MTH5104
Convergence and Continuity

Contents

Chapter 1

How to study mathematics

It is extremely important in this module, as in any mathematics module, to master the fine art of the mathematical proof. With this I mean to be able to prove a mathematical statement via a logical argument, stating clearly what is the starting point (hypothesis) and what is the final point (assertion or what we want to prove). To go from the first to the latter we need to use a rigorous mathematical argument where every step is carefully justified. It might be useful to follow the structure given below:

(A) write down clearly what is the hypothesis of your theorem/proposition, using the correct mathematical language (quantifiers, inequalities, etc..);

(B) write down the final assertion, i.e., what you want to prove using the correct mathematical language;

(C) compare (A) with (B) and find a way to go from (A) to (B). Often you need a very quick mathematical argument to pass from (A) to (B) (direct argument or a proof by contradiction) and then your proof is done!

If you work in this way you will become more and more independent in producing a mathematical proof and you will not need to rely on your lecture notes all the time. What is **not advisable** (almost illegal) in mathematics is to **learn by heart**. This will not help your understanding and will have a very limited effect in time, in the sense that you will forget everything you have learnt in a matter of few days. So, be wise and start from the beginning of this module to learn in the right way!

For the sake of completeness sometimes we refer to theorems proven in the 2021/22 lecture notes: these are not examinable. **Only the material of these lecture notes is examinable**.

Chapter 2

Elements of Logic and Set theory

We begin by building a common mathematical language. **As any language maths has its own alphabet** made of symbols. It is important to know them to be able to have a meaningful mathematical conversation. These are the building block of our mathematical background that we will construct during this module.

2.1 Symbols

In this section we collect the mathematical symbols that you will encounter all the time in your mathematical studies. We distinguish them into two classes: set theoretic symbols and logical symbols.

2.1.1 Set theoretic symbols

- $\in$ is in, is an element of
- $\subset$ is a subset of, is contained in
- $\subseteq$ subset (possibly equal)
- $\cup$ union
- $\cap$ intersection
- $\emptyset$ the empty set.

We can also reverse these symbols or negate them. For instance, the reverse of $\subset$ is $\supset$ which means contains and the negation of $\in$ is obtained by striking out the symbol, i.e., $\notin$. This last symbol stands for *is not an element of*.

2.1.2 Logical symbols

- $\forall$ for all
- $\exists$ there exists
- $\nexists$ there does not exist
- $\Rightarrow$ implies, is a sufficient condition for
- $\not\Rightarrow$ does not imply
- $\Leftrightarrow$ if and only if, iff

These symbols are quantifiers ($\forall$, $\exists$, ...) or express logical relations ($\Rightarrow$, $\Leftrightarrow$, ...). We will come back to them later on.

2.2 Set Theory

Every mathematical object belongs to a *set*. It is therefore important to have some foundations of set theory.

Definition 2.2.1. *A set is a collection of objects. The objects are referred to as* elements *of the set.*

We can represent a set by listing its elements or by stating a property that determines membership or in other words defines the set. It is common to use a capital letter to denote a set. In details, the set of numbers 1, 2, 3 is denoted as

$$S = \{1, 2, 3\}.$$

Here we have listed the elements of the set S. The set of all positive numbers (numbers greater than 0) can be represented by stating its defining property, i.e.,

$$T = \{x : x > 0\},$$
$$T = \{x | x > 0\}.$$

Note that : and | read as *such that*.

Some more definitions and notations:

- If x is an element of S then we write $x \in S$.
- If x does not belong to S then we write $x \notin S$.
- S is a subset of T if every element of S belongs to T, i.e.,

$$x \in S \Rightarrow x \in T.$$

- If S is a subset of T we write $S \subset T$. If the set might actually be equal then we can use the notation $S \subseteq T$. Analogously, we can write $T \supset S$ or $T \supseteq S$.
- The empty set $\emptyset$ is the set with no elements. It is a subset of every other set.

Definition 2.2.2. *Let A and B be two sets. The union of A and B is the set of all x such that $x \in A$* **or** *$x \in B$. In detail,*

$$A \cup B = \{x : \, x \in A \, or \, x \in B\}.$$

The intersection of A and B is

$$A \cap B = \{x : \, x \in A \, and \, x \in B\}.$$

The set of all elements of A which does not belong to B is denoted by $A \setminus B$.

Remark 2.2.3. It is clear from the definition above that $A \cap B$ could be empty. Moreover,

$$\begin{aligned} A \cap B &\subseteq A, \\ A \cap B &\subseteq B, \\ A &\subseteq A \cup B, \\ B &\subseteq A \cup B, \\ A \cap B &\subseteq A \cup B, \\ A \setminus B &\subseteq A. \end{aligned}$$

Example 2.2.4. Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 5\}$. Compute $A \cup B$, $A \cap B$ and $A \setminus B$. What is the relation between $(A \cup B) \setminus (A \cap B)$ and $A \setminus B$?

You are probably already very familiar with the following sets of numbers:

$$\begin{aligned} \mathbb{N} &= \{1, 2, 3, 4, \cdots\} \quad \text{natural numbers} \\ \mathbb{N}_0 &= \{0, 1, 2, 3, \cdots\} \quad \text{natural numbers with 0 included} \\ \mathbb{Z} &= \{0, \pm 1, \pm 2, \pm 3, \cdots\} \quad \text{integers} \\ \mathbb{Q} &= \{a/b : \, a \in \mathbb{Z}, \, b \in \mathbb{N}\} \quad \text{rational numbers} \end{aligned}$$

By definition of these sets we have immediately

$$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q}.$$

There exists a bigger set which contains all of these sets of numbers: it is the set $\mathbb{R}$ of real numbers. Intuitively it is the set of all points on a straight line extending indefinitely in both directions. We will give a rigorous definition later on but at the moment we can already write the following chain of inclusions:

$$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

2.3 Logic

As stated in the introduction this module is all about proving theorems. It is therefore important to have some notions of logic.

Definition 2.3.1. *For the scope of this module it is sufficient to define a* proposition *as a statement which is either true or false.*

Example 2.3.2. These are all examples of propositions:

1. All students in this room have dark hair.
2. $3 \leq 5$
3. 326 is an odd number.
4. 13 is a prime number.
5. 5 is not positive.
6. For all real numbers x, we have $x^2 \geq 0$.
7. There exists a real number a such that $a^2 = 2$.

A proposition can contain a *variable*, such as "The number x is positive". Here x is a variable, and the proposition can be true or false depending on the value of x. For example, it is true for $x = 1$ and false for $x = -2$.

2.3.1 Combining propositions

Using propositions with variables one can form new propositions in several ways:

(i) Using the quantifier $\forall$: from the proposition $b^2 \geq \frac{1}{2}$ one can form the proposition

$$\forall b \in \mathbb{N}, \text{one has } b^2 \geq \frac{1}{2}.$$

This proposition is true.

(ii) Using the quantifiers $\exists$: $\exists b \in \mathbb{Q}$ such that $b^2 \geq \frac{1}{2}$.

(iii) Using the implication $\Rightarrow$:

$$b^2 \geq \frac{1}{2} \quad \Rightarrow \quad b^2 \geq \frac{1}{3}.$$

(iv) Using the double implication $\Leftrightarrow$:

$$b^2 \geq \frac{1}{2} \quad \Leftrightarrow \quad -b^2 \leq -\frac{1}{2}.$$

Finally, one can combine all the above ways of forming propositions.

2.3.2 Negating propositions

It is very important to understand how to negate a proposition. If a proposition is true then its negation is false and vice versa. There is a basic rule that we need to follow:

When negating propositions containing symbols $\exists$ and $\forall$, one should replace $\exists$ by $\forall$ and vice versa.

When negating propositions containing the symbol $\Rightarrow$ one should replace $\Rightarrow$ by $\not\Rightarrow$.

Example 2.3.3. Let us write down a proposition A:

$$\forall x \in \mathbb{N}\ x \geq 2 \ \Rightarrow x+1 \geq 3.$$

A is true. Let us negate A. We use the notation $\neg A$. The proposition $\neg A$ is given by

$$\exists x \in \mathbb{N}: \quad x \geq 2 \not\Rightarrow x+1 \geq 3.$$

$\neg A$ can be equivalently written as follows:

$$\exists x \in \mathbb{N}, x \geq 2: \quad x+1 < 3.$$

2.3.3 Equivalence and implication

Let A and B be two propositions which contain the same variable. Then $A \Leftrightarrow B$ means that B is true if and only if A is true. In equivalences, you need to specify the range of the variable, so such statements will usually include the quantifier $\forall$. Examples:

1. $\forall x \in \mathbb{R}$: $x > 0 \Leftrightarrow x+1 > 1$
2. $\forall n \in \mathbb{Z}$: $n > \frac{1}{2} \Leftrightarrow n \geq 1$
3. $\forall k \in \mathbb{N}$: $k > 2 \Leftrightarrow k^2 > 4$
4. $\forall m \in \mathbb{Z}$: m is even $\Leftrightarrow m^2$ is even.

$A \Leftrightarrow B$ can be rephrased as 'A is a necessary and sufficient condition for B'.

Let A and B be two propositions which contain the same variable. $A \Rightarrow B$ means 'if A is true then B is true'. In implications, you need to specify the range of the variable, so such statements will usually include the quantifier $\forall$. Examples:

1. $\forall a \in \mathbb{R}$, one has: $a \geq 1 \Rightarrow a^2 \geq 1$;
2. $\forall x \in \mathbb{R}$: x is a solution of $(x^2-1) = 0 \Rightarrow x$ is a solution of $x^2 = 1$;
3. $\forall a, b \in \mathbb{R}$, one has: $a \in (0,1)$ and $b \in (0,1) \Rightarrow (a+b)/2 \in (0,1)$.

$A \Rightarrow B$ can be equivalently rephrased in one of the following ways:

1. A implies B.
2. If A then B.
3. A is a sufficient condition for B.
4. B is a necessary condition for A.

We say that $A \Rightarrow B$ is a conditional statement. Note that it can happen that $A \Rightarrow B$ but NOT $B \Rightarrow A$. Examples:

1. $k \in \mathbb{N} \Rightarrow k \in \mathbb{Z}$ but $k \in \mathbb{Z} \not\Rightarrow k \in \mathbb{N}$;
2. $\forall a \in \mathbb{R}$, one has $a \geq 1 \Rightarrow a^2 \geq 1$, but $a^2 \geq 1 \not\Rightarrow a \geq 1$.

Warning: It is a very common mistake to (explicitly or implicitly) confuse the statements $A \Rightarrow B$ and $B \Rightarrow A$.

2.3.4 Proof by contradiction

We will study many theorems in this module and we will need to prove them (provide a proof). A theorem is given by a first statement called hypothesis (a proposition A which is true) and a final statement B. The theorem states that A implies B. We need to prove with a rigorous mathematical argument, which will often requires a chain of implications, that $A \Rightarrow B$ is true. As an explanatory example let us consider the following theorem:

Theorem: There is no rational number that satisfies the equation $x^2 = 2$

Our hypothesis or starting point is

$$A : x^2 = 2.$$

Our final statement of what we want to prove is

$$B : x \notin \mathbb{Q}.$$

Proving the theorem in a direct way means to prove $A \Rightarrow B$. This is equivalent to prove that $\neg B \Rightarrow \neg A$. Indeed, $\neg B \Rightarrow \neg A$ is the contrapositive of $A \Rightarrow B$ and a contrapositive to an implication is true iff the implication itself is true. Proving that the contrapositive is true means to prove that $\neg B$ implies $\neg A$. Doing a **proof by contradiction** means to assume that A is true and that $\neg B$ holds and then to get a contradiction to A. For this reason we use the terminology proof by contradiction. In some cases, as this one, the proof by contradiction is easier than the direct proof. We will prove later on in the module that if x is rational then $x^2 \neq 2$.

