Actuarial Mathematics II
MTH5125

Revision: Insurance benefits
Chapter 4 (DHW)

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Spring Term

- Amicable Society for a Perpetual Assurance Office:
  - Founded in London, 1706.
  - First company offering life insurance.
- Society for Equittable Assurances on Lives and Survivorship:
  - Also known as Equittable Life, founded in London, 1762.
  - World's oldest mutual life insurer.

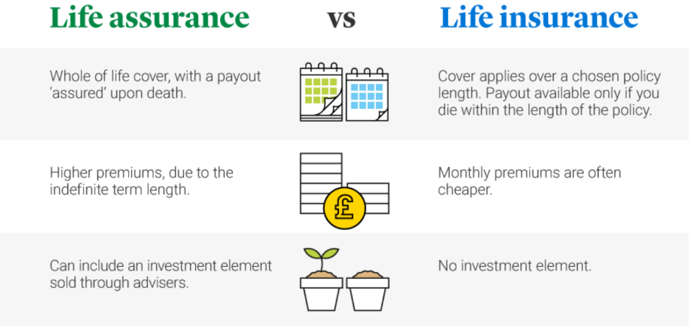
Introduction to life insurance

Samuel Huebner (1882-1964)

- One of the first insurance economists.
- The Economics of Life Insurance(1927):
  - *Not to insure adequately through life insurance is to gamble with the greatest economic risk confronting man. If understood, the gamble is a particularly selfish one, since the blow, in the event the gamble is lost, falls upon an innocent household whose economic welfare should have been the family head's first consideration.*

Life insurance vs life assurance

- https://www.legalandgeneral.com/insurance/life-insurance/definitions/assurance-vs-insurance/



- In DHW - insurance/assurance

Insurance benefits - assumptions

- Technical basis = a set of assumptions used for performing life insurance or pension calculations.
- Technical basis in this chapter (used in the examples):
  - The Standard Ultimate Survival Model:

$$\mu_x = 0.00022 + 2.7 \times 10^{-6} \times 1.124^x$$

  - A constant interest.
  - These are (pedagogically) convenient assumptions.
- Conventions:
  - Time $0 =$ now.
  - Time unit is 1 year.

Insurance benefits - assumptions

Some notions of financial algebra

- i = annual rate of interest.
- $i^{(p)}$ = nominal interest (compounded p times per year):

$$\left(1+\frac{i^{(p)}}{p}\right)^{p}=1+i$$

- δ = force of interest:

$$\delta=\ln\left(1+i\right)$$

- v = yearly discount factor:

$$v=\frac{1}{1+i}=e^{-\delta}$$

- d = discount rate per year:

$$d=1-v=i\ v=1-e^{-\delta}$$

- $d^{(p)}$ = nominal discount rate (compounded p times per year):

$$d^{(p)}=p\left(1-v^{\frac{1}{p}}\right)=i^{(p)}\ v^{1/p}$$

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Insurance benefits - assumptions

Cash flow notations

- The cash flow with payment c at time t is denoted by

$$(c, t)$$

- The cash flow $(\alpha c, t)$ is often denoted by

$$\alpha (c, t)$$

- In previous notations, c and t may be deterministic or random.

Whole life insurance - the continuous case (db payable at instant of death)

- Consider a life insurance underwritten on (x) at time 0, with a payment of 1 at T_x.
- Benefit cash flow:

$$(1, T_x)$$

- Present value:

$$Z = v^{T_x} = e^{-\delta T_x}$$

- Actuarial value (or EPV):

$$\boxed{\overline{A}_x \overset{\text{not.}}{=} \mathbb{E}\left[e^{-\delta T_x}\right] = \int_0^\infty e^{-\delta t}\, {}_t p_x\, \mu_{x+t}\, dt} \qquad (4.1)$$

