# Brief summary of MTH6113 Mathematical Tools for Asset Management

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## Contents

# 0 Revision of probability theory

Given random variables $X, Y$, we consider

- the expected value $\mathbb{E}(X)$:
  - $\sum_i x_i P(X = x_i)$ for discrete variables with possible values $x_i$;
  - $\int_{\mathbb{R}} x f_X(x)\ \mathrm{d}x$ for the probability density function (pdf) $f_X$.
- the variance $\mathrm{Var}(X) = \mathbb{E}\left((X - \mathbb{E}(X))^2\right)$;
- the standard deviation $\sigma_X = \sqrt{\mathrm{Var}(X)}$;
- the covariance $\mathrm{Cov}(X, Y) = \mathbb{E}\left((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\right)$;
- the correlation $\mathrm{corr}(X, Y) = \mathrm{Cov}(X, Y)/(\sigma_X \sigma_Y)$;
- the distribution function $F_X(x) = P(X \leq x)$;
  - for continuous variables, this is the integral of the density function: $F_X(x) = \int_{-\infty}^{x} f_X(\xi)\ \mathrm{d}\xi$.

With random variables $X, Y, Z$ and a deterministic scalar $a$, we frequently use:

- linearity of the expected value: $\mathbb{E}(aX + Y) = a\,\mathbb{E}(X) + \mathbb{E}(Y)$;
- variance as the covariance with itself: $\mathrm{Var}(X) = \mathrm{Cov}(X, X)$;
- symmetry and scaling of the covariance $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$ and $\mathrm{Cov}(aX, Y) = a\,\mathrm{Cov}(X, Y)$;
- as a result also $\mathrm{Var}(aX) = a^2\,\mathrm{Var}(X)$ and $\sigma_{aX} = a\sigma_X$;
- bilinearity of the covariance: $\mathrm{Cov}(aX + Y, Z) = a\,\mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z)$,
- which yields $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + 2\,\mathrm{Cov}(X, Y) + \mathrm{Var}(Y)$,
- and if $X$ and $Y$ are independent also $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.

# 1 Efficient Market Hypothesis (EMH)

Efficient markets restrict the possibility to strategically make profit that is larger than the market's average. We have discussed three main formulations of market efficiency:

- *Weak form* of the EMH:
  The current stock price reflects all the historical stock prices;
- *Semi-strong form* of the EMH:
  The current stock price reflects all public information;
- *Strong form* of the EMH:
  The current stock price reflects all public and private information.

The weaker formulations are contained in the stronger formulations, i.e.:

$$\text{strong form holds} \;\Rightarrow\; \text{semi-strong form holds} \;\Rightarrow\; \text{weak form holds.}$$

The implication on potential investments is of large interest for us:

1. Assuming *none of the hypothesis holds*, you can find investments, which are based on

   - patterns found in historical stock prices, or
   - any information concerning the company/the market

   and can consistently expect profits that are larger than the market average.

2. Assuming only the *weak form* is valid, you cannot find investments, which consistently yield superior profit and are based on

   - patterns found in historical stock prices,

   however, it can be based on

   - any further information concerning the company/the market.

3. Assuming the *semi-strong form* (hence also the weak form) is valid, you cannot find investments, which consistently yield superior profit and are based on

   - any public information,

   however, it can be based on

   - any private information concerning the company/the market.

4. Assuming the *strong form* (hence also the semi-strong and weak forms) is valid, you cannot find

   - any investment that consistently yield superior profit.

   The only way to increase the expected return is to

   - increasing the risk.

An overview on what can/cannot be used to design superior investment strategies is Table 1.

Empirical evidence of the weak formulation of efficient markets can be found, when considering the autocorrelation of the stock returns $\mathrm{corr}(R_{t+1}, R_t) \approx 0$. Graphically, this can seen in a scatter plot, where no correlation between subsequent returns is noted in Figure 1.

| investment based on | investor may believe in | investor *does not* believe in |
|---|---|---|
| historical stock prices | no EMH | weak form |
| public information | weak form | semi-strong form |
| private information | semi-strong form | strong form |
| investor needs to increase risk, to increase expected payoff | strong form | - |

