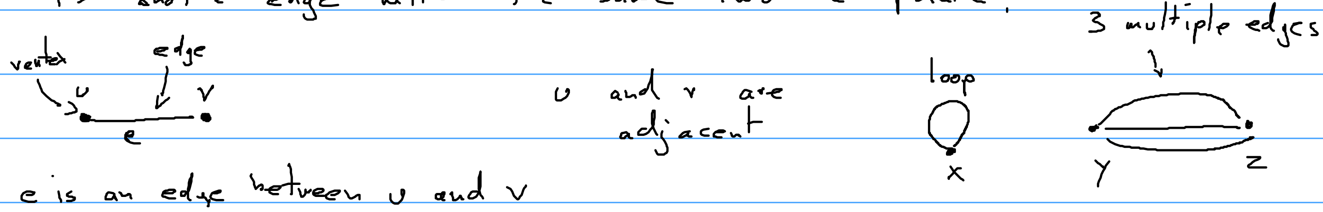
1. Introduction

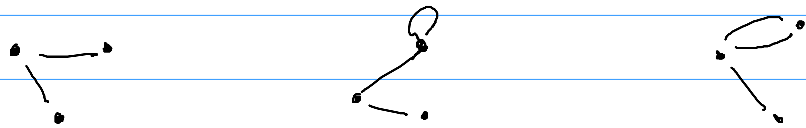
1.1 Graphs

<u>Definition</u> A <u>graph</u> is given by a finite set V of <u>vertices</u> and a finite set E of <u>edges</u>, where $V \cap E = \emptyset$. Each edge in E is associated with two vertices in V, called its <u>endpoints</u>. We will say that an edge is <u>incident</u> to its endpoints or <u>between</u> its endpoints. Two vertices are adjacent if there is an edge between them. An edge is a <u>loop</u> if both its endpoints are the same, and a multiple edge if there is another edge with the same two endpoints.



e is an edge between u and v
— " — incident to — " —

<u>Definition</u> A graph is <u>simple</u> if it does not contain any loops or multiple edges.



simple   not simple   not simple

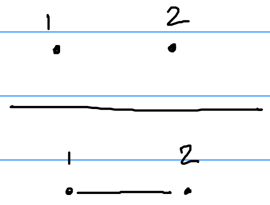
A graph G is simple if (i) every edge in $E(G)$ has two distinct endpoints and (ii) for every $u, v \in V(G)$ there is at most one edge in $E(G)$ between u and v.

For a simple graph it is convenient to identify an edge between u and v by a label uv. We will then talk about the "edge $\{u,v\}$" or the "edge uv" to refer to the edge with endpoints u and v.

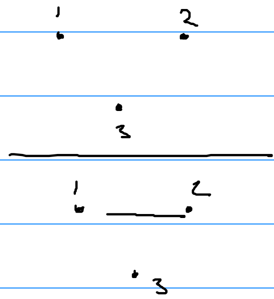
Theorem A simple graph with n vertices has at most $\binom{n}{2}$ edges.

Corollary For a finite set V, there exist $2^{\binom{|V|}{2}}$ distinct simple graphs with vertex set V.

$V = \{1,2\}$



$V = \{1,2,3\}$

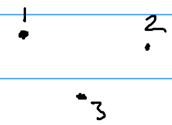


and 6 more...

Examples of simple graphs

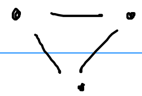
The empty graph with n vertices is the graph E_n with $V(E_n) = [n] = \{1,2,3,\dots\}$ and $E(E_n) = \emptyset$

E_3:



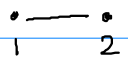
The complete graph K_n has $V(K_n) = [n]$ and $E(K_n) = \{uv : u,v \in [n] \text{ distinct}\}$

K_4

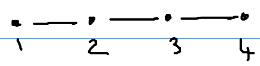


$K_3 = C_3$

The path P_n has $V(P_n) = [n]$ and $E(P_n) = \{uv : u,v \in [n], |u-v| = 1\}$

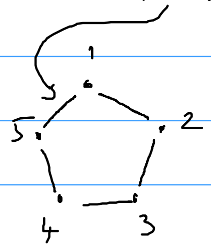


$P_2 = K_2$



P_4

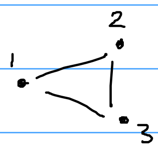
The cycle C_n has $V(C_n) = [n]$ and
$E(C_n) = E(P_n) \cup \{1n\}$



