + ... + v^{n-1} = \frac{1 - v^n}{d} \tag{5.1}$$

- Annuity-immediate:

$$a_{\overline{n}|} = v + ... + v^n = \frac{1 - v^n}{i}$$

- Continuous annuity:

$$\bar{a}_{\overline{n}|} = \int_0^n v^t dt = \frac{1 - v^n}{\delta} \tag{5.2}$$

- Annuity-due with $1/m$-thly payments:

$$\ddot{a}^{(m)}_{\overline{n}|} = \frac{1}{m}\left(1 + v^{\frac{1}{m}} + ... + v^{n-\frac{1}{m}}\right) = \frac{1 - v^n}{d^{(m)}}$$

- Annuity-immediate with $1/m$-thly payments:

$$a^{(m)}_{\overline{n}|} = \frac{1}{m}\left(v^{\frac{1}{m}} + ... + v^{n-\frac{1}{m}} + v^n\right) = \frac{1 - v^n}{i^{(m)}}$$

Annual life annuities: whole-life annuity-due

- Consider an annuity underwritten to (x) at time 0. It pays 1 annually in advance as long as (x) is alive.
- Benefit cash flow:

$$\sum_{k=0}^{K_x} (1, k)$$

- Present value:

$$Y = 1 + v + \dots + v^{K_x} = \ddot{a}_{\overline{K_x+1}|} = \frac{1 - v^{K_x+1}}{d}$$

- Actuarial value:

$$\boxed{\ddot{a}_x \overset{\text{not.}}{=} \mathbb{E}[Y] = \frac{1-\mathbb{E}\left[v^{K_x+1}\right]}{d} = \frac{1-A_x}{d}} \qquad (5.3)$$

Annual life annuities: whole-life annuity-due

- Benefit cash flow:

$$\sum_{t=0}^{K_x} (1,\ t) = \sum_{t=0}^{\infty} \left(1_{\{T_x>t\}}, t\right)$$

- Present value:

$$Y = \sum_{t=0}^{\infty} v^t\ 1_{\{T_x>t\}}$$

- Actuarial value:

$$\boxed{\ddot{a}_x = \mathbb{E}\left[Y\right] = \sum_{t=0}^{\infty} v^t\ {}_t p_x} \tag{5.5}$$

Annual life annuities: whole-life annuity-due

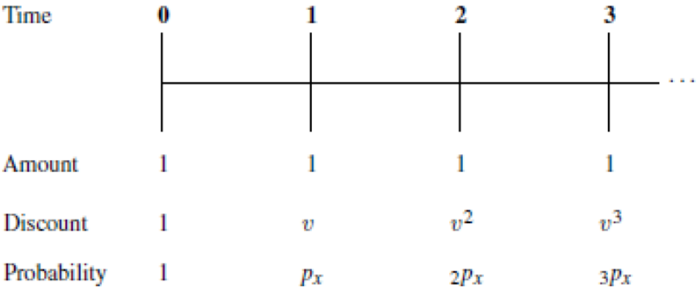

Figure 5.1 Time-line diagram for whole life annuity-due.

Annual life annuities: whole-life annuity-due

- Benefit cash flow:

$$\sum_{k=0}^{K_x} (1, k)$$

- Present value:

$$Y = \ddot{a}_{\overline{K_x+1}|}$$

- Actuarial value:

$$\boxed{\ddot{a}_x = \mathbb{E}[Y] = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} \times {}_{k|}q_x} \qquad (5.6)$$

Example 5.1

- Show algebraically that

$$\sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1|}} \times {}_{k|}q_x = \sum_{k=0}^{\infty} v^k \; {}_kp_x$$

- Proof:

$$\begin{aligned}
\sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1|}} \times {}_{k|}q_x &= \sum_{k=0}^{\infty} \left(\sum_{t=0}^{k} v^t \right) \times {}_{k|}q_x \\
&= \sum_{t=0}^{\infty} v^t \left(\sum_{k=t}^{\infty} {}_{k|}q_x \right) \\
&= \sum_{t=0}^{\infty} v^t \; {}_tp_x
\end{aligned}$$

Annual life annuities: term annuity-due

- Consider an annuity underwritten to (x) at time 0. It pays 1 at times $0, 1, \ldots, n-1$, provided (x) is alive.
- Benefit cash flow:

$$\sum_{t=0}^{\min(K_x, n-1)} (1, t)$$

- Present value:

$$Y = 1 + v + ... + v^{\min(K_x, n-1)} = \ddot{a}_{\overline{min(K_x+1,n)}|} = \frac{1 - v^{\min(K_x+1,n)}}{d}$$

