Queen Mary University of London

MTH5104: Convergence and Continuity 2023–2024
Problem Sheet 0 (Sets and Logic)

1. Let $S = \{2k : k \in \mathbb{N}\}$ and $T = \{2k+1 : k \in \mathbb{N}\}$.

   (i) Compute $S \cup T$ and $S \cap T$.

   (ii) Is $\mathbb{N}_0 \subseteq S \cup T$? Justify your answer.

   (iii) Prove that $\forall x \in S : x^2 \in S$.

   (iv) Does (iii) hold with S replaced by T? Justify your answer with a short proof or counterexample.

2. (Liebeck.) Consider the set

$$A = \{\alpha, \{1, \alpha\}, \{3\}, \{\{1, 3\}\}, 3\}.$$

   Which of the following statements are true and which are false?

   (a) $\alpha \in A$.

   (b) $\{\alpha\} \notin A$.

   (c) $\{1, \alpha\} \subseteq A$.

   (d) $\{3, \{3\}\} \subseteq A$.

   (e) $\{1, 3\} \in A$.

   (f) $\{\{1, 3\}\} \subseteq A$.

   (g) $\{\{1, \alpha\}\} \subseteq A$.

   (h) $\{1, \alpha\} \notin A$.

   (i) $\emptyset \subseteq A$.

3. Let A, B, C be three sets with $A, B \subseteq C$. Prove that

   (i) $A \subseteq B \Rightarrow (C \setminus B) \subseteq (C \setminus A)$;

   (ii) $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$;

   (iii) $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$.

   Important: to prove that two sets X and Y are equal you need to prove that $X \subseteq Y$ and $Y \subseteq X$.

4. (Liebeck.) Which of the following arguments are valid? For the valid ones, write down the argument symbolically.

   (a) I eat chocolate if I am sad. I am not sad. Therefore I am not eating chocolate.

   (b) I eat chocolate only if I am sad. I am not sad. Therefore I am not eating chocolate.

   (c) If a movie is not worth seeing, then it was not made in England. A movie is worth seeing only if critic Ivor Smallbrain reviews it. The movie "Cat on a Hot Tin Proof" was not reviewed by Ivor Smallbrain. Therefore "Cat on a Hot Tin Proof" was not made in England.

5. (Liebeck.) Which of the following statements are true, and which are false?

   (a) $n = 3 \Rightarrow n^2 - 2n - 3 = 0$.

   (b) $n^2 - 2n - 3 = 0 \Rightarrow n = 3$.

   (c) $\forall a, b \in \mathbb{Z}, ab$ is a perfect square $\Rightarrow a$ and b are perfect squares.

   (d) $\forall a, b \in \mathbb{Z}, a$ and b are perfect squares $\Rightarrow ab$ is a perfect square.

6. Prove that there exists a unique $x \in \mathbb{N}$ such that $x^2 = x$.

   Hint: Start by proving $\exists x \in \mathbb{N} : x^2 = x$. Then prove the uniqueness of x by contradiction.