Queen Mary
University of London

Main Examination period 2021 – May/June – Semester B
Online Alternative Assessments

MTH6105: Algorithmic Graph Theory

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

You have **24 hours** to complete and submit this assessment. When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are expected to spend about 2 hours to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: F. Fischer, R. Johnson

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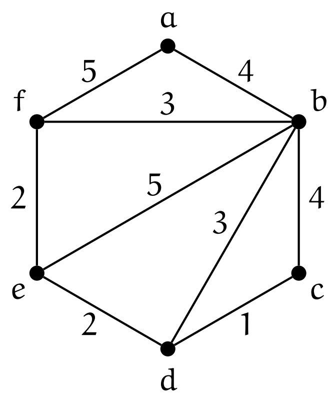
In this paper, $V(G)$ denotes the set of vertices of a graph or digraph G, $E(G)$ the set of edges of a graph G, and $A(G)$ the set of arcs of a digraph G. You may use any result from lecture notes and exercises without proving it, but you must state clearly which result you use.

Question 1 [22 marks]. Consider the following network (G, w), in which each vertex is labeled with its name and each edge $e \in E(G)$ with its weight $w(e)$.



(a) Use Prim's algorithm starting from vertex a to find a minimum spanning tree of (G, w). Show your working and give the minimum spanning tree and its weight. **[10]**

(b) Find another minimum spanning tree of (G, w). Show your working and give the minimum spanning tree and its weight. **[4]**

(c) Does there exist a minimum spanning tree of (G, w) that does **not** contain the edge de? Justify your answer. **[4]**

(d) Does there exist a minimum spanning tree of (G, w) that contains both of the edges bd and bf? Justify your answer. **[4]**

Question 2 [26 marks]. Consider a tree T in which the degree of each vertex is either 1 or 3. Let $n = |V(T)|$.

(a) Show that n is even. [**2**]

(b) Show that T has $\frac{n}{2} + 1$ leaves. [**6**]

(c) Determine the number of distinct simple graphs G such that T is a spanning tree of G. Explain your reasoning. [**6**]

Call a graph G a **tree-set** if every connected component of G is a tree.

(d) For each of the following graphs, determine if the graph is a tree-set. Justify your answer. [**6**]

(i) The graph G_1 with $V(G_1) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E(G_1) = \emptyset$

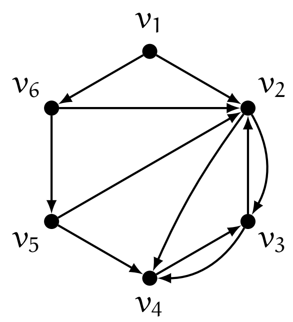
(ii) The graph G_2 with $V(G_2) = V(G_1)$ and $E(G_2) = \{v_1v_2, v_2v_4, v_2v_6, v_4v_6\}$

(iii) The graph G_3 with $V(G_3) = V(G_1)$ and $E(G_3) = \{v_1v_5, v_2v_6, v_3v_4, v_5v_6\}$

Now consider an arbitrary graph G.

(e) Give a polynomial-time algorithm to determine whether G is a tree-set. Show that the algorithm is indeed a polynomial-time algorithm. [**6**]

Question 3 [24 marks]. Consider the following digraph D, in which each vertex is labeled with its name.



(a) Determine the strongly connected components of D. Show your working. **[6]**

Now consider the directed network (D, w), where $w : A(D) \to \mathbb{R}$ with

$$
\begin{array}{llll}
w(v_1v_2) = 4, & w(v_1v_6) = -1, & w(v_2v_3) = 1, & w(v_2v_4) = -2 \\
w(v_3v_2) = 2, & w(v_3v_4) = 2, & w(v_4v_3) = 1, & w(v_5v_2) = 2 \\
w(v_5v_4) = 1, & w(v_6v_2) = 4, & w(v_6v_5) = 1. &
\end{array}
$$

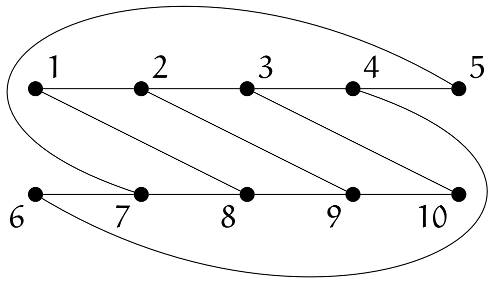
(b) Recall that Dijkstra's algorithm may fail to find shortest directed paths in the presence of negative weights. Illustrate this fact by giving vertices $u, v \in V(D)$ such that Dijkstra's algorithm fails to find a shortest directed u–v-path in (D, w). Explain your reasoning. **[6]**

(c) Use the Bellman-Ford algorithm to find a shortest directed v_1–v_3-path in (D, w). Show your working. **[12]**

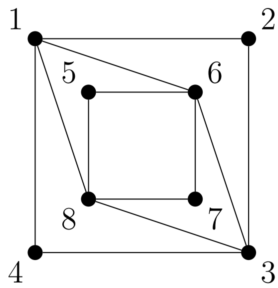
Question 4 [28 marks].

(a) For each of the following graphs, state if the graph is bipartite or not. Justify your answer. **[6]**

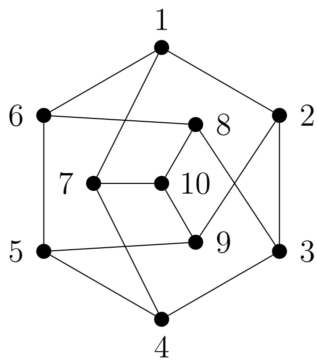
(i)



(ii)



(iii)



Now consider the bipartite graph G with

$$V(G) = \{u_1, u_2, u_3, u_4, u_5, u_6, v_1, v_2, v_3, v_4, v_5, v_6\}$$
$$E(G) = \{u_1v_2, u_1v_4, u_2v_1, u_2v_2, u_2v_3, u_2v_4, u_2v_5, u_2v_6, u_3v_2, u_3v_4, u_4v_2, u_4v_3, u_4v_4, u_5v_2, u_5v_4, u_6v_1, u_6v_2, u_6v_4, u_6v_5, u_6v_6\}.$$

(b) Let $U = \{u_1, u_3, u_5, v_2, v_4, v_6\}$. Draw $G[U]$, the induced subgraph of G with vertex set U. **[2]**

(c) For each of the following sets, state if the set is a matching of G or not. Justify your answer. **[6]**

(i) $M_1 = \{u_1v_2, u_2v_1, u_5v_2, u_6v_4\}$

(ii) $M_2 = \{u_1v_2, u_2v_1, u_3v_3, u_4v_4\}$

(iii) $M_3 = \{u_1v_4, u_2v_2, u_4v_3, u_6v_5\}$

(d) Find a maximum matching of G. Show your working. [**10**]

(e) Show that the matching you have found is indeed a maximum matching. [**4**]

End of Paper.