Whole life insurance - the continuous case

$$\overline{A}_x = \int_0^\infty e^{-\delta s}\, {}_sp_x\, \mu_{x+s}\, ds$$

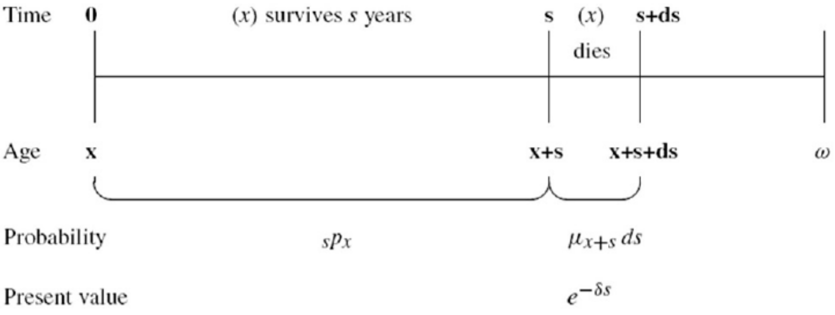

Figure 4.1 Time-line diagram for continuous whole life insurance.

Whole life insurance - the annual case (db payable at end of year of death)

- Consider a life insurance underwritten on (x) at time 0, with a payment of 1 at $K_x + 1$.
- Benefit cash flow:

$$(1, K_x + 1)$$

- Present value:

$$Z = v^{K_x+1}$$

- Actuarial value:

$$\boxed{A_x \overset{\text{not.}}{=} \mathbb{E}[v^{K_x+1}] = \sum_{k=0}^{\infty} v^{k+1} \; {}_{k|}q_x} \tag{4.4}$$

$$A_x = \sum_{k=0}^{\infty} v^{k+1} \; {}_{k|}q_x$$

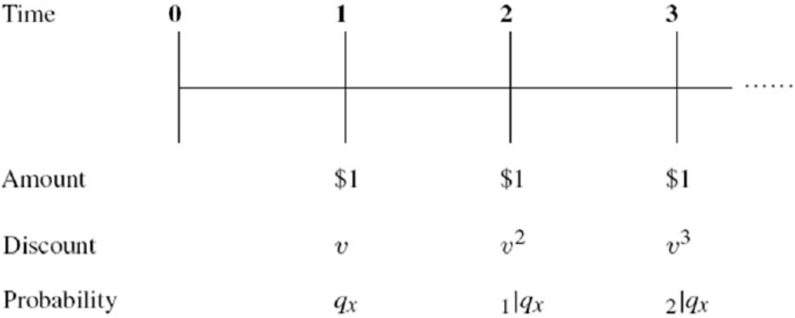

Figure 4.2 Time-line diagram for discrete whole life insurance.

Whole life insurance: the 1/m-thly case

- The floor function:

$$\lfloor r \rfloor = \text{ largest integer smaller than or equal to } r.$$

- The 1/m-thly curtate future lifetime of (x):

$$\boxed{K_x^{(m)} \overset{\text{not.}}{=} \tfrac{1}{m} \lfloor m T_x \rfloor} \tag{4.7}$$

  - $K_x^{(m)}$ = future lifetime of (x) in years, rounded down to the lower 1/m-thly of the year.

- The pdf of $K_x^{(m)}$: For $k = 0, 1, 2, \ldots$,

$$\begin{aligned} \mathbb{P}\left[K_x^{(m)} = \frac{k}{m}\right] &= \mathbb{P}\left[\frac{k}{m} \leq T_x < \frac{k+1}{m}\right] \\ &= {}_{\frac{k}{m}|\frac{1}{m}}q_x \\ &= {}_{\frac{k}{m}}p_x - {}_{\frac{k+1}{m}}p_x \end{aligned}$$

Whole life insurance: the 1/m-thly case

- Consider a life insurance underwritten on (x) at time 0, with a payment of 1 at $K_x^{(m)} + \frac{1}{m}$.
- Benefit cash flow:

$$\left(1, K_x^{(m)} + \frac{1}{m}\right)$$

- Present value:

$$Z = v^{K_x^{(m)} + \frac{1}{m}}$$

- Actuarial value:

$$\boxed{A_x^{(m)} \overset{\text{not.}}{=} \mathbb{E}[v^{K_x^{(m)} + \frac{1}{m}}] = \sum_{k=0}^{\infty} v^{\frac{k+1}{m}} {}_{\frac{k}{m}|\frac{1}{m}}q_x}$$

Whole life insurance: the 1/m-thly case

$$A_x^{(m)} = \sum_{k=0}^{\infty} v^{\frac{k+1}{m}} \; {}_{\frac{k}{m}|\frac{1}{m}}q_x$$