Table 1: Overview of the EMH

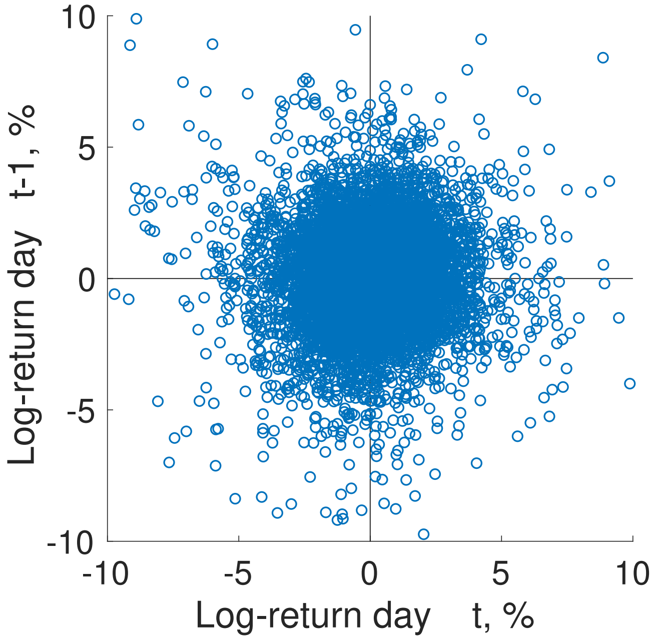


Figure 1: Scatter plot of subsequent plots for GE's returns within over 55 years. The empirical correlation $1,46\%$ is not statistically significant.

## 2 Stochastic models of long-term behaviour of security prices

A consequence of the efficient market hypothesis is the *random walk theorem*, stating that the returns on subsequent days are independent of each other. An important model is the *lognormal model*.

With $(S_t)_{t\in\mathbb{N}}$ the daily stock price, we consider $(X_t)_{t\in\mathbb{N}}$ the daily log-returns $X_t = \log(S_{t+1}/S_t)$. Log-returns for several days are obtained by summing up the daily log-returns:

$$\log(S_t) - \log(S_s) = \sum_{i=s}^{t-1} X_i,$$

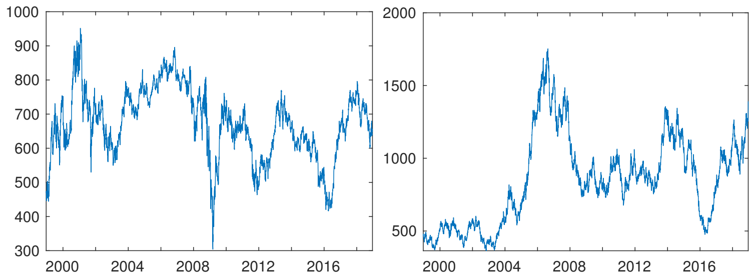


Figure 2: Stock prices. Left: Empirical data; Right: Lognormal model

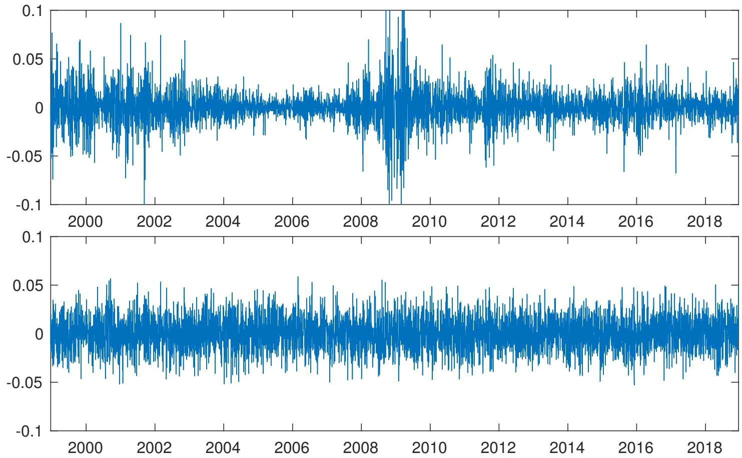


Figure 3: Log-returns Top: Empirical data; Bottom: Lognormal model

i.e. $S_t = S_s \exp(\sum_{i=s}^{t-1} X_i)$ for $s < t$.

The key-assumption for the lognormal model is that

- the daily log-returns $X_t$ are iid (i.e. independent and identically distributed), and that
- this distribution is a normal distribution $\mathcal{N}(\mu, \sigma)$.

An example of the stock price is given in Figure 2.

When comparing the model with empirical data, we see the limitations, in particular:

- Volatility clustering is observed (large squared daily returns are likely to follow each other), but not present in the lognormal model, see Figure 3
- Large losses are underestimated with the lognormal model, see Figures 3 and 4.

A better fit of the data is available with more complex models, e.g. the autoregressive AR(1) process. There the volatility (i.e. standard deviation of the log-returns) is a stationary autoregressive stochastic process:

$$X_t = \mu t + \sigma_t Z_t, \quad Z_t \sim \mathcal{N}(0,1) \text{ iid },$$
$$\sigma_t = \alpha + \beta\sigma_{t-1} + v\epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0,1) \text{ iid }, |\beta| < 1,$$

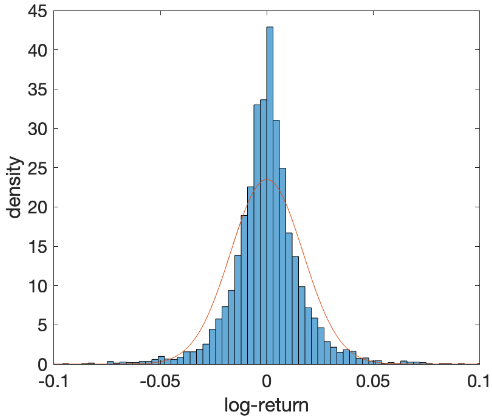


Figure 4: Histogram for log-returns and normal pdf

with $Z_t$ and $\epsilon_t$ being independent of each other and of $\sigma_{t-1}, X_{t-1}$. The autoregressive model introduces a positive correlation of the volatility and hence the magnitude of returns. This way volatility clusters are introduced. Challenges are the fitting the parameters and a more complex evaluation compared to the lognormal model.