2.3.5 Converse versus contrapositive

Let A and B be propositions depending on a variable. Then a *converse* to $A \Rightarrow B$ is $B \Rightarrow A$ and a *contrapositive* to $A \Rightarrow B$ is $\neg B \Rightarrow \neg A$. We have seen above that a contrapositive to an implication is true iff the implication itself is true. However, a converse may or may not be true regardless of whether the implication itself is true or not. This is illustrated by the following examples:

1. "If you live in London, then you live in England" (true) Converse: "If you live in England, then you live in London" (false). Contrapositive: "If you don't live in England, then you don't live in London" (true)
2. "$x > 1 \Rightarrow x^2 > 1$" (true). Converse: "$x^2 > 1 \Rightarrow x > 1$ (false). Contrapositive: "$x^2 \leq 1 \Rightarrow x \leq 1$" (true).

Chapter 3

The set of real numbers

The aim of this chapter is to introduce the set of real numbers in a *mathematical precise* way. Intuitively, $\mathbb{R}$ is the set of numbers which can be represented by the straight line, so extending to infinity in both directions ($\pm\infty$) and without "holes". let us first try to understand why we need $\mathbb{R}$, why we could not just work on $\mathbb{N}$ or $\mathbb{Z}$ for instance. It all boils down to *solving equations*.

3.1 Solving equations in $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$

Let us start by considering $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $a, b \in \mathbb{N}$ with $b < a$. The equation

$$a + x = b$$

cannot be solved in $\mathbb{N}$. As a consequence, we need to enlarge the set $\mathbb{N}$ and introduce $\mathbb{Z}$, where the solution $x = b - a$ belongs. Note that $<$ defines an order relation in $\mathbb{Z}$ and $\mathbb{Z}$ can be arranged in order as

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-a : \, a \in \mathbb{N}\}.$$

Note that the following theorem holds.

Theorem 3.1.1 (Fundamental Theorem of Arithmetic)**.** *Every integer greater than 1 either is a prime number itself or can be represented as the product of prime numbers. This representation is unique up to the order of the factors.*

Let us pass to the equation

$$ax = b,$$

where $a, b \in \mathbb{Z}$ with $a \neq 0$. In general, this equation is not solvable in $\mathbb{Z}$ so we need to enlarge $\mathbb{Z}$ to the set of rational numbers

$$\mathbb{Q} = \{a/b : \, a \in \mathbb{Z}, \, b \in \mathbb{N}\} = \{a/b : \, a \in \mathbb{Z}, \, b \in \mathbb{N}, hcf(a, b) = 1\},$$

where hcf stands for highest common factor. All the arithmetical operations of $\mathbb{Z}$ are easily extended to $\mathbb{Q}$ (addition, multiplication with their properties). At the same time we can introduce an order relation in $\mathbb{Q}$, as follows: let $b, d \in \mathbb{N}$. Then,

$$\frac{c}{d} < \frac{a}{b} \quad \Leftrightarrow \quad bc < ad.$$

Analogously,

$$\frac{a}{b} > \frac{c}{d} \quad \Leftrightarrow \quad ad > bc.$$

Unfortunately, $\mathbb{Q}$ is not enough to solve algebraic equations. Indeed, we will prove now that the equation

$$x^2 = 2$$

does not have a rational solution.

Theorem 3.1.2. *There is no rational number that satisfies the equation $x^2 = 2$.*

Proof. We prove this theorem by contradiction. Assume, that there exists $\frac{a}{b}$, where $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that

$$\frac{a^2}{b^2} = 2.$$

Assume that the fraction a/b is written in lowest terms (highest common factor is 1). From the equation it follows that

$$a^2 = 2b^2.$$

Hence, from the Fundamental Theorem of Arithmetic, a^2 is even ($a^2 = 2h$, $h \in \mathbb{Z}$) and necessarily a is even too (indeed a odd implies a^2 odd). It follows that, for some $k \in \mathbb{Z}$, $a = 2k$ and

$$2(2k^2) = 2b^2$$

which implies

$$2k^2 = b^2.$$

This means that b is even too. So, a and b are both even and have a common factor 2. This is in contradiction with them having highest common factor 1. $\square$

This theorem shows the need to have a new set of numbers which are not rational and they are therefore called *irrational*: $\sqrt{2}$, $\sqrt{3}$, π, e, etc... The set $\mathbb{R}$ contains all of these numbers. More precisely, $\mathbb{R} \setminus \mathbb{Q}$ is the set of irrational numbers.

3.2 The field of real numbers

In the previous section we have discussed the need to extend $\mathbb{N}$ to $\mathbb{Z}$ and $\mathbb{Z}$ to $\mathbb{Q}$. The rigorous construction of $\mathbb{R}$ from $\mathbb{Q}$ is rather complicated and lengthy so we will not treat it here. In this course we postulate the existence of the set of real numbers $\mathbb{R}$ and we summarised its properties in a collection of axioms. In the sequel $\exists!$ stands for *there exists a unique*. $\mathbb{R}$ is an algebraic field, i.e., it fulfils the following properties:

A.1 $\forall a, b \in \mathbb{R}$, $a + b \in \mathbb{R}$ (closed under addition).

A.2 $\forall a, b \in \mathbb{R}$, $a + b = b + a$ (commutativity of addition).

A.3 $\forall a, b, c \in \mathbb{R}$, $(a + b) + c = a + (b + c)$ (associativity of addition).

A.4 $\exists 0 \in \mathbb{R}$: $\forall a \in \mathbb{R}$, $0 + a = a$ (existence of additive identity).

A.5 $\forall a \in \mathbb{R}$ $\exists! x \in \mathbb{R}$: $a + x = 0$ (existence of additive inverse). We write $x = -a$

A.6 $\forall a, b \in \mathbb{R}$, $ab \in \mathbb{R}$ (closed under multiplication).

A.7 $\forall a, b \in \mathbb{R}$, $ab = ba$ (commutativity of multiplication).

A.8 $\forall a, b, c \in \mathbb{R}$, $(ab)c = a(bc)$ (associativity of multiplication).

A.9 $\exists 1 \in \mathbb{R}$: $\forall a \in \mathbb{R}$, $1a = a$ (existence of multiplicative identity).

A.10 $\forall a \in \mathbb{R} \setminus \{0\}$ $\exists! x \in \mathbb{R}$: $ax = 1$ (existence of multiplicative inverse). We write $x = 1/a$.

A.11 $\forall a, b, c \in \mathbb{R}$ $(a + b)c = ac + bc$ (distributive law).

Remark 3.2.1. The set of rational numbers $\mathbb{Q}$ is also an algebraic field because it fulfils the properties A.1-A.11. $\mathbb{Z}$ is not a field. Why? Think about an easy argument to show that $\mathbb{Z}$ is not a field.

We now add some *order axioms* to $\mathbb{R}$. We will therefore say that $\mathbb{R}$ is an ordered algebraic field.

O.1 $\forall a, b \in \mathbb{R}$, $a = b$ or $a < b$ or $a > b$. This is equivalent to say that if $a \geq b$ and $b \geq a$ then $a = b$ (trichotomy law).

O.2 $\forall a, b, c \in \mathbb{R}$ $a > b$ and $b > c$ imply $a > c$ (transitive law).

O.3 $\forall a, b, c \in \mathbb{R}$ $a > b$ implies $a + c > b + c$ (compatibility with addition).

O.4 $\forall a, b \in \mathbb{R}$, $\forall c > 0$, $a > b$ implies $ac > bc$ (compatibility with multiplication).

Note that both $\mathbb{R}$ and $\mathbb{Q}$ are ordered fields. However, there is a main property which distinguish $\mathbb{Q}$ from $\mathbb{R}$: the *completeness axiom.* $\mathbb{R}$ follows the completeness axioms but $\mathbb{Q}$ does not. We will formulate the completeness axiom later on. We will focus now on some important definitions and inequalities which are related to the the fact that $\mathbb{R}$ is an ordered field.

3.2.1 Absolute value and some important inequalities

We begin by defining the notion of *absolute value*

Definition 3.2.2. *Let $a \in \mathbb{R}$. The absolute value of a is defined as follows:*

$$|a| = \begin{cases} a, & a \geq 0, \\ -a, & a < 0. \end{cases}$$

It is common to call the absolute value of a also modulus of a. Note that for every real number a,

$$a > 0 \quad \Leftrightarrow -a < 0.$$

It follows that $|a| \leq 3$ iff $0 \leq a \leq 3$ and $-3 \leq a \leq 0$. This means $-3 \leq a \leq 3$. Analogously, $|a| \geq 3$ is equivalent to $a \geq 3$ or $a \leq -3$.

Proposition 3.2.3. *For all $a, b \in \mathbb{R}$,*

$$|a + b| \leq |a| + |b|.$$

Proof. Note that by definition of absolute value,

$$x \leq |x|, \qquad -x \leq |x|$$

for all $x \in \mathbb{R}$.

We distinguish between two cases.

Case 1: $a + b \geq 0$

$$|a + b| = a + b \leq |a| + |b|.$$

Here we have applied the axiom O.3.

Case 2: $a + b < 0$

We have

$$|a + b| = -(a + b) = (-a) + (-b) \leq |a| + |b|,$$

by the axiom O.3. □

To prove the following inequality we will employ the principle of **mathematical induction**. A **proof by induction** is used when we have a statement depending on a variable $n \in \mathbb{N}$ (or $n \in \mathbb{N}_0$). We prove the statement for $n = 1$ (or $n = 0$). Then we assume the statement valid for $n = k$ and we prove it for $n = k + 1$. This is the so-called induction step. Once concluded we have proven that our statement is true for every n.

Proposition 3.2.4 (Bernoulli's inequality)**.** *For all $n \in \mathbb{N}_0$ and for all $\alpha > -1$,*

$$(1 + \alpha)^n \geq 1 + n\alpha.$$

Proof. The inequality is trivially true for $n = 0, 1$ since $1 = 1$ and

$$1 + \alpha = 1 + \alpha,$$

for all $\alpha \in \mathbb{R}$. Assume that

$$(1 + \alpha)^k \geq 1 + k\alpha,$$

for $k > 1$ and for all $\alpha > -1$. We now consider $(1 + \alpha)^{k+1}$. We can write

$$(1 + \alpha)^{k+1} = (1 + \alpha)(1 + \alpha)^k.$$

By the induction hypothesis and using the fact that $1 + \alpha > 0$ we have

$$\begin{aligned}(1 + \alpha)^{k+1} = (1 + \alpha)(1 + \alpha)^k &\geq (1 + \alpha)(1 + k\alpha) \\ &= 1 + (k + 1)\alpha + k\alpha^2 \geq 1 + (k + 1)\alpha,\end{aligned}$$

since $k\alpha^2 \geq 0$. This concludes the proof by induction. □

3.2.2 Bounded and unbounded sets of real numbers

We now want to focus on the subsets of $\mathbb{R}$ and their properties.

Definition 3.2.5.

(i) A set $S \subseteq \mathbb{R}$ is bounded above *if there exists $c \in \mathbb{R}$ such that*

$$x \leq c,$$

for all $x \in S$. The number c is an upper bound *for S.*

(ii) A set $S \subseteq \mathbb{R}$ is bounded below *if there exists $c \in \mathbb{R}$ such that*

$$x \geq c,$$

for all $x \in S$. The number c is a lower bound *for S.*

(iii) A set $S \subseteq \mathbb{R}$ is bounded *iff it is bounded above and below.*

It follows that S is bounded iff there exists $c > 0$ such that $|x| \leq c$ for all $x \in S$.

Definition 3.2.6. *The following sets are intervals in $\mathbb{R}$:*

(i) $(a, b) = \{x \in \mathbb{R} : a < x < b\}$;

(ii) $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$;

(iii) $(a, b]$, $[a, b)$;

(iv) $(a, \infty) = \{x \in \mathbb{R} : x > a\}$, $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$;

(v) $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$, $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$.

(a, b) is an open interval of $\mathbb{R}$ as well as (a, ∞) and $(-\infty, b)$. $[a, b]$ is a closed interval as well as $[a, \infty)$ and $(-\infty, b]$. Note that these sets are open/closed because their complements are closed/open. The sets (a, b), $[a, b]$, $(a, b]$ and $[a, b)$ are bounded while (a, ∞), $[a, \infty)$, $(-\infty, b)$ and $(-\infty, b]$ are not bounded (unbounded).