- Actuarial value:

$$\boxed{\ddot{a}_{x:\overline{n}|} \stackrel{\text{not.}}{=} \mathbb{E}\left[Y\right] = \frac{1 - \mathbb{E}\left[v^{\min(K_x+1,n)}\right]}{d} = \frac{1 - A_{x:\overline{n}|}}{d}} \qquad (5.7)$$

Annual life annuities: term annuity-due

- Benefit cash flow:

$$\sum_{t=0}^{\min(K_x, n-1)} (1, t) = \sum_{t=0}^{n-1} \left(1_{\{T_x > t\}}, t\right)$$

- Present value:

$$Y = \sum_{t=0}^{n-1} v^t \; 1_{\{T_x > t\}}$$

- Actuarial value:

$$\boxed{\ddot{a}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}\left[Y\right] = \sum_{t=0}^{n-1} v^t \;\, {}_t p_x} \qquad (5.8)$$

Annual life annuities: term annuity-due

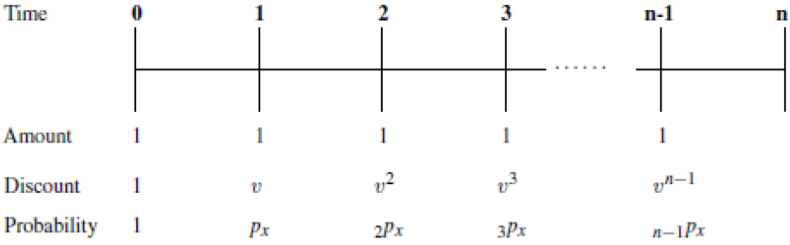

Figure 5.2 Time-line diagram for term life annuity-due.

Annual life annuities: term annuity-due

- Benefit cash flow:

$$\sum_{t=0}^{\min(K_x, n-1)} (1, t)$$

- Present value:

$$Y = \ddot{a}_{\overline{min(K_x+1, n)}|}$$

- Actuarial value:

$$\boxed{\ddot{a}_{x:\overline{n}|} = \mathbb{E}\left[Y\right] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} \times {}_{k|}q_x + {}_np_x \times \ddot{a}_{\overline{n}|}}$$

Annual life annuities: immediate life annuities

- Consider a **whole life immediate annuity** underwritten to (x) at time 0. It pays 1 annually in arrear, as long as (x) is alive.
- Benefit cash flow:

$$\sum_{t=1}^{K_x} (1, t) = \sum_{t=1}^{\infty} \left(1_{\{T_x > t\}}, t\right)$$

- Present value:

$$Y^* = \sum_{t=1}^{\infty} v^t \; 1_{\{T_x > t\}}$$

- Actuarial value:

$$\boxed{a_x \overset{\text{not.}}{=} \mathbb{E}\left[Y^*\right] = \ddot{a}_x - 1} \qquad (5.9)$$

Annual life annuities: immediate life annuities

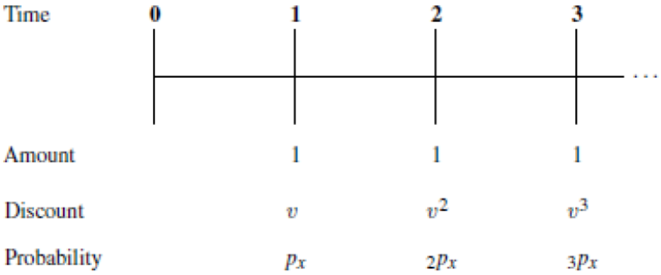

Figure 5.3 Time-line diagram for whole life immediate annuity.

Annual life annuities: immediate life annuities

- Consider a **n-year term immediate annuity** underwritten to (x) at time 0. It pays 1 at times $1, 2, \ldots, n$, provided (x) is alive.
- Benefit cash flow:

$$\sum_{t=1}^{\min(K_x,n)} (1, t) = \sum_{t=1}^{n} \left(1_{\{T_x>t\}}, t\right)$$

- Present value:

$$Y^* = \sum_{t=1}^{n} v^t \; 1_{\{T_x>t\}}$$

- Actuarial value:

$$\boxed{a_{x:\overline{n|}} \stackrel{\text{not.}}{=} \mathbb{E}\left[Y^*\right] = \sum_{t=1}^{n} v^t \;\, {}_t p_x} \tag{5.11}$$