# 3 Risk and return

Assessment of risk is one of the most important parts of mathematical finance. Quantified risk can be used to evaluate an investments, as well as optimize a portfolio of assets. We first discuss the dominance of assets based on mean and variance and then discuss various measures of risk.

For the investment evaluation based on mean and variance, we consider each investment as a pair $(\mu, \sigma)$ of the returns mean $\mu = \mathbb{E}(R)$ and standard deviation $\sigma = \sqrt{\mathrm{Var}(R)}$.

**Definition 1.** *An investments* $(\mu_1, \sigma_1)$ dominates *another investment* $(\mu_2, \sigma_2)$, *iff*

$$\mu_1 \geq \mu_2, \quad \sigma_1 \leq \sigma_2,$$

*and one of the inequalities is strict (i.e. not equal). We write* $(\mu_1, \sigma_1) \succ (\mu_2, \sigma_2)$.

An investment is dominated, when another investment has a higher expected payoff with less risk. Note that not all pairs can be ordered. Investments that are not dominated form the efficient subset

**Definition 2.** *Given a set of investments* $A = \{(\mu_i, \sigma_i), i \in \mathcal{I}\}$. *An investment* $(\widehat{\mu}, \widehat{\sigma}) \in A$ *is an element of the efficient subset* $A_{\text{eff}}$, *iff it is not dominated, i.e. there is no* $i \in \mathcal{I}$, *such that* $(\mu_i, \sigma_i) \succ (\widehat{\mu}, \widehat{\sigma})$.

We can use the efficient subset to determine reasonable investments. If we have a given set of investments and we want to invest according to the mean-variance analysis, only elements of the efficient subset are reasonable.

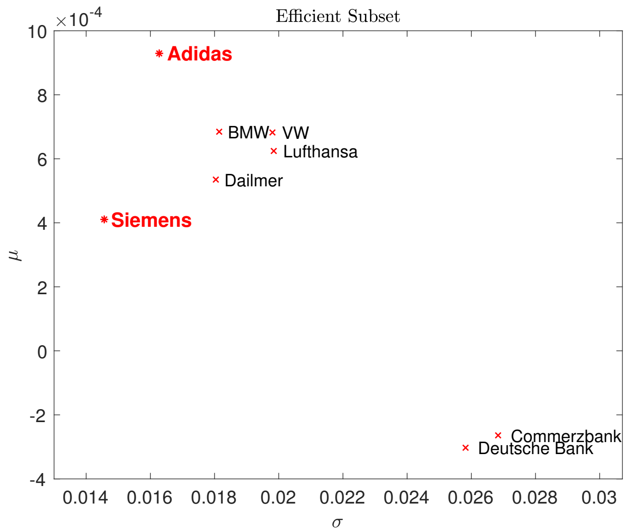


Figure 5: Several stocks in the $\sigma - \mu$-plane and their efficient subset.

We can evaluate the efficient subset by testing pairwise dominance and neglecting all elements that are dominated. Graphically dominance means that the dominating asset lies towards the top left in the $\sigma$-$\mu$ plane, see Figure 5.

## 3.1 Shortfall probability

The variance is a very simple measure of investment risk. While it enables us to easily compare stocks, for a more detailed investigation more advanced risk measures need to be considered. Shortcomings include:

- due to it's dependency on the expected value, assets with a larger expected value may seem riskier although they are not;
- unexpected large gains are valued the same as unexpected large losses;
- the variance does not give any information about the size of the risk or their probability. A likely small loss can have the same variance as a less likely huge loss.

To solve the problems, the *shortfall probability* and the *Value at Risk* can be considered. They answer the questions

- how likely are large losses (shortfall probability);
- how large are likely losses (Value at Risk).