Example 3.2.7. As observed above the interval $(a, b]$ is bounded because it has an upper bound and a lower bound. In particular,

$$a < x \leq b,$$

for all $x \in (a, b]$. So, a is a lower bound and b is an upper bound for this set. Since $b \in (a, b]$ we say that b is the maximum of the set $(a, b]$. Analogously, a is the minimum of the set $[a, b)$. Note that maximum and minimum of a set might not exist. Indeed, (a, b) does not have max and min but it has an upper bound and a lower bound. For a set S we use the notations $\max S$ and $\min S$ for its maximum and minimum, respectively.

3.2.3 The completeness axiom

We now introduce the notions of infimum and supremum of a set. These will be particularly useful when the maximum and minimum of a set fail to exist.

Definition 3.2.8. *Let $S \subseteq \mathbb{R}$ be bounded above. Then $a \in \mathbb{R}$ is called the* supremum *(or least upper bound) of S if*

(i) is an upper bound for S:

$$\forall x \in S, \qquad x \leq a;$$

(ii) it is smaller than any other upper bound (least upper bound):

$$\forall \varepsilon > 0, \exists b \in S : \quad b > a - \varepsilon.$$

We write $a = \sup S$.

The Completeness Axiom

Every nonempty subset of $\mathbb{R}$ that is bound above has a least upper bound.

This axiom ensures the existence of $\sup S$ for every nonempty $S \subseteq \mathbb{R}$. In addition, $\mathbb{R}$ can be characterised as follows:

$\mathbb{R}$ is an ordered algebraic field satisfying the Completeness Axiom.

We say that $\mathbb{R}$ is an ordered field fulfilling the Completeness Axiom. The difference between $\mathbb{R}$ and $\mathbb{Q}$ is that $\mathbb{R}$ is complete but $\mathbb{Q}$ is not.

Remark 3.2.9. To show that $\mathbb{Q}$ is not complete we need to find a subset of $\mathbb{Q}$ which is bounded above but does not have a least upper bound in $\mathbb{Q}$. This is quite immediate. Indeed,

$$S = \{x \in \mathbb{Q} : x^2 \leq 2\} = \{x \in \mathbb{Q} : -\sqrt{2} \leq x \leq \sqrt{2}\}.$$

the least upper bound of this set is $\sqrt{2}$ which does not belong to $\mathbb{Q}$.

Analogously, we can introduce the infimum of a set as the greatest lower bound.

Definition 3.2.10. *Let $S \subseteq \mathbb{R}$ be bounded below. Then $a \in \mathbb{R}$ is called the* infimum *(or greatest lower bound) of S if*

(i) is an lower bound for S:

$$\forall x \in S, \qquad x \geq a;$$

(ii) it is greater than any other lower bound (greatest lower bound):

$$\forall \varepsilon > 0, \exists b \in S : \quad b < a + \varepsilon.$$

We write $a = \inf S$.

Theorem 3.2.11. *Let S be a nonempty subset of $\mathbb{R}$ bounded below. Then, $\inf S$ exists and*

$$\inf S = -\sup(-S),$$

where $-S = \{-x : x \in S\}$.

Proof. We begin by analysing our hypothesis. S is bounded below, so there exists $c \in \mathbb{R}$ such that

$$x \geq c,$$

for all $x \in S$. This is equivalent to

$$-x \leq -c.$$

By definition of $-S$ we have that $-S$ is bounded above and by the Completeness Axiom it has a least upper bound $\xi = \sup(-S)$. We want to show that

$$-\xi = -\sup(-S) = \inf S.$$

This means to prove that $-\xi$ is the greatest lower bound for S, i.e., for all $\varepsilon > 0$, $-\xi + \varepsilon$ is not a lower bound:

$$\exists x \in S : \quad x < -\xi + \varepsilon.$$

Our assumption is that ξ is the least upper bound of $-S$, i.e.,

$$\forall \varepsilon > 0, \, \exists x \in S : \quad -x > \xi - \varepsilon.$$

This is equivalent to

$$\forall \varepsilon > 0, \, \exists x \in S : \quad x < -\xi + \varepsilon.$$

In other words we have proven that

$$-\xi = \inf S$$

as desired.

$\square$

Theorem 3.2.12 (The Archimedean Principle)**.** *Let $x \in \mathbb{R}$. Then there exists $n \in \mathbb{Z}$ such that $x < n$.*

Proof. We argue by contradiction. Assume that for all $n \in \mathbb{Z}$

$$x \geq n \;\Leftrightarrow\; n \leq x.$$

Therefore, the set

$$A = \{n : n \in \mathbb{Z}\}$$

is bounded above and by the Completeness Axiom $\sup A$ exists and

$$\sup A \leq x.$$

Note that

$$\sup A - 1 < \sup A$$

and it is not an upper bound for the set A. So, there exists $m \in \mathbb{Z}$ such that

$$m > \sup A - 1 \;\Leftrightarrow\; m + 1 > \sup A.$$

Since $m + 1 \in \mathbb{Z}$ it follows that $m + 1 \in A$ and therefore $\sup A$ is not un upper bound for A. This contradicts our hypothesis and completes the proof. $\square$

Theorem 3.2.13 (Well-ordering Principle)**.** *Any nonempty subset $S \subseteq \mathbb{Z}$ which is bounded below has a minimum.*

Proof. As S is nonempty and it is bounded below then there exists $s \in \mathbb{R}$ such that $s = \inf S$ (by the Completeness Axiom). Since $s + 1 > s$ there exists $m \in S$ such that

$$m < s + 1 \quad \Leftrightarrow \quad m - 1 < s.$$

It follows that

$$m - 1 < s \leq m.$$

We now want to prove that

$$m = \min S = \inf S.$$

This means to prove that $m \in S$ (trivial) and that

$$m \leq n,$$

for all $n \in S$. Let us do a proof by contradiction, i.e., assume that there exists $n \in S$ such that

$$m > n.$$

Hence

$$m > n \geq s > m - 1.$$

But this means

$$m > n \geq m \quad \Leftrightarrow \quad m \leq n < m.$$

This does not make sense so it is our contradiction. □

Analogously,

Theorem 3.2.14 (Well-ordering Principle)**.** *Any nonempty subset $S \subseteq \mathbb{Z}$ which is bounded above has a maximum.*

Theorem 3.2.15. *For any interval $(a, b) \subseteq \mathbb{R}$ there exists a rational $r \in (a, b)$.*

Proof. Since $a \neq b$ we have that $h = b - a > 0$. By the Archimedean Principle (see problems in the end of the chapter), there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < h.$$

Note that $1 < nh + n(b - a)$ or $na < nb - 1$. Let

$$S = \{m \in \mathbb{Z} : \, m < nb\}.$$

This set is non-empty and it is bounded above. Hence, by the well-ordering principle, there exists $m \in S$ such that $m = \max S$. From the definition of maximum we have

$$m < nb \leq m + 1.$$

Concluding,

$$na < nb - 1 \leq (m + 1) - 1 = m < nb.$$

Dividing by n we obtain,

$$a < \frac{m}{n} < b.$$

So, we have found $r = \frac{m}{n} \in \mathbb{Q}$ belonging to the interval (a, b). □

We conclude this chapter going back to the equation $x^2 = 2$. As a consequence of the completeness axiom one can prove that this equation is solvable in $\mathbb{R}$ (see for instance Theorem 1.14 in 2021/22 lecture notes). Hence, by Theorem 3.1.2 its solutions ($\pm\sqrt{2}$) are irrational.

Chapter 4

Sequences of real numbers and limits

The notions of sequence and limit are central in Analysis. They are also fundamental in applied mathematics when we want to represent a phenomena which is happening at time $n = 1, 2, 3, \cdots$, (in other words endlessly) and we want to understand what is happening at time large (at the end), i.e., as n tends to infinity. Sequences and limits are basically involved every time we talk about *approximations* and *behaviour at infinity*. In the sequel we will use the notion of function. A *function* f is a relation between two sets X and Y which links every element $x \in X$ to one and only one element $y \in Y$. We use the notations $f : X \to Y$ and $y = f(x)$.

4.1 Sequences

Definition 4.1.1. *A sequence of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$. We use the notation $a_n = f(n)$ for the elements of the sequence and $(a_n)_{n\in\mathbb{N}}$ (or briefly, (a_n)) for the whole sequence. So,*

$$(a_n) = (a_1, a_2, a_3, \cdots, a_n, \cdots).$$

Example 4.1.2. Some easy examples of sequences:

(i) The sequence $a_n = n$ is the sequence of natural numbers:

$$(a_n) = (1, 2, 3, 4, \cdots, n, \cdots).$$

(ii) The sequence $a_n = \frac{1}{n}$ is

$$(a_n) = \big(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots\big).$$

(iii) The sequence $a_n = \frac{(-1)^n}{n}$ is

$$(a_n) = \big(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \cdots, \frac{(-1)^n}{n}, \cdots\big).$$

4.1.1 Null sequences

In the Examples (i) and (iii) above we see that a_n becomes smaller and smaller as n becomes larger. These are two examples of *null sequences.*

Definition 4.1.3. (a_n) *is a null sequence if*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \ n > N \ \Rightarrow \ |a_n| < \varepsilon.$$

The previous line of mathematical symbols reads as follows: for all $\varepsilon > 0$ there exists a natural number N such that $|a_n| < \varepsilon$ for all $n > N$.

Exercise

We now want to prove that the sequence (a_n) with $a_n = \frac{1}{n}$ is a null sequence. This means to prove that

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \ n > N \ \Rightarrow \ |a_n| = \frac{1}{n} < \varepsilon.$$

In other words, given an arbitrary $\varepsilon > 0$ we need to find $N \in \mathbb{N}$ such that

$$\frac{1}{n} < \varepsilon,$$

for all $n > N$. Note that if $n > N$ then

$$\frac{1}{n} < \frac{1}{N},$$

so if we choose $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \varepsilon,$$

then

$$\frac{1}{n} < \frac{1}{N} < \varepsilon,$$

for all $n > N$ as desired. The question is: can I find $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$? The answer is yes. It is enough to choose $N \in \mathbb{N}$ bigger than $\frac{1}{\varepsilon}$. For instance,

$$N = \left[\frac{1}{\varepsilon}\right] + 1,$$

where $[1/\varepsilon]$ denotes the integer part of $1/\varepsilon$ (greatest integer less or equal to $1/\varepsilon$).

Proposition 4.1.4. (a_n) *is a null sequence iff* $(|a_n|)$ *is a null sequence.*

Proof. This is trivially true because

$$|a_n| = ||a_n||.$$

$\square$

Remark 4.1.5. From the definition of null sequence we see that we can rephrase the mathematical sentence

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}: \ n > N \ \Rightarrow \ |a_n| < \varepsilon.$$

in a more *geometrical* way.

We can equivalently say that (a_n) is a null sequence if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$a_n \in (-\varepsilon, \varepsilon)$$

for all $n > N$.

In other words in a null sequence for all $\varepsilon > 0$ only a finite number of elements of the sequence lays outside the open interval $(-\varepsilon, \varepsilon)$. These are the elements a_n with $n = 1, \cdots, N$. All the others belong to the interval $(-\varepsilon, \varepsilon)$.

4.1.2 Sequences converging to a limit

We now introduce a general notion of limit of a sequence.

Definition 4.1.6. *A sequence (a_n) converges to a limit $a \in \mathbb{R}$ if*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}: \ n > N \ \Rightarrow \ |a_n - a| < \varepsilon.$$

We write

$$\lim_{n \to \infty} a_n = a^1$$

or, equivalently,

$$a_n \to a \quad \textit{as} \quad n \to \infty.$$

We say that the sequence (a_n) converges to a.

