Actuarial Mathematics II
MTH5125

Revision: Survival Models
Chapter 2 (DHW)

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The future lifetime random variable

- **Status** (x):

$$(x) \overset{\text{not.}}{=} \text{a life aged } x, \qquad x \geq 0$$

- **Future lifetime** of (x):

$$T_x$$

  - Assumption: T_x is a *continuous* r.v. on $(0, +\infty)$.

- **Age-at-death** of (x):

$$x + T_x$$

- **Lifetime distribution** of (x):

$$F_x(t) = \mathbb{P}[T_x \leq t]$$

- **Survival function** of (x):

$$S_x(t) = 1 - F_x(t)$$

The future lifetime random variable

- Consider a person (x) with
  - Current future lifetime: T_x.
  - Future lifetime at birth: T_0.
  - Future lifetime at age $y \geq x$, given survival until age y: T_y.
- **Assumption**:
  For any $y \geq x$ and $t \geq 0$, we assume that

$$\boxed{\mathbb{P}[T_y \leq t] = \mathbb{P}[T_0 \leq y + t \mid T_0 > y]} \tag{2.1}$$

- **Interpretation**: Starting from the cdf of T_0, the only additonal information used to determine survival probabilties at age x and beyond is survival or not.
- **Corollary**: For any $t, u \geq 0$, we have that

$$\boxed{\mathbb{P}[T_{x+t} \leq u] = \mathbb{P}[T_x \leq t + u \mid T_x > t]}$$

The future lifetime random variable

- Lifetime distributions F_x and F_0:

$$F_x(t) = \frac{F_0(x+t) - F_0(x)}{S_0(x)} \tag{2.2}$$

- Survival functions S_x and S_0:

$$S_0(x+t) = S_0(x)\ S_x(t) \tag{2.4}$$

- Survival functions S_{x+t} and S_x:

$$S_x(t+u) = S_x(t)\ S_{x+t}(u) \tag{2.5}$$

The future lifetime random variable

- Consider (x) with continuous future lifetime T_x.
- $S_x(t)$ is a **survival function** for (x) if and only if the following conditions are satisfied:
  - Condition 1:
  $$S_x(0) = 1$$
  - Condition 2:
  $$\lim_{t \to +\infty} S_x(t) = 0$$
  - Condition 3:
  $$S_x(t) \text{ is a non-increasing continuous function of } t$$

The future lifetime random variable

- For all **survival functions** $S_x(t)$ in this course, we make the following assumptions:
  - Assumption 1:
$$\frac{d}{dt} S_x(t) \text{ exists for all } t > 0$$
  - Assumption 2:
$$\lim_{t \to +\infty} t \; S_x(t) = 0$$
  - Assumption 3:
$$\lim_{t \to +\infty} t^2 \; S_x(t) = 0$$

- Assumptions 2 and 3 ensure that the mean and the variance of the distribution of T_x exist.

The force of mortality

- Consider a person with survival function at birth $\mathbb{P}[T_0 > t]$.
- The **force-of-mortality** at age x:

$$\boxed{\mu_x \overset{\text{def.}}{=} \lim_{h\to 0+} \frac{\mathbb{P}[T_x \le h]}{h}}$$

- Other expression for μ_x:

$$\mu_x = \lim_{h\to 0+} \frac{\mathbb{P}[T_0 \le x+h \mid T_0 > x]}{h} \tag{2.6}$$

- Intuitive interpretation:

$$\boxed{\mu_x\, dx \approx \mathbb{P}[T_0 \le x+dx \mid T_0 > x]} \tag{2.8}$$

The force of mortality

- μ_x in terms of S_0:

$$\boxed{\mu_x = -\frac{1}{S_0(x)}\frac{d}{dx}S_0(x)} \qquad (2.9)$$

- The pdf of T_x:

$$f_x(t) = \frac{d}{dt}F_x(t) = -\frac{d}{dt}S_x(t)$$

- μ_x in terms of f_0 and S_0:

$$\boxed{\mu_x = \frac{f_0(x)}{S_0(x)}}$$

The force of mortality

- Suppose that x is fixed and t is variable.
- Expression for μ_{x+t}:

$$\boxed{\mu_{x+t} = \frac{f_x(t)}{S_x(t)}} \tag{2.10}$$

- Intuitive interpretation:

$$\boxed{\mu_{x+t}\, dt \approx \mathbb{P}[T_x \leq t + dt \mid T_x > t]}$$

- An expression for $S_x(t)$:

$$S_x(t) = \exp\left(-\int_0^t \mu_{x+s} ds\right) \tag{2.11}$$

Mortality Laws

- **Gompertz' law of mortality**:

$$\boxed{\mu_x = Bc^x, \qquad x > 0}$$

where B and c are constants such that $B > 0$ and $c > 1$.