Both are based on the realised loss $L = -R$ (note, that we can use either the return $R$ or the log-return $X$ in the definition of the loss, depending on the situation; the results will differ only slightly). The shortfall probability can best be evaluated using the distribution function of the return $R$: $F_R(x) = P(R \leq x)$:

$$\mathrm{SF}(b, R) = P(L \geq b) = F_R(-b),$$

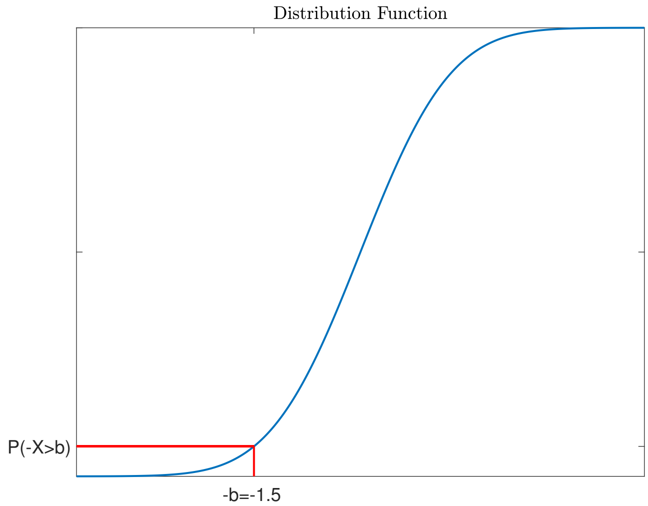


Figure 6: Evaluating the shortfall probability $\mathrm{SF}(X, 1.5)$ using the distribution function (here $X$ log-return)

see Figure 6 for an illustration. The Value at Risk is defined as

$$\mathrm{VaR}_\alpha = \inf\{b : P(L > b) < 1 - \alpha\}.$$

If the distribution function of the return $F_R$ is continuous and strictly increasing, we can use the inverse function to evaluate the value at risk:

$$\mathrm{VaR}_\alpha = -F_R^{-1}(1 - \alpha).$$

Note: we usually evaluate $\mathrm{VaR}_\alpha$ for $\alpha > 0.5$, e.g. 95% or 99%, which yields $1 - \alpha < 0.5$. See Figure 7 for the illustration of the evaluation using the density function.

# 4 Mean-variance portfolio theory

A portfolio is a combined investment into several securities. First, we consider two assets in the framework and evaluate them in terms of mean and variance. For simplicity only two values in time are available: now is t=0 and a future time of interest is t=1.

## 4.1 Notation

**Definition 3.** *In this chapter, we consider a portfolio based on two companies 1 and 2 . This means that we buy/own $x_1$ stocks of company 1 and $x_2$ stocks of company 2.*

*We denote $S^0(0)$ and $S^1(0)$ the current stock price of company 1 and 2, respectively. Since $t = 0$ refers to the present, these are known, deterministic values.*

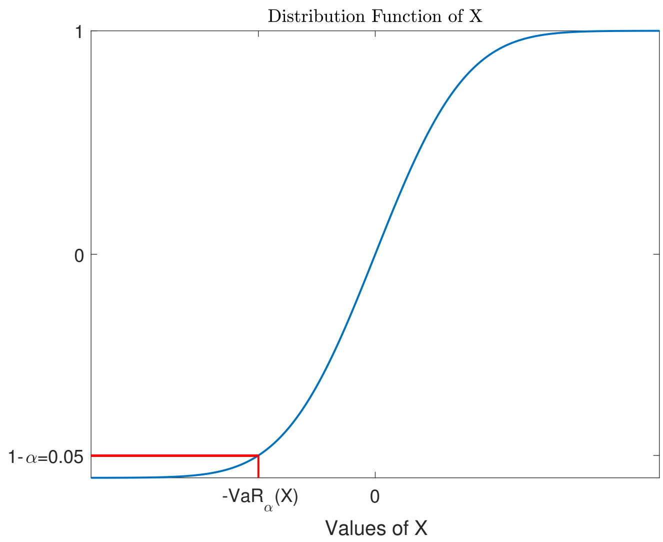


Figure 7: Evaluating the Value at Risk $\text{VaR}_{95\%}$ using the inverse of the distribution function (here $X$ log-return)

*The future stock price is denoted $S^0(1)$ and $S^1(1)$ and is a random variable. We denote the returns expected value by $\mu_1$ and $\mu_2$ and the standard deviations by $\sigma_1$ and $\sigma_2$.*

The portfolio has the current value $P(0) = x_1 S^1(0) + x_2 S^2(0)$ (deterministic) and the future value $P(1) = x_1 S^1(1) + x_2 S^2(1)$ (random variable). Besides the number of stocks it can be relevant which part of your wealth is invested in which stock:

$$w_1 = \frac{x_1 S^1(0)}{P(0)}, \quad w_2 = \frac{x_2 S^2(0)}{P(0)}$$

fulfil $w_1 + w_2 = 1$. This helps us to evaluate the random variable for the return

$$R_{\text{P}} = w_1 R^1 + w_2 R^2,$$

where $R^i = S^i(1)/S^i(0) - 1$, $i = 1, 2$, are the individual returns.