In other words when $a_n \to a$ the elements of the sequence a_n become closer to a as n gets larger. Note that in the definition above N depends on the choice of e > 0, i.e., $N = N(\varepsilon)$. Comparing Definition 4.1.3 with Definition 4.1.6 we easily see that (a_n) is a null sequence iff $a_n \to 0$ as $n \to \infty$.

Remark 4.1.7. A sequence does not necessarily has a limit. For instance the sequence

$$(a_n) = (1, 0, 1, 0, 1, 0, 1, \cdots ....),$$

does not have a limit because its entries are jumping between 1 and 0 and not getting closer to any specific value $a \in \mathbb{R}$. We say that $\lim_{n \to \infty} a_n$ does not exists.

However, if a sequence it is constantly equal to a certain number, i.e.,

$$a_n = a,$$

for all $n \in \mathbb{N}$, then trivially $a_n \to a$, as $n \to \infty$.

[1] This notation makes sense because the limit is unique, as we will prove later.

Exercise

We now want to show that the sequence

$$a_n = \frac{n}{n+1}$$

converges to 1 by using the definition of limit seen above. We need to prove that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left| \frac{n}{n+1} - 1 \right| < \varepsilon,$$

for all $n > N$. Since

$$|\frac{n}{n+1} - 1 = \frac{n+1-1}{n+1} - 1 = 1 - \frac{1}{n+1} - 1 = -\frac{1}{n+1},$$

we need to find $N \in \mathbb{N}$ such that

$$\frac{1}{n+1} < \varepsilon$$

for all $n > N$. This is equivalent to require that

$$n + 1 > \frac{1}{\varepsilon} \quad \Leftrightarrow n > \frac{1}{\varepsilon} - 1.$$

Choose $N \in \mathbb{N}$ such that

$$N > \frac{1}{\varepsilon} - 1.$$

Hence, for $n > N$ we have

$$n > N > \frac{1}{\varepsilon} - 1$$

as desired.

We now want to prove that when a sequence is convergent then its limit is unique.

Theorem 4.1.8. *If a sequence is convergent then its limit is unique.*

Proof. This is a proof by contradiction. Assume that there exists $a, b \in \mathbb{R}$ with $a \neq b$ such that $a_n \to a$ and $a_n \to b$ as $n \to \infty$. Hence, for all $\varepsilon > 0$ there exist $N_1, N_2 \in \mathbb{N}$ such that

$$|a_n - a| < \varepsilon,$$

for $n > N_1$ and

$$|a_n - b| < \varepsilon,$$

for $n > N_2$. It follows that for all $n > N = \max(N_1, N_2)$ both the inequalities above hold and by the triangle inequality we can write

$$|a - b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| = |a_n - a| + |a_n - b| < 2\varepsilon.$$

Since $a - b \neq 0$ we can choose

$$\varepsilon = \frac{|a-b|}{3} > 0.$$

This leads to

$$|a-b| < \frac{2}{3}|a-b|,$$

which is a contradiction. □

Definition 4.1.9.

(i) A sequence is divergent if it is not convergent.

(ii) A sequence is divergent to ∞ if for all $M > 0$ there exists $N \in \mathbb{N}$ such that

$$a_n > M,$$

for all $n > N$. We write $a_n \to \infty$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = \infty$.

(iii) A sequence is divergent to $-\infty$ if for all $M > 0$ there exists $N \in \mathbb{N}$ such that

$$a_n < -M,$$

for all $n > N$. We write $a_n \to -\infty$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = -\infty$.

Note that a divergent sequence not necessarily has limit ∞ or $-\infty$. It could happen that the limit does not exist. For instance,

$$(a_n) = (1, 0, 1, 0, 1, 0, \cdots)$$

is divergent and its limit does not exist.

4.1.3 Properties of convergent sequences

In the sequel we say that a sequence (a_n) is *bounded* iff the set of its elements is bounded, i.e., there exist $m, M \in \mathbb{R}$ such that

$$m \leq a_n \leq M,$$

for all $n \in \mathbb{N}$.

Proposition 4.1.10. *Every convergent sequence is bounded.*

Proof. We begin by stating our hypothesis. Let $a_n \to a$ as $n \to \infty$. Then, for all $\varepsilon > 0$ there exists $N \in |N$ such that

$$|a_n - a| < \varepsilon,$$

as $n > N$. It follows that for $\varepsilon = 1$ there exists $N \in \mathbb{N}$ such that

$$a - 1 < a_n < a + 1,$$

for all $n > N$. Let $M = \max\{a_1, a_2, \cdots, a_N, a+1\}$ and $m = \min\{a_1, a_2, \cdots, a_N, a-1\}$. Then,

$$m \leq a_n \leq M,$$

for all n. This shows that the sequence (a_n) is bounded. □

Remark 4.1.11. The converse is not true. Indeed, the sequence

$$(a_n) = (1, -1, 1, -1, 1, -1, \cdots)$$

is bounded but it is not convergent.

Theorem 4.1.12 (The Shift Rule)**.** *Let $n \in \mathbb{N}$. The sequence (a_n) is convergent iff the sequence (a_{n+N}) is convergent. Moreover,*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+N}.$$

Proof. Let $a_n \to a$ as $n \to \infty$. Hence, for all $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that

$$|a_n - a| < \varepsilon$$

for all $n > M$. Let $N \in \mathbb{N}$. Note that $n + N > M$ iff $n > M - N$. So, choosing $n > \max\{M - N, 1\}$ we have that

$$|a_{n+N} - a| < \varepsilon.$$

Analogously, if $a_{n+N} \to a$ as $n \to \infty$ then $a_n \to a$ as $n \to \infty$. □

The following theorem proves some algebraic properties of limits which are particularly useful when we do computations (you have already used these properties many times in Mathematical Methods).

Theorem 4.1.13.

(i) If $a_n \to a$ and $b_n \to b$ then $a_n + b_n \to a + b$ as $n \to \infty$;

(ii) let $\lambda \in \mathbb{R}$. If $a_n \to a$ then $\lambda a_n \to \lambda a$ as $n \to \infty$;

(iii) if $a_n \to a$ and $b_n \to b$ then $a_n b_n \to ab$ as $n \to \infty$;

(iv) if $a_n \to a$ and $b_n \to b$ with $b, b_n \neq 0$ for all $n \in \mathbb{N}$ then $\frac{a_n}{b_n} \to \frac{a}{b}$ as $n \to \infty$.

Proof. (i) Assume that $a_n \to a$ and $b_n \to b$ as $n \to \infty$. This means that for all $\varepsilon > 0$ there exists $N_1, N_2 \in \mathbb{N}$ such that

$$|a_n - a| < \varepsilon,$$

for all $n > N_1$ and

$$|b_n - b| < \varepsilon,$$

for $n > N_2$. Choosing $N = \max(N_1, N_2)$ we have that for all $n > N$ both the inequalities above hold. Hence, by applying the triangle inequality we have

$$|a_n + b_n - a - b| = |(a_n - a) + (b_{,n} - b)| \le |a_n - a| + |b_n - b| < \varepsilon + \varepsilon,$$

for all $n > N$. This shows that $a_n + b_n \to a + b$ as $n \to \infty$. (If we want to get e in the last inequality rather than 2ε we need to start with $\varepsilon/2$ in the definition of the limits of a and b.)

(ii) Our hypothesis is that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|a_n - a| < \varepsilon,$$

for all $n > N$. Let $\lambda \neq 0$. We have,

$$|\lambda a_n - \lambda a| = |\lambda||a_n - a| < |\lambda|\varepsilon,$$

for all $n > N$. Choosing $\varepsilon/|\lambda|$ rather than ε in the definition of the limit of a_n and the corresponding N we get that $\lambda a_n \to \lambda a$, as $n \to \infty$.

If $\lambda = 0$ then $\lambda a_n = 0$ for all n. This is the sequence identically 0 which has clearly limit $0 = \lambda a$ as desired.

(iii) Assume that $a_n \to a$ and $b_n \to b$ as $n \to \infty$. Hence, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ sufficiently large such that

$$|a_n - a| < \varepsilon, \qquad |b_n - b| < \varepsilon,$$

for all $n > N$. We need to estimate $|a_n b_n - ab|$. We write

$$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| \leq |b_n||a_n - a| + |a||b_n - b|.$$

Since the sequence (b_n) is convergent then by Proposition 4.1.10 it is bounded, i.e., there exists $c > 0$ such that $|b_n| \leq c$ for all n. It follows that for $n > N$,

$$|a_n b_n - ab| \leq |b_n||a_n - a| + |a||b_n - b| < c\varepsilon + |a|\varepsilon = (c + |a|)\varepsilon,$$

where $c + |a| > 0$. This proves that $\lim_{n\to\infty} a_n b_n = ab$.

(iv) To prove this final statement it is enough to show that under our hypotheses, $b_n \to b$ implies $\frac{1}{b_n} \to \frac{1}{b}$. Then, by applying (iii) we have immediately that

$$\frac{a_n}{b_n} \to \frac{a}{b},$$

as $n \to \infty$.

Let $b_n \to b$ as $n \to \infty$. Hence, for all $\varepsilon > 0$ there exists $N \in |N$ such that

$$|b_n - b| < \varepsilon,$$

for all $n > N$. Now,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{bb_n}\right| < \frac{\varepsilon}{|b||b_n|},$$

for all $n > N$. We need to estimate $|b_n|$ from below in order to be able to conclude our proof. We first note that since $b_n \to b$ then $|b_n| \to |b|$ as $n \to \infty$ (see problems at the end of the chapter). It follows that

$$|b_n| > |b| - \varepsilon.$$

So, if $\varepsilon < \frac{|b|}{2}$ we have that

$$|b_n| > \frac{|b|}{2},$$

for $n > N$. Summarising, for all $0 < \varepsilon < \frac{|b|}{2}$ there exists $N \in \mathbb{N}$ such that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{bb_n}\right| < \frac{\varepsilon}{|b||b_n|} < \varepsilon\frac{2}{|b|^2},$$

for all $n > N$. This shows that $\frac{1}{b_n} \to \frac{1}{b_n}$ as $n \to \infty$. Indeed, if we have $\varepsilon \geq \frac{|b|}{2}$ we can always find $0 < \varepsilon' < \varepsilon$ with $0 < \varepsilon' < \frac{|b|}{2}$. Applying the argument above we get

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon'\frac{2}{|b|^2} < \varepsilon\frac{2}{|b|^2},$$

for $n > N$.

□

Theorem 4.1.14 (The Sandwich Rule). *Let $a \in \mathbb{R}$ and let $(a_n), (b_n), (c_n)$ be three sequences with*

$$a_n \leq b_n \leq c_n,$$

for all $n \in \mathbb{N}$. If $a_n \to a$ and $c_n \to a$ as $n \to \infty$ then $b_n \to a$ as well as $n \to \infty$.

Proof. We begin by writing what it means that $a_n \to a$ and $c_n \to a$ as $n \to \infty$. It means that, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$a - \varepsilon < a_n < a + \varepsilon,$$
$$a - \varepsilon < c_n < a + \varepsilon,$$

for all $n > N$. Hence, since $a_n \leq b_n \leq c_n$ we have

$$a - \varepsilon < a_n \leq b_n \leq c_n < a + \varepsilon,$$

for all $n > N$. This proves that $b_n \to a$ as $n \to \infty$. □

4.2 Monotone sequences

In this section we study a particular class of sequences which often occur in applications.

Definition 4.2.1.

(i) A sequence (a_n) of real numbers is increasing *if*

$$a_n \leq a_{n+1},$$

for all $n \in \mathbb{N}$.

(ii) *A sequence* (a_n) *of real numbers is* decreasing *if*

$$a_n \geq a_{n+1},$$

for all $n \in \mathbb{N}$.

(iii) *A sequence is* monotone *if it is either increasing or decreasing.*

if the previous estimates hold with $<$ and $>$, respectively then we say that the sequence is strictly increasing and strictly decreasing, respectively.