- Relation:

$$\boxed{a_{x:\overline{n|}} = \ddot{a}_{x:\overline{n|}} - 1 + v^n \;\, {}_n p_x} \tag{5.12}$$

Annual life annuities: immediate life annuities

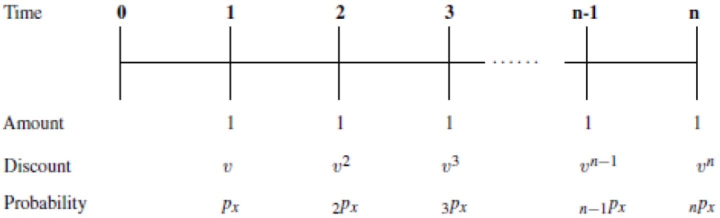

Figure 5.4 Time-line diagram for term life immediate annuity.

Annuities payable 1/m-thly

- <u>Recall</u>: Future lifetime of (x) in years, rounded down to the lower 1/m-th of the year:

$$K_x^{(m)} = \frac{1}{m} \lfloor m T_x \rfloor$$

- <u>Recall</u>: Annuity-due with $1/m$-thly payments:

$$\ddot{a}^{(m)}_{\overline{(j+1)/m|}} = \frac{1}{m} \sum_{k=0}^{j} v^{k/m} = \frac{1 - v^{\frac{j+1}{m}}}{d^{(m)}}$$

Annuities payable 1/m-thly: whole life annuities payable 1/m-thly

- Consider an annuity underwritten to (x) at time 0. It pays an amount of 1 per year, payable in advance m times per year, throughout the lifetime of (x).
- Benefit cash flow:

$$\sum_{k=0}^{m K_x^{(m)}} \left(\frac{1}{m}, \frac{k}{m} \right)$$

- Present value:

$$Y = \frac{1}{m} \sum_{k=0}^{m K_x^{(m)}} v^{k/m} = \ddot{a}^{(m)}_{\overline{K_x^{(m)}+1/m}|} = \frac{1 - v^{K_x^{(m)}+\frac{1}{m}}}{d^{(m)}}$$

- Actuarial value:

$$\boxed{\ddot{a}_x^{(m)} \overset{\text{not.}}{=} \mathbb{E}\left[Y\right] = \frac{1-\mathbb{E}\left[v^{K_x^{(m)}+\frac{1}{m}}\right]}{d^{(m)}} = \frac{1-A_x^{(m)}}{d^{(m)}}} \qquad (5.18)$$

Annuities payable 1/m-thly: whole life annuities payable 1/m-thly

- Actuarial value:

$$\boxed{\ddot{a}_x^{(m)} = \frac{1}{m}\sum_{t=0}^{\infty} v^{t/m} \; {}_{\frac{t}{m}}p_x} \tag{5.19}$$

- Annuity-immediate vs. annuity-due:

$$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m} \tag{5.20}$$

Annuities payable 1/m-thly: whole life annuities payable 1/m-thly

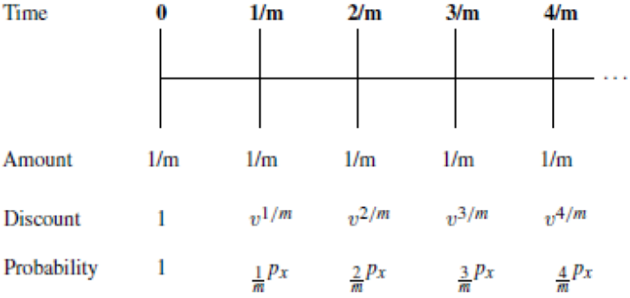

Figure 5.6 Time-line diagram for whole life $1/m$thly annuity-due.

Annuities payable 1/m-thly: term annuities payable 1/m-thly

- Consider an annuity underwritten to (x) at time 0, paying 1 per year, payable in advance m times per year, throughout the lifetime of (x), limited to a maximum of n years.
- Benefit cash flow:

$$\sum_{k=0}^{\min\left(m K_x^{(m)},\ mn-1\right)} \left(\frac{1}{m}, \frac{k}{m}\right)$$

- Present value:

$$Y = \ddot{a}^{(m)}_{\overline{min(K_x^{(m)}+1/m,n)}|} = \frac{1 - v^{\min\left(K_x^{(m)}+1/m,\ n\right)}}{d^{(m)}}$$