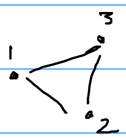
C_5

1.2 Representations of Graphs

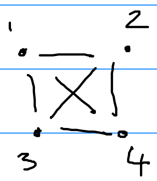
The position of vertices and edges as we draw graphs in the plane does not matter,



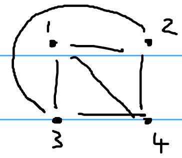
and



are both representations of K_3

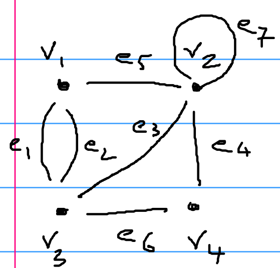


K_4



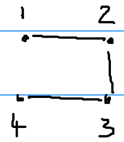
also K_4

Definition The *incidence matrix* of a graph G is a matrix with $|V(G)|$ rows corresponding to the vertices of G and $|E(G)|$ columns corresponding to the edges of G. The entry in row $v \in V(G)$ and column $e \in E(G)$ is 2 if e is a loop with endpoint v, 1 if v is one of two distinct endpoints of e, and 0 otherwise.



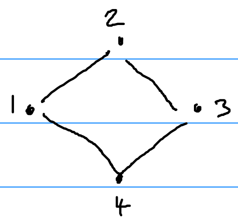
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

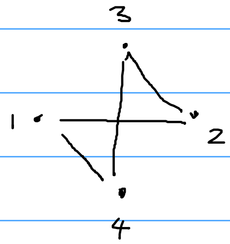
<u>Definition</u> The adjacency matrix of a simple graph G is a $|V(G)| \times |V(G)|$ matrix. The entry in row $u \in V(G)$ and column $v \in V(G)$ is equal to 1 if $uv \in E(G)$ and 0 otherwise.

P_4 

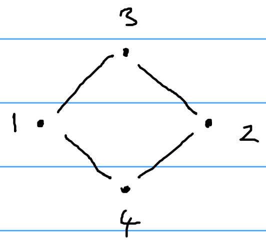
$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

1.3 Graph isomorphism



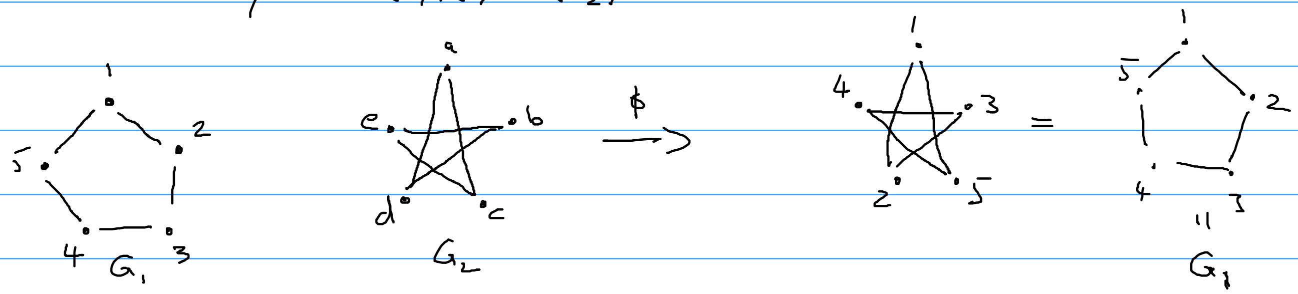


are both representations of C_4



is a different graph, not C_4

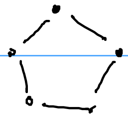
<u>Definition</u> Two simple graphs G_1 and G_2 are <u>isomorphic</u> if there exists a bijection $\phi: V(G_1) \to V(G_2)$ such $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$.