- **Makeham's law of mortality**:

$$\boxed{\mu_x = A + Bc^x, \qquad x > 0}$$

where A, B and c are constants such that A, $B > 0$ and $c > 1$.

- Both models often provide a good fit to mortality data over certain age ranges, particularly from middle age to early old age.

Actuarial Notations

- **Survival rates**:

$$ {}_t p_x \overset{\text{not.}}{=} \mathbb{P}[T_x > t] = S_x(t) \tag{2.13} $$

- **Mortality rates**:

$$ {}_t q_x \overset{\text{not.}}{=} \mathbb{P}[T_x \leq t] = F_x(t) \tag{2.14} $$

- Deferred mortality rates:

$$ {}_{u|t} q_x \overset{\text{not.}}{=} \mathbb{P}[u < T_x \leq u + t] = S_x(u) - S_x(u+t) \tag{2.15} $$

- Simplified notations for 1 - year probabilities:

$$ p_x \overset{\text{not.}}{=} {}_1 p_x $$

$$ q_x \overset{\text{not.}}{=} {}_1 q_x $$

$$ {}_{u|} q_x \overset{\text{not.}}{=} {}_{u|1} q_x $$

Actuarial notations

- Survival and mortality rate add to 1:

$$ {}_t p_x + {}_t q_x = 1 $$

- Survival rates at different ages:

$$ {}_{t+u} p_x = {}_t p_x \times \ {}_u p_{x+t} \tag{2.16} $$

- Survival rates in terms of one-year survival rates:

$$ {}_n p_x = p_x \times p_{x+1} \times \ldots \times \ p_{x+n-1} $$

- Deferred mortality rates:

$$ {}_{u|t} q_x = {}_u p_x - {}_{u+t} p_x = {}_u p_x \times {}_t q_{x+u} $$

Actuarial notations

- Force-of-mortality at age x:

$$\mu_x = \lim_{h \to 0+} \frac{{}_h q_x}{h} = -\frac{1}{{}_x p_0} \frac{d}{dx} {}_x p_0 \tag{2.17}$$

- Force-of-mortality at age $x + t$:

$$\mu_{x+t} = -\frac{1}{{}_t p_x} \frac{d}{dt} {}_t p_x \tag{2.18}$$

- Density function of T_x:

$$f_x(t) = {}_t p_x \, \mu_{x+t} \tag{2.19}$$

- Survival rate in terms of forces-of-mortality:

$${}_t p_x = \exp\left(-\int_0^t \mu_{x+s} ds \right) \tag{2.20}$$

Actuarial notations

- Death rates and forces-of-mortality:

$$ {}_tq_x = \int_0^t {}_sp_x \; \mu_{x+s} ds \qquad (2.21) $$

- Graphical interpretation:

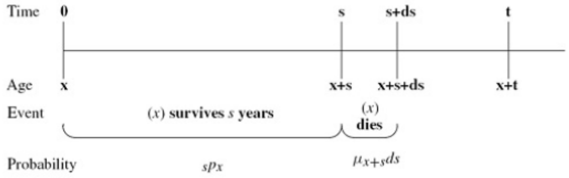

Figure 2.3 Time-line diagram for ${}_tq_x$

- Approximation:

$$ q_x \approx \mu_{x+\frac{1}{2}}, \qquad \text{when } q_x \text{ is small} $$

Actuarial notations

- **Complete expectation of life**:

$$\mathring{e}_x \stackrel{\text{def.}}{=} \mathbb{E}[T_x]$$

- Evaluating $\mathring{e}_x$:

$$\boxed{\mathring{e}_x = \int_0^\infty {}_tp_x\, dt} \quad (2.23)$$

- Second moment of T_x:

$$\mathbb{E}[T_x^2] = 2\int_0^\infty t\ {}_tp_x\, dt \quad (2.24)$$

- Variance of T_x:

$$V[T_x] := \mathbb{E}[T_x^2] - (\mathring{e}_x)^2$$

Curtate future lifetime

- **Curtate future lifetime**:

$$K_x \stackrel{\text{def.}}{=} \lfloor T_x \rfloor$$

- Probability function of K_x:

$$\mathbb{P}\left[K_x = k\right] = {}_k p_x \; q_{x+k}, \qquad k = 0, 1, 2, \ldots$$

- **Curtate expectation of life**:

$$e_x \stackrel{\text{not.}}{=} \mathbb{E}\left[K_x\right]$$

- Evaluating e_x:

$$e_x = \sum_{k=1}^{\infty} {}_k p_x \tag{2.25}$$

- Second moment of K_x:

$$\mathbb{E}[K_x^2] = 2 \sum_{k=1}^{\infty} k \, {}_k p_x - e_x$$