**Theorem 1.** *Based on the return's expectation and variance for both stocks $\mu_1 = \mathbb{E}(R^1)$, $\mu_2 = \mathbb{E}(R^2)$ and $\sigma_1^2 = \text{Var}(R^1)$, $\sigma_2^2 = \text{Var}(R^2)$, we can compute these properties for the portfolio's return:*

$$\mu_{\text{P}} = w_1\mu_1 + w_2\mu_2$$
$$\sigma_{\text{P}}^2 = w_1^2\sigma_1^2 + 2\rho w_1 w_2 \sigma_1\sigma_2 + w_2^2\sigma_2^2$$

An overview of the notation is given in Table 2.

| | Asset 1 | Asset 2 | Portfolio |
|---|---|---|---|
| Nr. of stocks | $x_1$ | $x_2$ | - |
| Portion of total wealth | $w_1$ | $w_2$ | $1 = w_1 + w_2$ |
| Current value | $S^0(0)$ | $S^1(0)$ | $P(0) = x_1 S^1(0) + x_2 S^2(0)$ |
| Future value (stochastic) | $S^1(1)$ | $S^2(1)$ | $P(1) = x_1 S^1(1) + x_2 S^2(1)$ |
| Return (stochastic) | $R^1$ | $R^2$ | $R_{\text{P}} = w_1 R^1 + w_2 R^2$ |
| Expected return | $\mu_1$ | $\mu_2$ | $\mu_{\text{P}} = w_1 \mu_1 + w_2 \mu_2$ |
| Variance of return | $\sigma_1^2$ | $\sigma_2^2$ | $\sigma_{\text{P}}^2 = w_1^2 \sigma_1^2 + 2\rho w_1 w_2 \sigma_1 \sigma_2 + w_2^2 \sigma_2^2$ |
| Correlation (between assets 1 and 2) | $\rho$ | $\rho$ | - |

Table 2: Most important notation and properties of portfolios with two assets

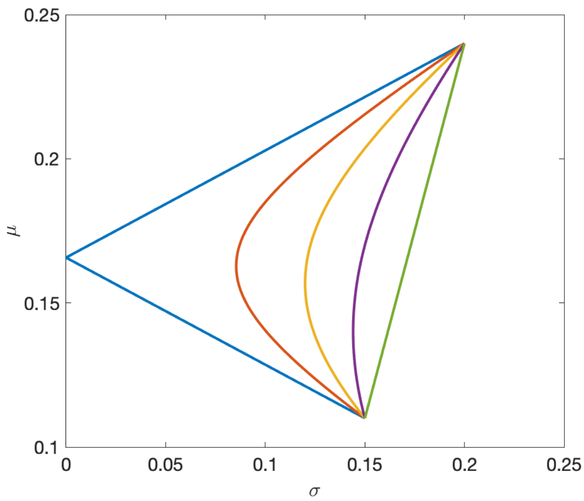


Figure 8: Attainable set for several values of the correlation without short-selling

## 4.2 Analysis in the $\sigma - \mu$-plane

The attainable set is a curve (for $\rho \in (-1, 1)$ a hyperbola) in the $\sigma - \mu$-plane consisting of all possible portfolios:

$$A^{\text{att}} = \Big\{ \big(\sigma_{\text{P}}(w_1), \mu_{\text{P}}(w_1)\big) : w_1 \in [0, 1] \Big\},$$

where $\sigma_{\text{P}} = \sqrt{w_1^2 \sigma_1^2 + 2\rho w_1 (1 - w_1) \sigma_1 \sigma_2 + (1 - w_1)^2 \sigma_2^2}$ and $\mu_{\text{P}} = w_1 \mu_1 + (1 - w_1)\mu_2$. An example, including the influence of $\rho$ is given in Figure 8. In markets with short-selling, the restriction $w \in [0, 1]$ can be dropped:

$$A^{\text{att}}_{\text{short}} = \Big\{ \big(\sigma_{\text{P}}(w_1), \mu_{\text{P}}(w_1)\big) : w_1 \in \mathbb{R} \Big\},$$

see Figure 9 for an example.

The Minimal Variance Portfolio (MVP) is the portfolio which has the smallest value for the variance. It can be found by solving a minimisation problem.

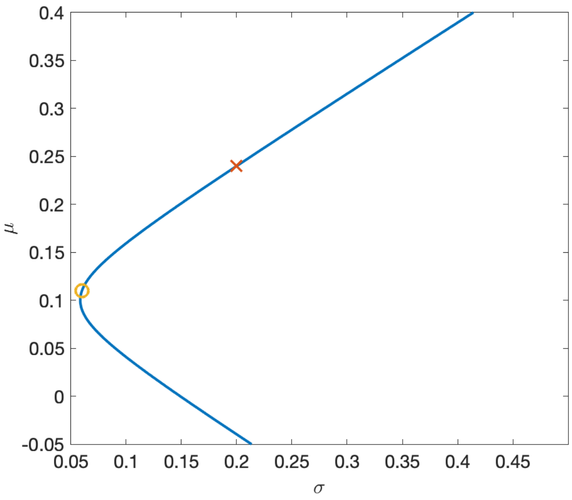


Figure 9: Attainable set for a fixed correlation with short-selling.