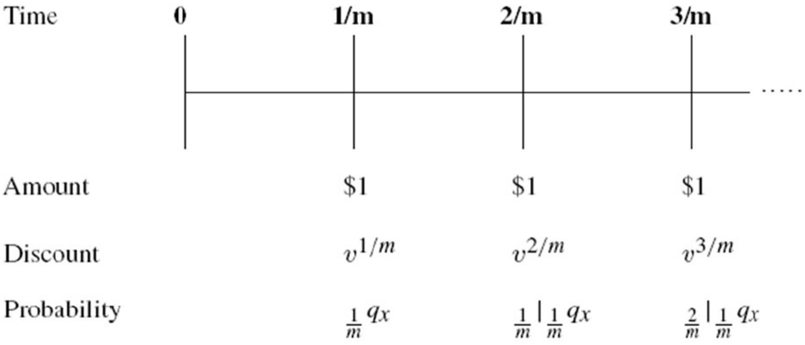

Figure 4.3 Time-line diagram for *m*thly whole life insurance.

Term insurance (policies of duration n): continuous case

Continuous case

- Consider a term life insurance underwritten on (x) at time 0, with a payment of 1 at T_x, provided $T_x \leq n$.
- Benefit cash flow:

$$\left(1_{\{T_x \leq n\}}, T_x\right)$$

- Present value:

$$Z = e^{-\delta T_x} \, 1_{\{T_x \leq n\}}$$

- Actuarial value:

$$\boxed{\overline{A}^{\,1}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}[Z] = \int_0^n e^{-\delta t} \, {}_t p_x \, \mu_{x+t} \, dt} \qquad (4.9)$$

Term insurance (policies of duration n): annual case

Annual case

- Consider a term life insurance underwritten on (x) at time 0, with a payment of 1 at K_x+1, provided $K_x+1 \leq n$.
- Benefit cash flow:

$$\left(1_{\{K_x+1\leq n\}},\ K_x+1\right)$$

- Present value:

$$Z = v^{K_x+1}\ 1_{\{K_x+1\leq n\}}$$

- Actuarial value:

$$\boxed{A^{1}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}[Z] = \sum_{k=0}^{n-1} v^{k+1}\ {}_{k|}q_x} \qquad (4.10)$$

Term insurance:the 1/m-thly case

- Consider a term life insurance underwritten on (x) at time 0, with a payment of 1 at $K_x^{(m)} + \frac{1}{m}$, provided $K_x^{(m)} + \frac{1}{m} \leq n$.
- Benefit cash flow:

$$\left(1_{\left\{K_x^{(m)}+\frac{1}{m}\leq n\right\}},\ K_x^{(m)} + \frac{1}{m}\right)$$

- Present value:

$$Z = v^{K_x^{(m)}+\frac{1}{m}}\ 1_{\left\{K_x^{(m)}+\frac{1}{m}\leq n\right\}}$$

- Actuarial value:

$$\boxed{A^{(m)1}_{\ x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}[Z] = \sum_{k=0}^{mn-1} v^{(k+1)/m}\ {}_{\frac{k}{m}|\frac{1}{m}}q_x} \tag{4.11}$$

Pure endowment

- Consider a pure endowment insurance underwritten on (x) at time 0, with a payment of 1 at time n, provided $T_x > n$.
- Benefit cash flow:

$$\left(1_{\{T_x>n\}},\ n\right)$$

- Present value:

$$Z = v^n \ 1_{\{T_x>n\}}$$

- Actuarial value:

$$\boxed{{}_nE_x \overset{\text{not.}}{=} \mathbb{E}[Z] = v^n \, {}_np_x} \qquad (4.13)$$

- Alternate notation:

$$A_{x:\overset{1}{\overline{n}|}}$$

Continuous case

- Consider an endowment insurance underwritten on (x) at time 0, with a payment of 1 at time T_x, provided $T_x \leq n$, and a payment of 1 at time n, provided $T_x > n$.
- Benefit cash flow:

$$(1, \min(T_x, n)) = \left(1_{\{T_x \leq n\}}, T_x\right) + \left(1_{\{T_x > n\}}, n\right)$$

- Present value:

$$Z = v^{\min(T_x, n)}$$

- Actuarial value:

$$\boxed{\overline{A}_{x:\overline{n}|} \stackrel{\text{not.}}{=} \mathbb{E}[Z] = \overline{A}^{\,1}_{x:\overline{n}|} + {}_nE_x} \tag{4.17}$$

