MTH786U/P, Semester A, 2023/24

Coursework 0 solution **N. Perra**

Problem 1

1. Compute the gradient ∇L of the function $L : \mathbb{R}^2_+ \to \mathbb{R}$ defined as

$$L(x, y) = \frac{x}{y} - 1 - \log\left(\frac{x}{y}\right) .$$

   Here $\mathbb{R}^2_+$ is the space of all real two-dimensional vectors with positive entries.

2. Show that L from Question 1 is scalar-invariant, i.e. $L(x, y) = L(c\,x, c\,y)$ for any scalar $c > 0$ and all arguments $x > 0, y > 0$.

Solution:

1. For the first partial derivative we obtain

$$\frac{\partial L}{\partial x} = \frac{1}{y} - \frac{1}{x} = \frac{x - y}{xy}$$

   and for the second one

$$\frac{\partial L}{\partial y} = -\frac{x}{y^2} + \frac{1}{y} = \frac{y - x}{y^2} .$$

   Hence, the entire gradient reads as

$$\nabla L(x, y) = \frac{1}{y}\begin{pmatrix} \frac{x-y}{x} \\ \frac{y-x}{y} \end{pmatrix} .$$

2. We simply observe

$$\begin{aligned} L(cx, cy) &= \frac{cx}{cy} - 1 - \log\left(\frac{cx}{cy}\right) \\ &= \frac{x}{y} - 1 - \log\left(\frac{x}{y}\right) \\ &= L(x, y) , \end{aligned}$$

   and, consequently, the function L is scalar-invariant.

Problem 2

1. Compute the expected value $\mathbb{E}_x$ of a (discrete) Poisson-distributed random variable X with probability

$$\rho_x := \exp(-\lambda)\frac{\lambda^x}{x\,!}\,, \quad x = 0, 1, 2, \ldots, s \tag{1}$$

   for a constant $\lambda > 0$. What is the solution for $s \to \infty$?
   Hint: Make use of the identity $\exp(\lambda) = \sum_{x=0}^{\infty} \frac{\lambda^x}{x\,!}$.

2. For a uniform (and absolutely continuous) random variable X in $[0, 1]$ compute the expectation of $f(X)$ for

$$f(x) := \begin{cases} -\log(x) & x \in [0, 1/5] \\ 0 & \text{otherwise} \end{cases}\,,$$

   Make use of the convention $0 \log(0) = 0$.

Solution:

1. The expectation for a discrete Poisson-distributed random variable X reads

$$\begin{aligned} \mathbb{E}_x[x] &= \sum_{x=0}^{s} x\rho_x = \sum_{x=0}^{s} x \exp(-\lambda)\frac{\lambda^x}{x\,!} \\ &= \lambda \exp(-\lambda) \sum_{x=1}^{s} \frac{\lambda^{x-1}}{(x-1)\,!} = \lambda \exp(-\lambda) \sum_{x=0}^{s-1} \frac{\lambda^x}{x\,!}\,. \end{aligned}$$

   Taking the limit $s \to \infty$ therefore yields

$$\begin{aligned} \mathbb{E}_x[x] &= \lambda \exp(-\lambda) \sum_{x=0}^{\infty} \frac{\lambda^x}{x\,!} \\ &= \lambda \exp(-\lambda) \exp(\lambda) = \lambda\,. \end{aligned}$$

1. For an absolutely continuous uniform random variable on the interval $[a, b]$ the probability density function reads

$$\rho(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{otherwise} \end{cases} .$$

Hence, for $[a, b] = [0, 1]$ we simply have $\rho(x) = 1$ for $x \in [0, 1]$, and we compute

$$\begin{aligned} \mathbb{E}_x[f(x)] &= \int_0^1 f(x)\, dx = -\int_0^{\frac{1}{5}} \log(x)\, dx = -\left[x \log x - x\right]_0^{\frac{1}{5}} \\ &= \frac{1}{5} - \frac{1}{5}(\log(1) - \log(5)) = \frac{1}{5}\left(1 + \log(5)\right) \approx 0.5218875825 . \end{aligned}$$

Problem 3

1. Let X be a random variable with expectation μ and variance σ^2. Show that the variance of $aX + b$, where $a, b \in \mathbb{R}$, is

$$\mathrm{Var}_x[ax + b] = a^2\sigma^2 .$$

Solution:

1. With the definition of the variance we compute

$$\begin{aligned} \mathrm{Var}_x[ax + b] &= \mathbb{E}_x\left[(ax + b - \mathbb{E}_x[ax + b])^2\right] \\ &= \mathbb{E}_x\left[(ax + b - \mathbb{E}_x[ax] - \mathbb{E}_x[b])^2\right] \\ &= \mathbb{E}_x\left[(ax + b - a\mathbb{E}_x[x] - b\mathbb{E}_x[1])^2\right] \\ &= \mathbb{E}_x\left[(ax + b - a\mathbb{E}_x[x] - b)^2\right] \\ &= \mathbb{E}_x\left[(ax - a\mathbb{E}_x[x])^2\right] \\ &= \mathbb{E}_x\left[a^2\left(x - \mathbb{E}_x[x]\right)^2\right] \\ &= a^2\mathbb{E}_x\left[(x - \mathbb{E}_x[x])^2\right] \\ &= a^2\, \mathrm{Var}_x[x] \\ &= a^2\sigma^2 . \end{aligned}$$