- Actuarial value:

$$\boxed{\ddot{a}^{(m)}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}\left[Y\right] = \frac{1-\mathbb{E}\left[v^{\min\left(K_x^{(m)}+1/m,\ n\right)}\right]}{d^{(m)}} = \frac{1-A^{(m)}_{x:\overline{n}|}}{d^{(m)}}} \qquad (5.21)$$

Annuities payable 1/m-thly: term annuities payable 1/m-thly

- Actuarial value:

$$\boxed{\ddot{a}_{x:\overline{n}|}^{(m)} = \frac{1}{m}\sum_{r=0}^{mn-1} v^{r/m} \; {}_{\frac{r}{m}}p_x} \tag{5.22}$$

- Annuity-immediate vs. annuity-due:

$$a_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} - \frac{1}{m}\left(1 - v^n \; {}_np_x\right) \tag{5.23}$$

Annuities payable 1/m-thly: term annuities payable 1/m-thly

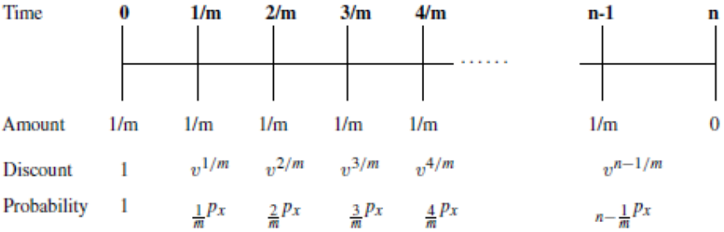

Figure 5.7 Time-line diagram for term life $1/m$thly annuity-due.

Comparison of annuities by payment frequency

Table 5.1. *Values of* a_x , $a_x^{(4)}$, $\bar{a}_x$, $\ddot{a}_x^{(4)}$ *and* $\ddot{a}_x$.

x	a_x	$a_x^{(4)}$	$\bar{a}_x$	$\ddot{a}_x^{(4)}$	$\ddot{a}_x$
20	18.966	19.338	19.462	19.588	19.966
40	17.458	17.829	17.954	18.079	18.458
60	13.904	14.275	14.400	14.525	14.904
80	7.548	7.917	8.042	8.167	8.548

- Technical basis:
  Standard Ultimate Survival Model and interest of 5%.
- Ordering:

$$a_x < a_x^{(4)} < \bar{a}_x < \ddot{a}_x^{(4)} < \ddot{a}_x$$

- Reasons for this ordering:
  - Time value of money.
  - Payments only due upon survival.

Deferred annuities

- Consider an annuity underwritten to (x) at time 0, with lifelong anual payments of 1 in advance, commencing at age $x+u$ (u is a non-negative integer).
- Benefit cash flow:

$$\sum_{k=u}^{K_x} (1, k)$$

- Actuarial value:

$$\boxed{{}_{u|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{u|}}} \tag{5.25}$$

- Relation via actuarial discounting:

$${}_{u|}\ddot{a}_x = {}_{u}E_x \; \ddot{a}_{x+u} \tag{5.26}$$

Deferred annuities

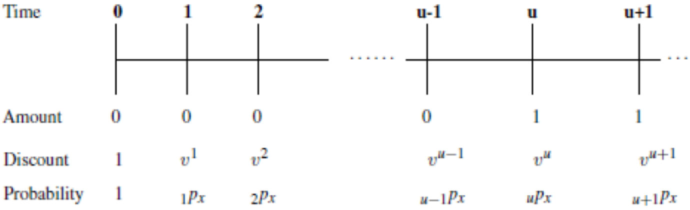

Figure 5.8 Time-line diagram for deferred annual annuity-due.

Deferred annuities

- Deferred term immediate annuity:

$$ {}_{u|}a_{x:\overline{n}|} = {}_{u}E_x \times a_{x+u:\overline{n}|} $$

- Deferred annuity-due payable $1/m$-thly:

$$ {}_{u|}\ddot{a}_x^{(m)} = {}_{u}E_x \times \ddot{a}_{x+u}^{(m)} \tag{5.27} $$

- Term annuity-due:

$$ \ddot{a}_{x:\overline{n}|} = \ddot{a}_x - {}_{n}E_x \times \ddot{a}_{x+n} \tag{5.28} $$

- Term annuity-due payable $1/m$-thly:

$$ \ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_x^{(m)} - {}_{n}E_x \times \ddot{a}_{x+n}^{(m)} \tag{5.29} $$

- Term-annuity with continuous payments:

$$ \bar{a}_{x:\overline{n}|} = \sum_{u=0}^{n-1} {}_{u|}\bar{a}_{x:\overline{1}|} \tag{5.31} $$