Example 4.2.2. The sequence $a_n = n^2$ is increasing. The sequence $a_n = \frac{1}{n}$ is decreasing. The sequence

$$a_n = (-1)^n \frac{1}{n}$$

is not monotone. Indeed,

$$(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \cdots)$$

is neither increasing nor decreasing.

Theorem 4.2.3.

(i) *If a sequence* (a_n) *is increasing and bounded above then it is convergent and*

$$\lim_{n\to\infty} a_n = \sup_n a_n.$$

(ii) *If a sequence* (a_n) *is decreasing and bounded below then it is convergent and*

$$\lim_{n\to\infty} a_n = \inf_n a_n.$$

This theorem shows that a monotone bounded sequence is convergent.

Proof.

(i) Let (a_n) be increasing and bounded above. Then $l = \sup_n a_n \in \mathbb{R}$ and by definition of supremum, for all $\varepsilon > 0$ there exists a_N such that

$$a_N > l - \varepsilon.$$

For all $n > N$ we have that

$$a_n \geq a_N > l - \varepsilon.$$

Combining this inequality with the definition of supremum we conclude that, for all $n > N$,

$$l - \varepsilon < a_N \leq a_n \leq l < l + \varepsilon.$$

This means that $a_n \to l$ as $n \to \infty$.

(ii) Let (a_n) be decreasing and bounded below. Then $l = \inf_n a_n \in \mathbb{R}$ and by definition of infimum, for all $\varepsilon > 0$ there exists a_N such that

$$a_N < l + \varepsilon.$$

For all $n > N$ we have that

$$a_n \leq a_N < l + \varepsilon.$$

Combining this inequality with the definition of infimum we conclude that, for all $n > N$,

$$l - \varepsilon < l \leq a_n \leq a_N < l + \varepsilon.$$

This means that $a_n \to l$ as $n \to \infty$.

□

Example 4.2.4. Let us consider the following sequence defined recursively by

$$\begin{aligned} a_1 &= \sqrt{2}, \\ a_n &= \sqrt{a_{n-1} + 2}, \qquad n \geq 2. \end{aligned}$$

We will prove that this sequence is convergent by showing that it is bounded and increasing.

(i) We prove by induction that

$$0 < a_n \leq 2,$$

for all $n \in \mathbb{N}$. This is true for $n = 1$. Assume that for $n = k$,

$$0 < a_k = \sqrt{a_{k-1} + 2} \leq 2.$$

Hence,

$$0 < a_{k+1} = \sqrt{a_k + 2} \leq \sqrt{2 + 2} = 2.$$

So, our sequence is bounded.

(ii) We now prove that it is increasing, i.e., $a_{n+1} \geq a_n$, for all $n \in \mathbb{N}$. This is equivalent to prove that

$$\sqrt{a_n + 2} \geq a_n \quad \Leftrightarrow a_n + 2 \geq a_n^2 \quad \Leftrightarrow \quad a_n^2 - a_n - 2 = (a_n - 2)(a_n + 1) \leq 0.$$

The last inequality holds because $0 < a_n \leq 2$.

By Theorem 4.2.3 the sequence (a_n) is convergent because it is bounded and increasing and

$$\lim_{n\to\infty} a_n = \sup_n a_n = x$$

By the shift rule,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n-1} = x.$$

So, taking the limit at both sides of the equality

$$a_n^2 = a_{n-1} + 2,$$

we get

$$x^2 - x - 2 = 0.$$

This gives $x = 2$ or $x = -1$. Since x must be positive we have that $x = 2$. Concluding, $a_n \to 2$ as $n \to \infty$.

4.3 Subsequences and the Bolzano-Weierstrass Theorem

In this section we deal with subsequences, i.e., the sequences obtained from a given sequence (a_n) by selecting only some of its entries.

Definition 4.3.1. *Let n_k be a strictly increasing sequence of natural numbers (i.e. $1 \leq n_1 < n_2 < n_3 < \cdots$). Then a_{n_k} is called a* subsequence *of a_n.*

Example 4.3.2. Let $a_n = \frac{1}{n}$ and $n_k = k^2$. Then $a_{n_k} = a_{k^2} = \frac{1}{k^2}$.

If a sequence is convergent then all its subsequences are convergent too.

Proposition 4.3.3. *Let (a_n) be a convergent sequence and let (a_{n_k}) be a subsequence of (a_n). Then,*

$$\lim_{n\to\infty} a_n = \lim_{k\to\infty} a_{n_k}.$$

Proof. Since $a_n \to a$ as $n \to \infty$ we have that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|a_n - a| < \varepsilon,$$

for all $n > N$. Since (n_k) is a strictly increasing sequence, there exists $K \in \mathbb{N}$ such that

$$n_k > n_K > N,$$

for all $k > K$. Hence, for all $k > K$,

$$|a_{n_k} - a| < \varepsilon.$$

This shows that $a_{n_k} \to a$ as $k \to \infty$, as desired. □

It follows from this result that if a sequence has a divergent subsequence then necessarily the whole sequence is not convergent. We now prove a non trivial result about sequences which will lead us to the proof of the Bolzano-Weierstrass Theorem.

Theorem 4.3.4. *Every sequence has a monotone subsequence.*

Proof. Let (a_n) be our sequence and let C be the subset of the natural numbers defined by

$$C = \{n \in \mathbb{N} : \text{ if } m > n \text{ then } a_m < a_n\},$$

i.e., those indices for which all following sequence elements are smaller.

Now C is either infinite or finite (possibly empty).

If C is infinite there exist $n_1 < n_2 < n_3 < \cdots$ in C, and thus $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ is a decreasing subsequence of a_n.

If C is finite there exists n_1 such that none of n_1, $n_1 + 1$, $n_1 + 2, \ldots$ are in C. Since $n_1 \notin C$ there exists $n_2 > n_1$ with $a_{n_2} \geq a_{n_1}$. Similarly $n_2 \notin C$ so there exists $n_3 > n_2$ with $a_{n_3} \geq a_{n_2}$. Proceeding in this way we see that we have an increasing subsequence of a_n.

In each case we have shown there exists a monotone subsequence. If $C = \emptyset$ then $m > n$ implies $a_m \geq a_n$. This means that the sequence (a_n) is increasing. □

Example 4.3.5. Let $a_n = (-1)^n n$. Then the subsequence a_{n_k}, where $n_k = 2k$, increases.

The following corollary of Theorem 4.3.4 is the famous The Bolzano-Weierstrass Theorem.

Corollary 4.3.6 (The Bolzano-Weierstrass Theorem)**.** *Every bounded sequence has a convergent subsequence.*

Proof. By Theorem 4.3.4 we have that our sequences has a bounded monotone subsequence. So, by Theorem 4.2.3 this subsequence is convergent. □

Example 4.3.7. Let $a_n = \sin \frac{\pi n}{2}$. Then the subsequence a_{n_k}, $n_k = 4k$, converges to zero.

The Bolzano-Weierstrass Theorem appears in other contexts too. It is so important that it is worth giving another, different, proof that is generalisable to sequences other than real ones. It is based on the idea of bisection that one also encounters in numerical analysis.

Another proof of the Bolzano-Weierstrass Theorem. Let (s_n) be a bounded sequence. Hence, there exist $a_0 < b_0$ such that $a_0 \leq s_n \leq b_0$ for all n. If we bisect the closed interval $[a_0, b_0]$, at least one half must contain s_n for infinitely many n. Call such a half $[a_1, b_1]$. If we now bisect $[a_1, b_1]$, then, again, at least one half must contain s_n for infinitely many n. Call such a half $[a_2, b_2]$. We can repeat this process indefinitely to obtain a nested sequence $[a_k, b_k]$ of closed intervals with the following properties.

(i) $a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq b_k \leq \cdots \leq b_2 \leq b_1 \leq b_0$.

(ii) $b_k - a_k = \frac{1}{2^k}(b_0 - a_0)$.

(iii) For every k, $[a_k, b_k]$ contains s_n for infinitely many n.

From (i) we see that the sequence a_k is increasing and bounded above, therefore convergent, say to ℓ. Likewise, the sequence b_k is decreasing and bounded below, therefore convergent, say to ℓ'. Taking limits in (ii) gives $\ell' - \ell = 0$, i.e., $\ell = \ell'$.

To construct a convergent subsequence s_{n_k} we simply choose $s_{n_k} \in [a_k, b_k]$ for each k, in such a way that $n_{k+1} > n_k$ for all k. Condition (iii) clearly ensures that this is possible. We then have $a_k \leq s_{n_k} \leq b_k$ for all k, and therefore by the Sandwich Theorem we must have s_{n_k} convergent to ℓ. □

4.4 Cauchy sequences

In this section we will prove that convergent sequences can be characterised by the fact that their entries a_n and a_m become closer and closer as the indexes n, m become larger and larger. This is quite an intuitive idea that we will state now precisely in mathematical terms.

Definition 4.4.1. *We say that a sequence (a_n) of real numbers is a Cauchy sequence if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$|a_n - a_m| < \varepsilon,$$

for all $n, m > N$.

The sequence $a_n = \frac{1}{n}$ is a Cauchy sequence. Indeed, given an arbitrary $\varepsilon > 0$ we have that

$$\left| \frac{1}{n} - \frac{1}{m} \right| \leq \frac{1}{n} + \frac{1}{m} < 2\varepsilon$$

for all $n, m > N$ with

$$\frac{1}{N} < \varepsilon.$$

More in general we can prove the following proposition.

Proposition 4.4.2. *Every convergent sequence is a Cauchy sequence.*

Proof. Let (a_n) be a convergent sequence with limit $a \in \mathbb{R}$. Hence, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|a_n - a| < \varepsilon,$$

for all $n > N$. Let us now take any $n, m > N$. We have,

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a_m - a| < 2\varepsilon.$$

This show that our convergent sequence is a Cauchy sequence. □

Proposition 4.4.3. *Every Cauchy sequence is bounded.*

Proof. Let (a_n) be a Cauchy sequence. Then, for $\varepsilon = 1$ we have that there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < 1,$$

for all $n, m > N$. Hence, for all $n \in \mathbb{N}$,

$$|a_n| \leq \max\{|a_1|, |a_2|, \cdots, |a_N|, 1 + |a_{N+1}|\}.$$

This shows that the sequence (a_n) is bounded. □

The most important part of this section is the following theorem where we prove that every Cauchy sequence is convergent. It follows that convergent sequences are characterised by the Cauchy property, or in other words being a Cauchy sequence is a sufficient and necessary condition for being a convergent sequence. In general, we say that a space where every Cauchy sequence is convergent is a *complete space*. So, $\mathbb{R}$ is *complete*. Please, do not confuse the fact that $\mathbb{R}$ is complete with the axiom of completeness seen earlier!