For $\rho \in (-1, 1)$ it can be computed using

$$w_1 = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

in a market wit short-selling. Without short-selling, the value is $w_1 = 0$ if the above formula is negative and $w_1 = 1$ if it is larger than one.

The efficient subset of the attainable set is called *efficient frontier*. It consists of the half-line starting at the MVP with growing expectation. With short selling this is a closed set and special cases can appear when the MVP would involve short-selling (e.g. whole attainable set is efficient, or only one portfolio). The efficient frontier for several values of $\rho$ can be found in Figure 10.

## 4.3 Adding a risk-free interest

A risk-free asset is denoted as $(0, \mu_0)$. The attainable line considering the risk-free asset and a risky asset is a line combining both assets. This way, we can construct the attainable set, which is now a surface, see Figure 11 for a sampling of some of these lines. The final surfaces are given in Figure 12 and 13.

An important portfolio is the *market portfolio*, which defines the efficient subset when short-selling is allowed. For $\mu_0 < \mu_{\text{MVP}}$ and $\rho \in (0, 1)$, it is given as

$$w_1 = \frac{c}{c + d},$$

with $c = \sigma_2^2(\mu_1 - \mu_0) - \rho\sigma_1\sigma_2(\mu_2 - \mu_0)$ and $d = \sigma_1^2(\mu_2 - \mu_0) - \rho\sigma_1\sigma_2(\mu_1 - \mu_0)$.

The line connecting the risk-free portfolio with the market portfolio is called *capital market line* and is the efficient subset. When short-selling restrictions apply several special cases need to be distinguished.

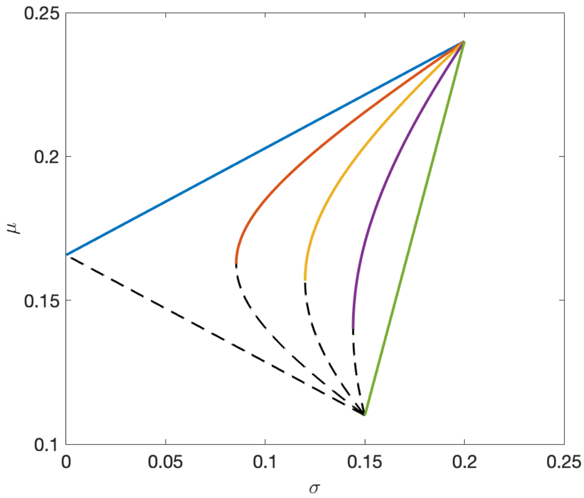


Figure 10: Efficient frontiers for several values of the correlation without short-selling

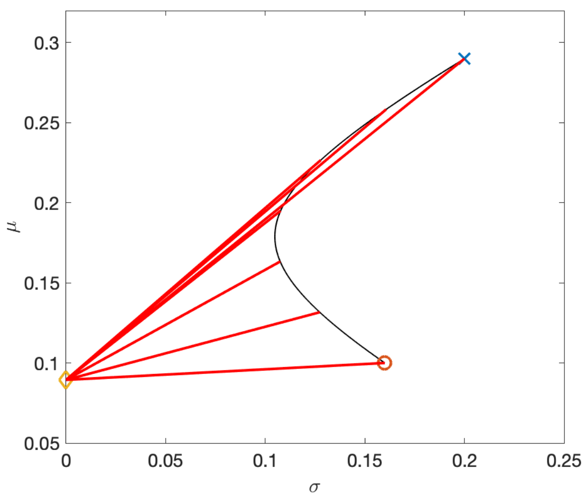


Figure 11: Some attainable portfolios including a risk-free interest.

# 5 Multi-factor models

Multi-factor models are stochastic stock price models of several stocks. Instead of modelling each of the pairwise correlations, we assume the dependency of a set of *factors*:

