Random Processes – 2023/24 Solutions 1

1.

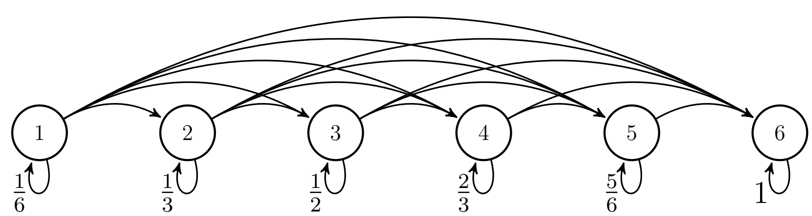
(a) The largest number seen after $t+1$ rolls depends only on the largest number we have seen after t rolls and the value of the $(t+1)$th roll. This means that the Markov property is satisfied: if I know X_t, the distribution of X_{t+1} is determined and it does not depend on $X_{t-1}, X_{t-2}, \ldots$.

(b) The state space is $\{1, 2, 3, 4, 5, 6\}$ (these are the possible values that each random variable X_t can take).

Suppose that $X_t = i$. If my next roll is one of the i numbers in $\{1, \ldots, i\}$ then $X_{t+1} = i$. If my next roll is j for some j in $\{i+1, \ldots, 6\}$ then $X_{t+1} = j$. So the transition probabilties are:

$$p_{ij} = \begin{cases} \frac{i}{6} & \text{if } i = j \\ \frac{1}{6} & \text{if } i < j \\ 0 & \text{if } i > j \end{cases}$$

The transition graph is



(Where all of the unlabelled arrows have transition probability $\frac{1}{6}$.)

The transition matrix is

$$\begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(c) The only thing that changes in the analysis is that the transition probabilities when I am using the biased die may be different. This process is still a Markov chain but it is not homogeneous.

(d) Suppose that $X_t = i$. Then I know that the number rolled on roll t was one of the numbers in $\{1, \ldots, i\}$. If I roll one of the other $i-1$ numbers in this set then $X_{t+1} = i$. If my next roll is j for some j in $\{i+1, \ldots, 6\}$ then $X_{t+1} = j$. None of this depends on $X_{t-1}, X_{t-2}, \ldots$ so this is still a (homogeneous) Markov chain. The transition probabilties are:

$$p_{ij} = \begin{cases} \frac{i-1}{5} & \text{if } i = j \\ \frac{1}{5} & \text{if } i < j \\ 0 & \text{if } i > j \end{cases}$$

(e) This time the Markov property will fail. For example, if $X_1 = 1$ and $X_2 = 2$ then my first two rolls must have been a 1 followed by a 2. Under these circumstances the third roll cannot be a 4. So,

$$\mathbb{P}(X_3 = 4 \mid X_2 = 2, X_1 = 1) = 0$$

However, if $X_1 = 2$ and $X_2 = 2$ then my first two rolls may have been a 2 followed by a 1. Under these circumstances the third roll can be a 4:

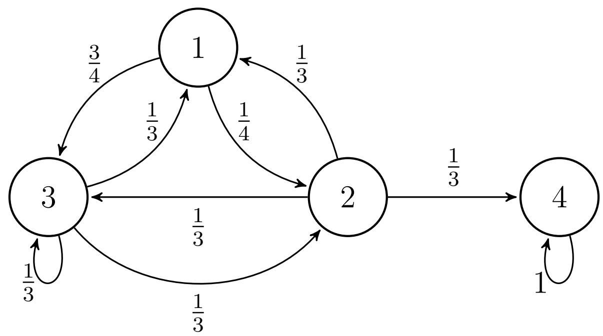
$$\mathbb{P}(X_3 = 4 \mid X_2 = 2, X_1 = 2) \neq 0$$

This is enough to show that the Markov property fails but if you want to work out the last probability exactly it is:

$$\begin{aligned} \mathbb{P}(X_3 = 4 \mid X_2 = 2, X_1 = 2) &= \frac{\mathbb{P}(X_3 = 4, X_2 = 2, X_1 = 2)}{\mathbb{P}(X_2 = 2, X_1 = 2)} \\ &= \frac{\mathbb{P}(\text{first three rolls are } 2, 1, 4)}{\mathbb{P}(\text{first two rolls are } 2, 1 \text{ or } 2, 2)} \\ &= \frac{1/6 \times 1/5 \times 1/5}{2 \times 1/6 \times 1/5} = \frac{1}{10} \end{aligned}$$

2.