Theorem 4.4.4 (Cauchy's Convergence Criterion)**.** *Every Cauchy sequence of real numbers is convergent.*

Proof. Let (a_n) be a Cauchy sequence. Hence, by Proposition 4.4.3 the sequence (a_n) is bounded. The Bolzano-Weierstrass Theorem tell us that (a_n) has a convergent subsequence $(a_{n_k})_k$. We now want to prove that (a_n) is convergent to the same limit a of $(a_{n_k})_k$. We write down our hypotheses in mathematical terms. We have:

$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N}, \, \forall n, m > N, \qquad |a_n - a_m| < \varepsilon;$$

$$\forall \varepsilon > 0 \, \exists K \in \mathbb{N}, \, \forall k > K, \qquad |a_{n_k} - a| < \varepsilon.$$

Hence, for all $n > N$ there exists $n_k > N$ with $k > K$ such that

$$|a_n - a_{n_k}| < \varepsilon,$$

and

$$|a_{n_k} - a| < \varepsilon.$$

Concluding, for all $n > N = N(\varepsilon)$,

$$|a_n - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < 2\varepsilon.$$

This shows that $a_n \to a$ as $n \to \infty$. □

4.5 Some fundamental limits

We conclude this chapter with some fundamental limits that you have probably encountered already in Mathematical Methods or your previous maths studies. We will make use of the following binomial theorem, where, for $n, k \in \mathbb{N}$,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

and

$$h! = 1 \cdot 2 \cdot 3 \cdots (h-1)h, \qquad 0! = 1.$$

Theorem 4.5.1 (The binomial theorem)**.** *Let $x, y > 0$ and $n \in \mathbb{N}$. Hence,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof. We perform a proof by induction. For $n = 1$ we have

$$x + y = \sum_{k=0}^{1} \binom{1}{k} x^k y^{1-k} = y + x.$$

Assume that our equality holds for n. We want to prove it for $n + 1$. We have

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)(x+y)^n = (x+y)\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} \\
&= \sum_{k=0}^{n}\binom{n}{k}x^{k+1}y^{n-k} + \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k+1} \\
&= \binom{n}{0}xy^n + \binom{n}{1}x^2y^{n-1} + \binom{n}{2}x^3y^{n-2} + \cdots + \binom{n}{n}x^{n+1} \\
&+ \binom{n}{0}y^{n+1} + \binom{n}{1}xy^n + \binom{n}{2}x^2y^{n-1} + \cdots + \binom{n}{n}x^n y \\
&= \binom{n}{0}y^{n+1} + \left(\binom{n}{0} + \binom{n}{1}\right)xy^n + \left(\binom{n}{1} + \binom{n}{2}\right)x^2y^{n-1} + \\
&+ \left(\binom{n}{2} + \binom{n}{3}\right)x^3y^{n-2} + \cdots + \left(\binom{n}{n-1} + \binom{n}{n}\right)x^n y + \binom{n}{n}x^{n+1}.
\end{aligned}$$

Since, for $h \leq n$,

$$\binom{n}{h-1} + \binom{n}{h} = \binom{n+1}{h},$$

$\binom{n}{0} = \binom{n+1}{0}$ and $\binom{n}{n} = \binom{n+1}{n+1}$, we conclude that

$$(x+y)^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}x^k y^{n+1-k},$$

as desired. $\square$

Example 4.5.2.

$$\lim_{n\to\infty} a^n = \begin{cases} \infty, & a > 1, \\ 1, & a = 1, \\ 0, & |a| < 1. \end{cases}$$

If $a > 1$ then $a = 1 + x$, where $x > 0$. By the binomial theorem we have

$$\begin{aligned}
a^n = (1+x)^n &= \sum_{k=0}^{n}\binom{n}{k}1^k x^{n-k} = \sum_{k=0}^{n}\binom{n}{k}1^{n-k}x^k \\
&= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n.
\end{aligned}$$

Since all the summands are positive we have that

$$a^n > 1 + nx.$$

So, for all $M > 0$ if $n > N$ with

$$Nx > M$$

we conclude that

$$a^n > 1 + nx > 1 + Nx > 1 + M > M.$$

This shows that $\lim_{n\to\infty} a^n + \infty$ if $a > 1$. If $a = 1$ then $a^n = 1$ for all n and therefore its limit is 1 as well. Let now $|a| < 1$. We have $\frac{1}{|a|} > 1$ and therefore

$$\frac{1}{|a|^n} \to \infty,$$

as $n \to \infty$. This means that $|a|^n \to 0$ as $n \to \infty$. As a consequence $a^n \to 0$ as well.

Example 4.5.3. The Euler's number e is defined as the limit of a sequence. Precisely,

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e}.$$

It is an irrational number and also a transcendental number (it is not the solution of an algebraic equation with rational coefficients). Let us convince ourselves that the sequence above is convergent. By using the binomial theorem again we can write, for all $n \in \mathbb{N}$,

$$\begin{aligned} &\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!}\frac{1}{n^k}, \\ &= 1 + \frac{n}{n} + \frac{n(n-1)}{n^2}\frac{1}{2!} + \frac{n(n-1)(n-2)}{n^3}\frac{1}{3!} + \cdots + \frac{n(n-1)(n-2)\cdots 2\cdot 1}{n^n}\frac{1}{n!} \end{aligned}$$

Since all the fractions of the type

$$\frac{n(n-1)(n-2)\cdots(n-h)}{n^{h+1}}$$

tends to 1 as $n \to \infty$ we have that

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.$$

Let us now consider the sequence

$$a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.$$

This is an increasing sequence and it is also a convergent sequence (this fact will become clear in the next chapter). Concluding,

$$\mathrm{e} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!}.$$

Chapter 5

Series of real numbers

In the first part of the module you have seen sequences of real numbers. Sequences are an important concept in pure as well as applied mathematics. For instance, when applied mathematicians use a mathematical model to study a physical phenomenon they often make use of approximations which basically means to deal with a lot of sequences and their limits. They might also need to make the sum of the elements of the sequence, for instance when they need to measure a certain area. In other words, they need to give a mathematical meaning to the following object:

$$\sum_{n=1}^{\infty} a_n.$$

This is possible thanks to the concept of series that we will study in detail in this chapter.

5.1 Series of real numbers: definition and main properties

Definition 5.1.1. *Let (a_n) be a sequence of real number. Let $k \in \mathbb{N}$ and let S_k be the finite sum*

$$\sum_{n=1}^{k} a_n.$$

If the sequence (S_k) of the partial sums *converges to a real number S we say that the series or infinite sum $\sum_{n=1}^{\infty} a_n$ is convergent, i.e.,*

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k = S.$$

If the sequence (S_k) is divergent we say that the series $\sum_{n=1}^{\infty} a_n$ is divergent as well.

Recall that for a sequence being divergent means not being convergent. So, S could be equal to $-\infty$, ∞ or S could simply not exists as a limit. For instance the sequence given by $0, 1, 0, 1, 0, 1, 0, \cdots$ is divergent because it has no limit.

Remark 5.1.2. The expressions $\sum_{n=1}^{\infty} a_n$ and $\sum_{m=1}^{\infty} a_m$ are equivalent, in the sense that we can change the sub-index letter. If we start to count the elements a_n from $n = n_0$ then we use the expression

$$\sum_{n=n_0}^{\infty} a_n.$$

In this case,

$$S_k = \sum_{n=n_0}^{k} a_n,$$

for $k \geq n_0$.

In the sequel we focus on convergent series. We discuss their main properties and we give some important examples. We will also prove some theorems and propositions.

5.1.1 Convergent series of real numbers

The next theorem explains some important properties of convergent series.

Theorem 5.1.3. *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Then,*

(i) the sequence (a_n) converges to 0;

(ii) the tail of the series, i.e., the sequence

$$t_k = \sum_{n=k}^{\infty} a_n$$

also converges to 0;

(iii) for all $c \in \mathbb{R}$, the series $\sum_{n=1}^{\infty} c a_n$ is convergent and

$$\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n;$$

(iv) if $\sum_{n=1} b_n$ is convergent then $\sum_{n=1}(a_n + b_n)$ is convergent as well and

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

Proof.

(i) Our hypothesis is that the series $\sum_{n=1}^{\infty} a_n$ is convergent, i.e., there exists $l \in \mathbb{R}$ such that $S_k \to S$:

$$(A) \qquad \forall \varepsilon > 0,\ \exists M \in \mathbb{N} : \forall k \geq M \qquad \left| \sum_{n=1}^{k} a_n - l \right| < \varepsilon.$$

We want to prove that the sequence a_n converges to 0, i.e.,

$$(B) \qquad \forall \varepsilon > 0\, \exists N \in \mathbb{N} : \forall n \geq N \qquad |a_n| < \varepsilon.$$

To go from (A) to (B) we need to write a_n in terms of the sequence of the partial sums and the sum l. Note that, for $n > 1$,

$$a_n = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} a_i = \left(\sum_{i=1}^{n} a_i - l \right) - \left(\sum_{i=1}^{n-1} a_i - l \right).$$

Hence, by (A), choosing $n - 1 \geq M$ and applying the triangle inequality ($|x + y| \leq |x| + |y|$) we have that

$$|a_n| \leq \left| \sum_{i=1}^{n} a_i - l \right| + \left| \sum_{i=1}^{n-1} a_i - l \right| \leq 2\varepsilon,$$

for all $n \geq M + 1$. This shows that a_n converges to 0. Note that if you want to get ε rather than 2ε you should state (A) with $\varepsilon/2$ rather than ε.

(ii) We now want to prove that also the tail of the series, i.e., the sequence

$$t_k = \sum_{n=k}^{\infty} a_n.$$

First of all, note that

$$t_k = \lim_{N \to \infty} \sum_{n=k}^{N} a_n = \lim_{N \to \infty} \left(\sum_{n=1}^{N} a_n - \sum_{n=1}^{k-1} a_n \right) = \lim_{N \to \infty} \sum_{n=1}^{N} a_n - \sum_{n=1}^{k-1} a_n = l - \sum_{n=1}^{k-1} a_n.$$

We now take the limit as $k \to \infty$ in both side of this equality. We have

$$\lim_{k \to \infty} t_k = l - \lim_{k \to \infty} \sum_{n=1}^{k-1} a_n.$$

Since for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $k \geq N$

$$\left| \sum_{n=1}^{k} a_n - l \right| < \varepsilon$$

it follows that $\lim_{k \to \infty} \sum_{n=1}^{k-1} a_n = l$. It is sufficient to take $k \geq N + 1$ given $\varepsilon > 0$. Hence,

$$\lim_{k \to \infty} t_k = l - \lim_{k \to \infty} \sum_{n=1}^{k-1} a_n = l - l = 0,$$

as desired.

(iii) We now want to prove that for all $c \in \mathbb{R}$, the series $\sum_{n=1}^{\infty} ca_n$ is convergent and

$$\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n.$$

By assumption, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $k \geq N$

$$\left| \sum_{n=1}^{k} a_n - l \right| < \varepsilon.$$

Hence,

$$\left| \sum_{n=1}^{k} ca_n - cl \right| < |c|\varepsilon. \tag{5.1.1}$$

This shows that the series $\sum_{n=1}^{\infty} ca_n$ converges to $c\sum_{n=1}^{\infty} a_n$. In order to get ε in (5.1.1) one can choose $\varepsilon/|c|$ and the corresponding N in the definition of the convergent series $\sum_{n=1}^{\infty} a_n$.

(iv) Assume that both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1} b_n$ are convergent with sums l and m, respectively. Hence, for all $\varepsilon > 0$ there exists $N_1 > 0$ such that for all $k \geq N_1$.

$$\left| \sum_{n=1}^{k} a_n - l \right| < \frac{\varepsilon}{2}$$

and there exists $N_2 \in \mathbb{N}$ such that for all $k \geq N_2$

$$\left| \sum_{n=1}^{k} b_n - m \right| < \frac{\varepsilon}{2}.$$

It follows that for all $k \geq N = \max\{N_1, N_2\}$,

$$\left| \sum_{n=1}^{k} (a_n + b_n) - l - m \right| \leq \left| \sum_{n=1}^{k} a_n - l \right| + \left| \sum_{n=1}^{k} b_n - m \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves what we wanted, i.e.,

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

□

the following corollary is an easy test for divergent series.

Corollary 5.1.4. *If $a_n \not\to 0$ then the series $\sum_{n=0}^{\infty} a_n$ is divergent.*

This corollary is the negation of the first assertion of Theorem 5.1.3.

We now discuss some famous examples of series.