$$\mathbf{X} = \mathbf{a} + \mathbf{BF} + \varepsilon,$$

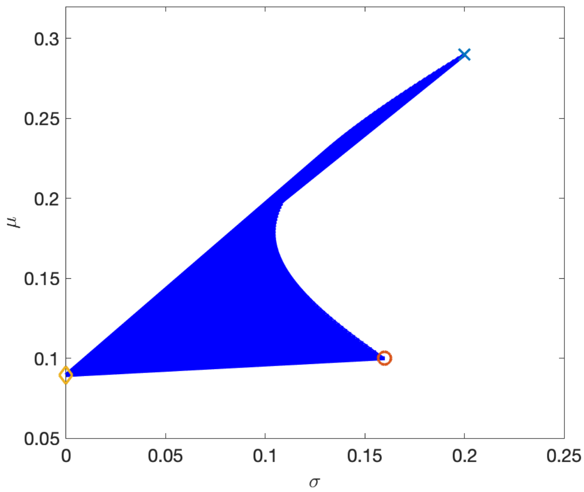


Figure 12: Attainable set including a risk-free interest but no short-selling

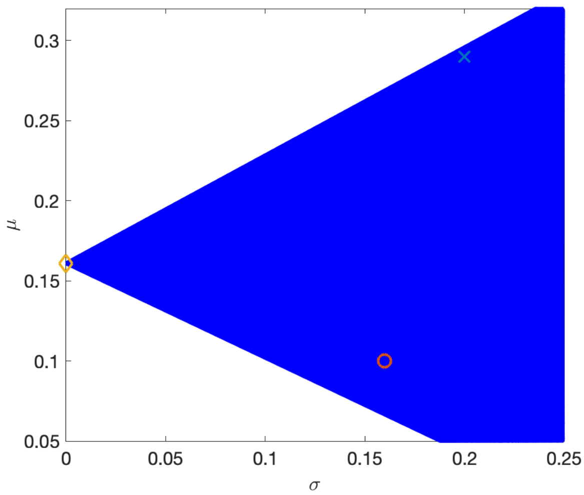


Figure 13: Attainable set including a risk-free interest and short-selling

where $\mathbf{X} = (X^i)_{i=1,\dots,n}$ is the vector of the modelled log-returns on the assets. $\mathbf{F} = (F_1, \dots, F_p)^\top \in \mathbb{R}^p$ is a vector of random variables modelling the $p$-factors, where $\mathbf{B} \in \mathbb{R}^{d \times p}$ is a deterministic matrix of the factor loadings. $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_d)^\top \in \mathbb{R}^d$ denotes the (stochastic) idiosyncratic risk, i.e. the individual randomness of each stock. Finally $\mathbf{a} \in \mathbb{R}^d$ is a constant. We assume $\operatorname{corr}(F_j, \varepsilon_i) = 0$.

The factor loadings $\mathbf{B}$ describe the influence of each factor on each of the considered stocks. We have considered three types of factors

1. Macroeconomical factors;
2. Fundamental factors;
3. Statistical factors.

# 6 Asset Pricing

## 6.1 CAPM

The Capital Asset Pricing Model (CAPM) is based on an equilibrium assumption, that all investors hold the same portfolio (the market portfolio). Then any investment can be priced based on its correlation with the market portfolio.

Heart of the CAPM is the CAPM formula

$$\mu_{\mathrm{P}} - \mu_0 = \beta_{\mathrm{P}} \left(\mu_{\mathrm{MP}} - \mu_0\right),$$

where $\mu_{\mathrm{P}}$ is the expected return of portfolio P, $\mu_{\mathrm{MP}}$ is the expected return of the market portfolio and $\mu_0$ the risk-free return.

The beta of the portfolio is given as

$$\beta_{\mathrm{P}} = \frac{\mathrm{Cov}(R_{\mathrm{P}}, R_{\mathrm{MP}})}{\sigma_{\mathrm{MP}}^2}.$$

Using the correlation $\mathrm{corr}(R_{\mathrm{P}}, R_{\mathrm{MP}}) = \frac{\mathrm{Cov}(R_{\mathrm{P}}, R_{\mathrm{MP}})}{\sigma_{\mathrm{P}}\sigma_{\mathrm{MP}}}$, we can rewrite this as

$$\beta_{\mathrm{P}} = \mathrm{corr}(R_{\mathrm{P}}, R_{\mathrm{MP}}) \frac{\sigma_{\mathrm{P}}}{\sigma_{\mathrm{MP}}}.$$

Within the CAPM, the expected return $\mu_{\mathrm{P}}$ and the beta $\beta_{\mathrm{P}}$ lie on one line, called the *security market line* (SML).

### 6.1.1 Beta as the rewarded market risk

As $\beta$ is defined using the correlation with the market portfolio, it is a measure of how much the asset is exposed to market risk. We have

$$\begin{aligned} \sigma_{\mathrm{P}}^2 &= \beta_{\mathrm{P}}^2 \sigma_{\mathrm{MP}}^2 + \mathrm{Var}\, \varepsilon_{\mathrm{P}}, \\ \mu_{\mathrm{P}} - \mu_0 &= \beta \left(\mu_{\mathrm{MP}} - \mu_0\right), \end{aligned}$$

where $\varepsilon_{\mathrm{P}}$ is uncorrelated to the market return.