(a) The transition graph is



(b) (i) This is a transition probability so can be read off the matrix (the $(2,3)$-entry)

$$\mathbb{P}(X_1 = 3 \mid X_0 = 2) = p_{2,3} = \frac{1}{3}.$$

(ii) This is the same transition probability (because the chain is homogeneous)

$$\mathbb{P}(X_2 = 3 \mid X_1 = 2) = p_{2,3} = \frac{1}{3}.$$

(iii) By the Markov property

$$\mathbb{P}(X_2 = 3 \mid X_1 = 2, X_0 = 1) = \mathbb{P}(X_2 = 3 | X_1 = 2) = p_{2,3} = \frac{1}{3}$$

(iv) We need the probability that starting from state 0, the chain follows the trajectory given (first step to state 2, next step to state 3). This is the product of transition probabilities:

$$\mathbb{P}(X_2 = 3, X_1 = 2 \mid X_0 = 1) = p_{1,2}p_{2,3} = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}.$$

(v) Thinking about the possible 2-step paths from state 3 to state 3, this is:

$$\begin{aligned}\mathbb{P}(X_2 = 3 \mid X_0 = 3) &= p_{3,1}p_{1,3} + p_{3,2}p_{2,3} + p_{3,3}p_{3,3} \\ &= \frac{1}{3} \times \frac{3}{4} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} = \frac{17}{36}\end{aligned}$$

Looking at the calculation you did, this is the same as working out the $(3,3)$-entry of the square of the transition matrix. We will see later that this is not a coincidence.

(vi) There is no way that the chain can ever leave state 4. So if we ever have $X_k = 4$ then certainly $X_t = 4$ for all $t \geqslant k$. So

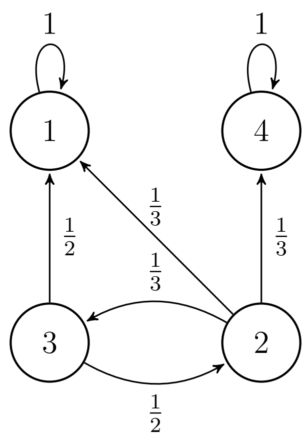
$$\mathbb{P}(X_{1000} = 4 \mid X_0 = 4) = 1.$$

and so

$$\mathbb{P}(X_{1000} = 1 \mid X_0 = 4) = 0.$$

3.

(a) The transition graph is



(b) The states 1 and 4 have the property that once we reach one of them, the chain can never leave. We will see in Week 2 lectures that these states are called absorbing states.

(c) (i) The only way in which we can have $X_n \neq 1, 4$ is for the process to alternate between states 2 and 3 for n steps. It follows that if n is even

$$\mathbb{P}(X_n \neq 1, 4 \mid X_0 = 2) = p_{2,3}p_{3,2} \dots p_{2,3}p_{3,2} = (p_{2,3})^{\frac{n}{2}}(p_{3,2})^{\frac{n}{2}} = \left(\frac{1}{6}\right)^{\frac{n}{2}}$$

while if n is odd

$$\mathbb{P}(X_n \neq 1, 4 \mid X_0 = 2) = p_{2,3}p_{3,2} \dots p_{2,3} = (p_{2,3})^{\frac{n+1}{2}}(p_{3,2})^{\frac{n-1}{2}} = \frac{1}{3}\left(\frac{1}{6}\right)^{\frac{n-1}{2}}.$$

(ii) If $X_t = 4$ for any $t \leqslant n$ then $X_n = 4$ since once the process reaches state 4 it can never leave it. It follows that

$$\mathbb{P}(X_n = 4 \mid X_0 = 2) = \sum_{t=1}^{n} \mathbb{P}(\text{the process reaches 4 for the first time at step } t)$$

We have that

$$\mathbb{P}(\text{the process reaches 4 for the first time at step } t) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \left(\frac{1}{6}\right)^{\frac{n-1}{2}} \frac{1}{3} & \text{if } n \text{ is odd} \end{cases}$$

So

$$\mathbb{P}(X_n = 4 \mid X_0 = 2) = \frac{1}{3} + \frac{1}{6}\frac{1}{3} + \left(\frac{1}{6}\right)^2 \frac{1}{3} + \cdots + \left(\frac{1}{6}\right)^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{1}{3}.$$

Using the formula for the sum of a geometric progression we get

$$\mathbb{P}(X_n = 4 \mid X_0 = 2) = \begin{cases} \frac{2}{5}\left(1 - \left(\frac{1}{6}\right)^{\frac{n}{2}}\right) & \text{if } n \text{ is even} \\ \frac{2}{5}\left(1 - \left(\frac{1}{6}\right)^{\frac{n+1}{2}}\right) & \text{if } n \text{ is odd} \end{cases}$$

(d) As $n \to \infty$ we have that

$$\begin{aligned} \mathbb{P}(X_n \neq 1, 4 \mid X_0 = 2) &\to 0 \\ \mathbb{P}(X_n = 4 \mid X_0 = 2) &\to \frac{2}{5}. \end{aligned}$$

So the process will eventually leave states 2 and 3 (ending up in either 1 or 4). The probability that it ends up in state 4 is 2/5.

The calculations in this question ended up being rather fiddly. Part of the point of this question was to get used to doing calculations with transition probabilities in a direct way. We will see shortly (Week 2 lectures) a slicker way of working out things like the probability that we end up in state 4 in this example which is both simpler and can be used in more complicated examples.

Please let me know if you have any comments or corrections

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