5.1.2 The geometric series and the telescopic series

Proposition 5.1.5. *The geometric series*

$$\sum_{n=0}^{\infty} x^n$$

diverges for $|x| \geq 1$ *and converges for* $|x| < 1$ *with sum*

$$\frac{1}{1-x}.$$

Proof. Let us compute the sequence of the partial sums of the geometric series. For $k \in \mathbb{N}$ we have

$$S_k = \sum_{n=0}^{k} x^n = 1 + x + \cdots + x^k.$$

Note that

$$S_k = \sum_{n=0}^{k} x^n = 1 + x + \cdots + x^k = \frac{1 - x^{k+1}}{1-x}.$$

Hence,

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{1 - x^{k+1}}{1-x}.$$

The result of this limit depends on x. Indeed, if $|x| < 1$ then $x^{k+1} \to 0$ and

$$\lim_{k\to\infty} S_k = \frac{1}{1-x}.$$

If $x \geq 1$ then $S_k \to \infty$. If $x \leq -1$ then the sequence S_k does not have a limit. It follows that the geometric series is convergent for $|x| < 1$ and divergent for $|x| \geq 1$. □

Proposition 5.1.6. *The telescopic series*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

converges and has infinite sum 1*.*

Proof. By direct computations

$$\begin{aligned} S_k = \sum_{n=1}^{k} \frac{1}{n(n+1)} &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{(k-1)k} + \frac{1}{k(k+1)} \\ &= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{k-1} - \frac{1}{k}) + (\frac{1}{k} - \frac{1}{k+1}) \\ &= 1 - \frac{1}{k+1}. \end{aligned}$$

Taking the limit as $k \to \infty$ we easily see that S_k converges to 1. □

5.2 Simple tests for convergence and divergence

We now want to prove some simple tests which will allow us to understand if a series is convergent or not. We begin with the following shift rule for series which is analogous to the ones seen for sequence.

Proposition 5.2.1 (Shift rule for series). *Let $N \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} a_n$ converges iff*[1] *$\sum_{n=1}^{\infty} a_{N+n}$ converges.*

Proof. By definition of convergent series we have that $\sum_{n=1}^{\infty} a_n$ is convergent iff the corresponding sequence $S_k = \sum_{n=1}^{k} a_n$ of the partial sums is convergent. Analogously, $\sum_{n=1}^{\infty} a_{N+n}$ is convergent iff $T_k = \sum_{n=1}^{k} a_{N+n}$ is convergent. Note that

$$T_k = S_{N+k} - \sum_{n=1}^{N} a_n = a_{N+1} + \cdots + a_{N+k},$$

and

$$\sum_{n=1}^{N} a_n = c \in \mathbb{R}$$

because it is a finite sum, i.e.,

$$T_k = S_{N+k} - c.$$

By the shift theorem for sequences we have that S_k is convergent iff S_{N+k} is convergent iff T_k is convergent. This proves that $\sum_{n=1}^{\infty} a_n$ is convergent iff $\sum_{n=1}^{\infty} a_{N+n}$ is convergent. □

The following test is based on comparison and it is stated for series of non-negative numbers.

Proposition 5.2.2 (Comparison test for series). *Let (a_n) and (b_n) be two sequences such that $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$.*

(i) If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges as well.

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges as well.

Proof. (i) We assume that the series $\sum_{n=1}^{\infty} b_n$ converges. This means that its sequence of the finite sums

$$S_k = \sum_{n=1}^{k} b_n$$

is convergent. In particular, the sequence (S_k) is increasing ($S_k \leq S_{k+1}$) and bounded ($S_k \leq \sup_h S_h \in \mathbb{R}$) and

$$S_k \to \sum_{n=1}^{\infty} b_n = \sup_h S_h \in \mathbb{R}.$$

[1] if and only if

Since, $0 \le a_n \le b_n$ we have that, for all $k \in \mathbb{N}$,

$$0 \le \sum_{n=1}^{k} a_n \le \sum_{n=1}^{k} b_n \le \sum_{n=1}^{\infty} b_n.$$

So, the sequence of the partial sums of the series $\sum_{n=1}^{\infty} a_n$ is increasing and bounded and therefore convergent (see a previous result about sequences). This proves the first assertion of our proposition.

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges then the sequence of its partial sums is not bounded (if it were it would converge). This means that

$$\sup_k \sum_{n=0}^{k} a_n = \infty.$$

Since, $0 \le a_n \le b_n$ for all $n \in \mathbb{N}$ it follows that

$$\sum_{n=1}^{k} b_n \ge \sum_{n=1}^{k} a_n \ge 0$$

and therefore

$$\sup_k \sum_{n=1}^{k} b_n \ge \sum_{n=1}^{\infty} a_n = \infty.$$

This shows that the sequence of the partial sums of $\sum_{n=1}^{\infty} b_n$ is divergent.

□

It is now a good moment to introduce the notion of absolutely convergent series.

Definition 5.2.3. *We say that a series $\sum_{n=0}^{\infty} a_n$ is* absolutely convergent *if the series $\sum_{n=0}^{\infty} |a_n|$ is convergent.*

Note that the terms a_n of the series are not necessarily positive while the series $\sum_{n=0}^{\infty} |a_n|$ has only non-negative entries. To prove the following proposition we make use of the comparison test (Proposition 5.2.2) above.

Proposition 5.2.4. *Every absolutely convergent series is convergent.*

Proof. Let us define the positive and negative parts of the terms a_n, i.e.,

$$a_n^+ = \begin{cases} a_n, & \text{if } a_n \ge 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$a_n^- = \begin{cases} -a_n, & \text{if } a_n \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

Both the sequences a_n^+ and a_n^- are non-negative and by construction

$$\begin{aligned} a_n &= a_n^+ - a_n^-, \\ |a_n| &= a_n^+ + a_n^-, \\ 0 \le a_n^+ &\le |a_n|, \\ 0 \le a_n^- &\le |a_n|. \end{aligned}$$

Since the series $\sum_{n=1}^{\infty} |a_n|$ is convergent then by the comparison test both series $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are convergent. Then, by Theorem 5.1.3 (iii) and (iv) their difference

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-$$

is convergent as well. This proves that the series $\sum_{n=1}^{\infty} a_n$ is convergent. □

Absolutely convergent series can be re-ordered without losing convergence.

Theorem 5.2.5. *Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series and $\phi : \mathbb{N} \to \mathbb{N}$ a bijection. Then, the re-ordered series $\sum_{n=1}^{\infty} a_{\phi(n)}$ converges to the same sum.*

The proof of this theorem can be found in the 2021/22 lecture notes (Theorem 3.16) and it is not examinable.

If a series converges then it does not necessarily converges absolutely. This motivates the following definition.

Definition 5.2.6. *We say that the series $\sum_{n=1}^{\infty} a_n$ converges conditionally if it is convergent but not absolutely convergent.*

Example 5.2.7. The series

$$\sum_{n=1}^{\infty} a_n = 1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \cdots$$

is conditionally convergent. Indeed it is convergent but not absolutely convergent. If we compute the sequence of the finite sums we have that $S_k = 0$ if k is even and

$$\begin{aligned} S_1 &= 1, \\ S_3 &= \frac{1}{2}, \\ S_5 &= \frac{1}{3}, \\ S_7 &= \frac{1}{5}, \\ &\vdots \end{aligned}$$

It follows that $S_k \to 0$ as $k \to \infty$. However, the series $\sum_{n=1}^{\infty} |a_n|$ is not convergent. Indeed,

$$\sum_{n=1}^{\infty} a_n = 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \cdots$$

and

$$\begin{aligned} S_1 =& 1, \\ S_2 =& 2, \\ S_3 =& \frac{5}{2}, \\ S_4 =& 3, \\ S_5 =& \frac{10}{3}, \\ &\vdots \end{aligned}$$

This is an increasing sequence so $\lim_{k\to\infty} S_k = \infty$.

The following test is very frequently used in problems and exercises.

Theorem 5.2.8 (Ratio test for series)**.** *Let $a_n > 0$ for all $n \in \mathbb{N}$ and let $(a_{n+1}/a_n) \to l$ as $n \to \infty$.*

(i) If $l < 1$ then the series $\sum_{n=0}^{\infty} a_n$ converges;

(ii) if $l > 1$ or $l = \infty$ then $\sum_{n=0}^{\infty} a_n$ diverges.

Proof. (i) Assume that $l < 1$. Then, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left| \frac{a_{n+1}}{a_n} - l \right| < \varepsilon,$$

for all $n \geq N$. This means

$$l - \varepsilon < \frac{a_{n+1}}{a_n} < l + \varepsilon,$$

hence

$$0 < a_{n+1} < a_n(l + \varepsilon),$$

for all $n \geq N$. To prove that the series $\sum_{n=0}^{\infty} a_n$ is convergent we will prove that its tail

$$\sum_{n=N+1}^{\infty} a_n$$

is convergent. Indeed, by iteration

$$a_{N+1} < a_N(l + \varepsilon),$$

$$a_{N+2} < a_{N+1}(l + \varepsilon) < a_N(l + \varepsilon)^2$$

and

$$a_{N+k} < a_N(l+\varepsilon)^k,$$

for all $k \in \mathbb{N}$. It follows that

$$0 \leq \sum_{n=N+1}^{\infty} a_n = \sum_{k=1}^{\infty} a_{N+k} < \sum_{k=1}^{\infty} a_N(l+\varepsilon)^k < a_N \sum_{k=0}^{\infty}(l+\varepsilon)^k.$$

The last series is a geometric series. Since $l < 1$ we can choose $\varepsilon > 0$ small enough such that $l + \varepsilon < 1$ and therefore the corresponding series is convergent. By the comparison test we have that the series $\sum_{n=N+1}^{\infty} a_n$ is convergent too and therefore the series $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) Assume now that $l > 1$. From the definition of limit we have that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon,$$

for all $n \geq N$. This means

$$l - \varepsilon < \frac{a_{n+1}}{a_n} < l + \varepsilon,$$

hence

$$a_{n+1} > a_n(l-\varepsilon),$$

for all $n \geq N$. In analogy to (i), we have

$$a_{N+1} > a_N(l-\varepsilon)$$

and by iteration as in (i),

$$a_{N+2} > a_{N+1}(l-\varepsilon) > a_N(l-\varepsilon)^2$$

and in general

$$a_{N+k} > a_N(l-\varepsilon)^k,$$

for all $k \in \mathbb{N}$. It follows that

$$\sum_{n=0}^{\infty} a_n \geq \sum_{k=1}^{\infty} a_{N+k} > \sum_{k=1}^{\infty} a_N(l-\varepsilon)^k = a_N \sum_{k=1}^{\infty}(l-\varepsilon)^k.$$

Hence, by choosing $\varepsilon > 0$ small enough such that $l - \varepsilon > 1$ we have that the geometric series

$$\sum_{k=1}^{\infty}(l-\varepsilon)^k$$

is divergent and therefore by the comparison test the series $\sum_{n=0}^{\infty} a_n$ is divergent as well.

Finally, if $l = \infty$ then for all $c > 0$ there exists $N \in \mathbb{N}$ such that

$$\frac{a_{n+1}}{a_n} > c,$$

for all $n \geq N$. We can therefore repeat the argument above with $l - \varepsilon > 1$ replaced by $c > 1$ and conclude that the series is divergent.

$\square$

Remark 5.2.9. What can I say about the series $\sum_{n=0}^{\infty} a_n$ when $l = 1$? Not much, in the sense that the previous proof will not work and we might have a convergent series or a divergent series. For instance, if $a_n = 1$ for all n, we have that $a_{n+1}/a_n = 1$ for all n and the corresponding series is divergent. However, the telescopic series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

is convergent as seen in Proposition 5.1.6 but

$$\frac{a_{n+1}}{a_n} = \frac{n(n+1)}{(n+1)(n+2)} = \frac{n}{n+2} \to 1.$$

The ratio test for series with positive elements can be adapted to general series if stated in terms of absolute convergence. Note that it is not restrictive to assume that $a_n \neq 0$ for all n (if $a_n = 0$ then it does not contribute to the sum).

Corollary 5.2.10 (Extended version of the ratio test)**.** *Let $a_n \neq 0$ for all $n \in \mathbb{N}$ and let $(|a_{n+1}|/|a_n|) \to l$ as $n \to \infty$.*

(i) If $l < 1$ then the series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent and therefore convergent;

(ii) if $l > 1$ or $l = \infty$ then $\sum_{n=0}^{\infty} a_n$ is divergent and $|a_n| \to \infty$;

(iii) if $l = 1$ then the test is inconclusive.

Proof.