$\phi: V(G_2) \to V(G_1)$

$\phi(a) = 1$, $\phi(b) = 3$, $\phi(c) = 5$, $\phi(d) = 2$, $\phi(e) = 4$

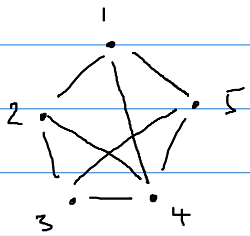
We can think of graph isomorphism in terms of unlabeled graphs



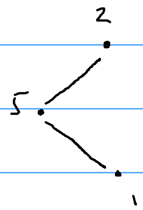
represents all graphs isomorphic to C_5

1.4 Subgraphs

Definition Let G be a graph. Then a graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

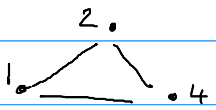


G

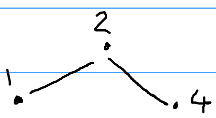


not a subgraph of G because $\{2,5\} \notin E(G)$

Definition Let G be a graph and $A \subseteq V(G)$. Then the induced subgraph of G on vertex set A, denoted $G[A]$, is the subgraph H of G whee $V(H) = A$ and $E(H) \subseteq E(G)$ is the set of all edges in G with endpoints in A.



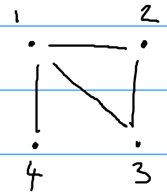
is the induced subgraph of G on vertex set $\{1,2,4\}$



is a subgraph of G but not an induced subgraph

1.5 Neighbourhood and Degrees

Definition Let G be a graph and $v \in V(G)$. Then the neighbourhood $N_G(v)$ of v in G is the set of all vertices adjacent to v in G. The degree of v in G, denoted $d_G(v)$ is the number of edges incident to v, where loops are counted twice.

$G =$ 

$$N_G(1) = \{2,3,4\}$$
$$d_G(2) = 2$$

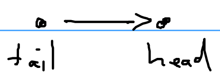
Lemma For any graph G, $\sum_{v \in V(G)} d_G(v) = 2 \cdot |E(G)|$.

Proof. Consider an edge $e \in E(G)$ with endpoints $u, v \in V(G)$. If e is a loop, it contributes 2 to $d_G(u) = d_G(v)$ and therefore to $\sum_{w \in V(G)} d_G(w)$. If e is not a loop, it contributes 1 each to $d_G(u)$ and $d_G(v)$ and therefore 2 to $\sum_{w \in V(G)} d_G(w)$. This is true for every edge, so the claim follows.

Corollary In any graph, the number of vertices of odd degree is even.

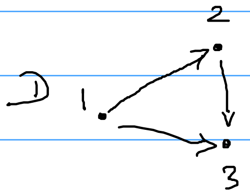
1.6 Digraphs and Networks

Definition A directed graph or digraph D is a graph where each edge e has been given a direction from one endpoint u to the other endpoint v. In this case u will be called the tail of e and v the head of e. Then e will be called an arc.



For a digraph D, we write $A(D)$ for the set of its arcs.

Definition Let D be a digraph and $v \in V(D)$. Then the outdegree $d_D^+(v)$ of v in D is the number of arcs in $A(D)$ that have v as its tails. The indegree $d_D^-(v)$ of v in D is the number of arcs in $A(D)$ that have v as their head.

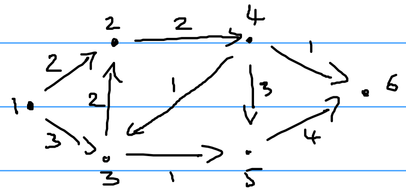


$$d_D^+(1) = 2$$
$$d_D^-(1) = 0$$
$$d_D^-(2) = d_D^+(2) = 1$$

Lemma For any digraph D,

$$\sum_{v \in V(D)} d_D^+(v) = \sum_{v \in V(D)} d_D^-(v) = |A(D)|.$$

Definition A network is a graph or digraph G together with a function $w: E(G) \to \mathbb{R}$ that assigns a weight $w(e)$ to each edge or arc e of G



$$w((1,2)) = 2$$

2. Paths, Cycles, Tree

Definition A walk is an alternating sequence of vertices and edges in which each edge is preceded by one of its endpoints and succeeded by the other. The length of a walk is the number of edges in the sequence. A u-v-walk is a walk that starts at u and ends at v. A walk is closed if it starts and ends with the same vertex. A trail is a walk where all edges are distinct. A path is a trail where all vertices are distinct. A tour is a closed trail. A cycle is a tour that contains at least one edge and in which all vertices are distinct.