As the expected return grows with $\beta$, we get rewarded for taking more market risk. However if $\mathrm{Var}\, \varepsilon_{\mathrm{P}}$ is large, we have a higher risk without a higher expected return. This means that the idiosynchratic risk $\varepsilon_{\mathrm{P}}$ is not rewarded, because we can eliminate it by *diversification.*

### 6.1.2 For efficient portfolios

For efficient portfolio it holds $\mathrm{corr}(R_{\mathrm{P}}, R_{\mathrm{MP}}) = 1$, thus

$$\beta_{\mathrm{P}} = \frac{\sigma_{\mathrm{P}}}{\sigma_{\mathrm{MP}}}.$$

Joint with the CAPM formula $\mu_P - \mu_0 = \beta_P(\mu_{MP} - \mu_0)$, we can relate the values of

$$\mu_P, \sigma_P \text{ and } \beta_P,$$

and compute the missing values once we know one of these values.

As shown in Chapter 4 all efficient portfolios are a combination of the risk-free asset and the market portfolio. The value of $\beta_P$ *(for efficient portfolios)* gives the portion invested in the market portfolio, with $1 - \beta_P$ being invested in the risk-free asset. If $\beta_P > 1$, we to borrow money from the bank to buy more of the market portfolio than we originally could. This leverage results in an amplified market risk (i.e. a risk larger than for the market portfolio).

### 6.1.3 For general portfolios

For general portfolios, the variance can be arbitrarily large. However, we can compute a lower bound to the variance based on the value of beta:

$$\beta_P \leq \frac{\sigma_P}{\sigma_{MP}}.$$

Also we cannot say anything about the composition of the portfolio in general.

## 6.2 APT

The arbitrage pricing theory is more general than CAPM as it does not require an equilibrium. Instead, it assumes several factors that determine the price of the assets. The only restriction is that the factors must not allow for any arbitrage possibilities. Arbitrage possibilities are risk-free returns that are larger than the risk-free rate $\mu_0$.

It is shown that no arbitrage possibilities exist, when the constant term is the same for all assets, i.e. for

$$\mu_i = \mu_0 + \lambda_1 b_{i1} + \ldots + \lambda_L b_{iL},$$

where $\lambda_k$ are the expected values of the $L$ factors, $b_{ik}$ the factor loadings and $\mu_0$ is the risk-free interest rate.

# 7 Utility theory

Utility theory is based on the observation that most people have a non-linear perception of money. Instead of considering the expected amount of money gained, we measure the expected utility.

A utility function is a function that maps payoff (in terms of money) to the utility of this payoff, i.e. a function $u : \mathbb{R} \to \mathbb{R}$ which is monotonic increasing.

We call $u$

- *risk-averse*, if $u$ is *strictly concave*, i.e.

  $$u(t\,x + (1-t)\,y) > t\,u(x) + (1-t)\,u(y), \quad t \in (0,1),$$

  for all $x, y \in \mathbb{R}$, $x \neq y$.

  To test if $u$ is convex, we check whether $u''(x) < 0$ for $x \in \mathbb{R}$.

- *risk-seeking*, if $u$ is *strictly convex*, i.e.

  $$u(t\,x + (1-t)\,y) < t\,u(x) + (1-t)\,u(y), \quad t \in (0,1),$$

  for all $x, y \in \mathbb{R}$, $x \neq y$.

  To test if $u$ is convex, we check whether $u''(x) > 0$ for $x \in \mathbb{R}$.

- *risk-neutral*, if $u$ is *linear*, i.e.

  $$u(x) = a\,x,$$

  with $a > 0$.

We can use utility theory to decide the preference between two lotteries. Each lottery is given as a discrete random variable of the payoff, i.e.

$$L = \begin{cases} l_1, & \text{probability } p_1, \\ l_2, & \text{probability } p_2, \\ \vdots & \vdots \\ l_n, & \text{probability } p_n. \end{cases}$$

Then the utility of the lottery is given by evaluating the utility of each case

$$u(L) = \begin{cases} u(l_1), & \text{probability } p_1, \\ u(l_2), & \text{probability } p_2, \\ \vdots & \vdots \\ u(l_n), & \text{probability } p_n. \end{cases}$$

When we have a choice of lotteries, we choose the one with the largest expected utility

$$\mathbb{E}\big(u(L)\big) = p_1\,u(l_1) + p_2\,u(l_2) + \cdots + p_n\,u(l_n).$$

Where no specific utility function is given, but the risk-attitude of the investor, we might be able to use the concavity/convexity of the utility function for the decision. An Example is Exercise II-A-2 of Coursework 10.

# 8 Behavioural Finance

Behavioural finance is a modern field of finance studying the influence of psychology on the behaviour of market participants. We have only touched the huge field by looking at three effects:

1. different perception of small probabilities:

   - when unlikely events are involved people may have a clear choice that contradicts the choice that would be made in the utility theory. The reason is that people tend to overvalue small probabilities compared to larger ones (i.e. the difference between 0% and 1% is more significant than between 89% and 90%);

2. dependency on a reference point:

- although the final outcome is exactly the same, people make different choices depending on a given reference point, used to define *losses* and *gains*.

3. different risk attitude for losses and profits, resulting in an S-shaped utility function:

    - risk-seeking for losses, and
    - risk-averse for profits.