Endowment insurance: annual case

- Consider an endowment insurance underwritten on (x) at time 0, with a payment of 1 at time $K_x + 1$, provided $K_x + 1 \leq n$, and a payment of 1 at time n, provided $K_x + 1 > n$.
- Benefit cash flow:

$$(1, \min(K_x + 1, n)) = \left(1_{\{K_x+1 \leq n\}},\ K_x + 1\right) + \left(1_{\{K_x+1 > n\}},\ n\right)$$

- Present value:

$$Z = v^{\min(K_x+1,n)}$$

- Actuarial value:

$$\boxed{A_{x:\overline{n|}} \overset{\text{not.}}{=} \mathbb{E}[Z] = A^{\ 1}_{x:\overline{n|}} + {}_nE_x} \tag{4.19}$$

Endowment insurance: 1/m-thly case

- Consider an endowment insurance underwritten on (x) at time 0, with a payment of 1 at time $K_x^{(m)} + \frac{1}{m}$, provided $K_x^{(m)} + \frac{1}{m} \leq n$, and a payment of 1 at time n, provided $K_x^{(m)} + \frac{1}{m} > n$.
- Benefit cash flow:

$$\left(1, \min\left(K_x^{(m)} + \frac{1}{m}, n\right)\right)$$

- Present value:

$$Z = v^{\min\left(K_x^{(m)} + \frac{1}{m},\ n\right)}$$

- Actuarial value:

$$\boxed{A^{(m)}{}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}[Z] = A^{(m)1}{}_{x:\overline{n}|} + {}_nE_x} \qquad (4.20)$$

Recursive relations

- whole life insurance: $A_x = vq_x + vp_x\, A_{x+1}$
- term insurance: $A^{1}_{x:\overline{n|}} = vq_x + vp_x\, A^{\;\;1}_{x+1:\overline{n-1|}}$
- endowment insurance: $A_{x:\overline{n|}} = vq_x + vp_x\, A_{x+1:\overline{n-1|}}$

Deferred insurance

- Consider a deferred term insurance underwritten on (x) at time 0, with a payment of 1 at T_x, provided $u < T_x \leq u + n$.
- Benefit cash flow:

$$\left(1_{\{u<T_x\leq u+n\}},\, T_x\right)$$

- Present value:

$$Z = e^{-\delta T_x}\, 1_{\{u<T_x\leq u+n\}}$$

- Actuarial value:

$$\boxed{{}_{u|}\overline{A}^{\,1}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}[Z] = \int_u^{u+n} e^{-\delta t}\, {}_tp_x\, \mu_{x+t}\, dt} \qquad (4.21)$$

Deferred insurance

- Deferred term insurance and actuarial discounting:

$$ {}_{u|}\overline{A}^{1}_{x:\overline{n}|} = {}_{u}E_{x} \times \overline{A}^{\;\,1}_{x+u:\overline{n}|} \tag{4.22} $$

- Deferred vs. immediate insurance:

$$ {}_{u|}\overline{A}^{1}_{x:\overline{n}|} = \overline{A}^{1}_{x:\overline{u+n}|} - \overline{A}^{1}_{x:\overline{u}|} \tag{4.23} $$

- Term insurance in terms of yearly insurances:

$$ \overline{A}^{1}_{x:\overline{n}|} = \sum_{r=0}^{n-1} {}_{r|}\overline{A}^{1}_{x:\overline{1}|} \tag{4.24} $$

Deferred insurance

- Deferred term insurance in terms of yearly insurances:

$$ {}_{u|}\bar{A}^{1}_{x:\overline{n}|} = \sum_{r=u}^{u+n-1} {}_{r|}\bar{A}^{1}_{x:\overline{1}|} $$

- Whole life insurance in terms of yearly insurances:

$$ \bar{A}_x = \sum_{r=0}^{\infty} {}_{r|}\bar{A}^{1}_{x:\overline{1}|} $$

- Term insurance in terms of whole life insurances:

$$ \begin{aligned} A^{1}_{x:\overline{n}|} &= A_x - {}_{n|}A_x \\ &= A_x - {}_{n}E_x \times A_{x+n} \end{aligned} \qquad (4.25) $$