(i) If $l < 1$ we can apply the ratio test to the series $\sum_{n=0}^{\infty} |a_n|$. We conclude that this series is convergent and therefore $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) if $l > 1$ by applying the ratio test to $\sum_{n=1}^{\infty} |a_n|$ we have that this series is divergent. In particular, looking at the proof of Theorem 5.2.8 we have that

$$|a_{N+k}| > |a_N|(l - \varepsilon)^k,$$

where N is chosen large enough, $\varepsilon > 0$ is chosen small enough such that $l - \varepsilon > 1$ and k varies in $\mathbb{N}$. Taking the limit as $k \to \infty$ we therefore have that $|a_n| \to \infty$ as $n \to \infty$. It follows that the series $\sum_{n=1}^{\infty} a_n$ cannot be convergent because if it were then $a_n \to 0$ which is equivalent to $|a_n| \to 0$.

$\square$

We now prove the so-called limit comparison test.

Theorem 5.2.11 (Limit comparison test)**.** *Let $a_n, b_n > 0$ for all $n \in \mathbb{N}$ and let the sequence*

$$\frac{a_n}{b_n}$$

converge to $l \neq 0$ as $n \to \infty$. Then, the series $\sum_{n=0}^{\infty} a_n$ converges iff the series $\sum_{n=0}^{\infty} b_n$ converges.

Proof. From the definition of limit we have that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left|\frac{a_n}{b_n} - l\right| < \varepsilon$$

for all $n \geq N$. Choose now $\varepsilon = l/2$ (l can only be positive because $a_n, b_n > 0$). It follows that, for all $n \geq N$,

$$\frac{l}{2} < \frac{a_n}{b_n} < \frac{3l}{2}$$

which is equivalent to

$$\frac{l}{2} b_n < a_n < \frac{3l}{2} b_n,$$

for all $n \geq N$. Since $l \neq 0$ we have that if $\sum_{n=1}^{\infty} b_n$ is convergent then both the series $\sum_{n=1}^{\infty} \frac{l}{2} b_n$ and $\sum_{n=1}^{\infty} \frac{3l}{2} b_n$ are convergent and viceversa. We can now apply the comparison test. We have that

$$\sum_{n=1}^{\infty} b_n \text{ convergent} \Rightarrow \sum_{n=1}^{\infty} \frac{3l}{2} b_n \text{ convergent} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ convergent},$$

and

$$\sum_{n=1}^{\infty} a_n \text{ convergent} \Rightarrow \sum_{n=1}^{\infty} \frac{l}{2} b_n \text{ convergent} \Rightarrow \sum_{n=1}^{\infty} b_n \text{ convergent}.$$

□

Under the hypotheses of Theorem 5.2.11 it is clear that $\sum_{n=1}^{\infty} a_n$ is divergent iff $\sum_{n=1}^{\infty} b_n$ is divergent.

5.3 Some important positive series

In this section we want to investigate series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}},$$

where $\alpha > 0$.

Theorem 5.3.1. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}},$$

converges iff $\alpha > 1$.

Proof. **Case 1.** Assume that $\alpha \leq 1$. Since $n \geq 1$ we have that $n^{\alpha} \leq n$. Let us take N large, for instance $N = 2^M$ with M large as well. We will now compute the finite sum S_N and show that it tends to ∞ as $N \to \infty$. It will follows that when $\alpha \leq 1$ the series $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ is divergent. We have that

$$\begin{aligned}
S_N &= \sum_{n=1}^{N} \frac{1}{n^{\alpha}} \\
&\geq \sum_{n=1}^{N} \frac{1}{n} \\
&= 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + \cdots + (\frac{1}{2^{M-1}+1} + \cdots + \frac{1}{2^M}) \\
&\geq 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \cdots + \frac{2^{M-1}}{2^M} \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\
&= 1 + \frac{M}{2}.
\end{aligned}$$

Hence, if $M \to \infty$ then $N \to \infty$ and $S_N \to \infty$. This shows that the series $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ is divergent for $\alpha \leq 1$.

Case 2. We now assume $\alpha > 1$. Then $n^{\alpha} > n$ and $\frac{1}{n^{\alpha}} < \frac{1}{n}$. Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent we cannot argue by comparison test. As in case 1 we want to take a value of N which is large. This time we take $N = 2^M - 1$ with M large enough. We have

$$\begin{aligned}
S_N &= \sum_{n=1}^{N} \frac{1}{n^{\alpha}} \\
&= 1 + (\frac{1}{2^{\alpha}} + \frac{1}{3^{\alpha}}) + (\frac{1}{4^{\alpha}} + \frac{1}{5^{\alpha}} + \frac{1}{6^{\alpha}} + \frac{1}{7^{\alpha}}) + \cdots + (\frac{1}{2^{(M-1)\alpha}} + \cdots + \frac{1}{(2^M-1)\alpha}) \\
&\leq 1 + \frac{2}{2^{\alpha}} + \frac{4}{4^{\alpha}} + \cdots + \frac{2^{(M-1)}}{2^{(M-1)\alpha}} \\
&= 1 + \frac{1}{2^{\alpha-1}} + \frac{1}{4^{\alpha-1}} + \cdots + \frac{1}{2^{(M-1)(\alpha-1)}} \\
&= \sum_{n=0}^{M-1} \frac{1}{2^{(\alpha-1)n}}.
\end{aligned}$$

Note that since $\alpha > 1$ we have that $\frac{1}{2^{\alpha-1}} < 1$ and therefore the last sum is the finite sum of a convergent geometric series. Hence,

$$S_N \leq \sum_{n=0}^{M-1} \frac{1}{2^{(\alpha-1)n}} \leq \sum_{n=0}^{\infty} \frac{1}{2^{(\alpha-1)n}}.$$

This shows that our series can be bounded by a convergent series so by the comparison test it is convergent as well.

$\square$

Remark 5.3.2. The sum

$$\zeta(\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha},$$

is called the *Riemann-Zeta Function* and has some important application in Mathematics (in number theory it is related to the distribution of the prime numbers, it has applications to probability theory, physics, statistics, etc.).

The divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is called the *Harmonic series.*

5.4 Alternating series

We now investigate series where the sign of the terms alternates, i.e., series of the type

$$\sum_{n=1}^{\infty} (-1)^n a_n,$$

where $a_n \geq 0$. In detail,

$$\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + \cdots$$

Note that one could also start from $n = 0$ if a_0 is defined.

The following test tells us when an alternating series is convergent.

Proposition 5.4.1. *Let a_n be a decreasing non-negative sequence such that $a_n \to 0$ as $n \to \infty$. Then the corresponding alternating series*

$$\sum_{n=1}^{\infty} (-1)^n a_n,$$

is convergent.

Note that this is a sufficient condition for the series $\sum_{n=1}^{\infty}(-1)^n a_n$ to be convergent. There might exist alternating series which are convergent but do not fulfil the hypotheses of the test above.

Proof. Let us take $N \in \mathbb{N}$ and write the partial sums S_{2N} and S_{2N-1}. We have

$$\begin{aligned} S_{2N} &= \sum_{n=1}^{2N}(-1)^n a_n = -a_1 + a_2 - a_3 + \cdots - a_{2N-1} + a_{2N}, \\ S_{2N-1} &= \sum_{n=1}^{2N-1}(-1)^n a_n = -a_1 + a_2 - a_3 + \cdots + a_{2N-2} - a_{2N-1}. \end{aligned}$$

We easily see that we have two sequences of partial sums, one with even indexes and one with odd indexes. We now investigate the properties of these two sequences.

Claim 1: the sequence (S_{2N}) is decreasing.

Indeed,

$$S_{2(N+1)} - S_{2N} = -a_{2N+1} + a_{2N+2} \leq 0$$

because the sequence a_n is non-negative and decreasing.

Claim 2: the sequence (S_{2N-1}) is increasing.

Indeed,

$$S_{2(N+1)-1} - S_{2N-1} = a_{2N} - a_{2(N+1)-1} = a_{2N} - a_{2N+1} \geq 0$$

because the sequence $a_n \geq 0$ is decreasing.

Claim 3: $S_{2N} \geq S_{2N-1}$ for all $N \in \mathbb{N}$.

Indeed,

$$S_{2N} - S_{2N-1} = a_{2N} \geq 0$$

for all $N \in \mathbb{N}$.

By combining all these claims we can write

$$S_2 \geq S_4 \geq \cdots S_{2N} \geq S_{2N-1} \geq S_{2N-3} \cdots \geq S_1.$$

Note that the decreasing sequence S_{2N} is bounded from below by S_1 so it is convergent (result about sequences). Hence, there exists $L_1 \in \mathbb{R}$ such that $S_{2N} \to L_1$. The sequence S_{2N-1} is increasing and bounded from above by S_2. So, it is convergent, i.e., there exists $L_2 \in \mathbb{R}$ such that $S_{2N-1} \to L_2$. If we prove that L_1 coincides with L_2 we will have that the sequence S_{2N} and S_{2N-1} converge to the same limit and therefore the sequence S_N will be convergent as well. This will imply that the series $\sum_{n=1}^{\infty}(-1)^n a_n$ is convergent. Combining Claim 3 with $a_n \to 0$ we have that

$$S_{2N} - S_{2N-1} = a_{2N} \to 0.$$

Hence, by the algebraic properties of limits,

$$\begin{aligned} \lim_{N\to\infty} S_{2N} - \lim_{N\to\infty} S_{2N-1} &= 0 \\ L_1 - L_2 &= 0, \end{aligned}$$

which gives $L_1 = L_2$ as desired. The proof is complete. $\square$

Example 5.4.2. The series

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$$

is conditionally convergent. Indeed, the sequence $a_n = \frac{1}{n}$ is positive, decreasing and tends to 0. So by the alternating series test it is convergent. However,

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{1}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$$

is not convergent as seen in Theorem 5.3.1.

5.5 Conclusion

When we have a series $\sum_{n=1}^{\infty} a_n$ and we want to decide if it is convergent or not we should follow this train of thought:

- if $a_n \not\to 0$ then the series is divergent;
- if $a_n \to 0$ and $a_n \geq 0$ then we can use some tests (shift rule, comparison test, ratio test, limit comparison test);
- if not all a_n are positive then we can check whether the series is absolutely convergent;
- if the series $\sum_{n=1}^{\infty} |a_n|$ is divergent then we can try the alternating series test.

It is very useful to remember the properties of some important series: geometric series, telescopic series, harmonic series, etc.

Chapter 6

Power series

In this chapter we investigate series of the type

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where (a_n) is a sequence of real numbers, x_0 is a fixed real number and x is varying in $\mathbb{R}$. This is a power series centred at x_0. Power series are particularly important in connections with functions as you will see in Analysis 2. In general they can be used to provide an approximation of a given function in one real variable. Our aim in this chapter is to understand for which values of x is a given power series convergent. It will be very useful to apply to the power series the following root test that we will prove in the general context of series of real numbers.

6.1 The root test

Theorem 6.1.1. *Let $\sum_{n=0}^{\infty} a_n$ be a series of real numbers such that*

$$|a_n|^{\frac{1}{n}} \to l.$$

(i) If $l < 1$ then the series converges absolutely.

(ii) if $l > 1$ then the series diverges.

(iii) if $l = 1$ the test is inconclusive.

Proof.

(i) Since $|a_n|^{\frac{1}{n}} \to l < 1$ by definition of limit we know that

$$|a_n|^{\frac{1}{n}} \leq k < 1,$$

for n large enough, let's say $n \geq N$. Hence,

$$\sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n| \leq \sum_{n=0}^{N-1} |a_n| + \sum_{n=N}^{\infty} k^n.$$

Since the last series can be bounded by a convergent geometric series with $k < 1$, by the comparison test we conclude that the series $\sum_{n=0}^{\infty} |a_n|$ is convergent and therefore $\sum_{n=0}^{\infty} a_n$ is absolutely convergent.

(ii) if $|a_n|^{\frac{1}{n}} \to l > 1$ then for n large enough, i.e. $n \geq N$, we have that

$$|a_n|^{\frac{1}{n}} \geq k > 1.$$

Hence

$$|a_n| \geq k^n > 1$$

for $n \geq N$. This implies that $a_n \not\to 0$ and therefore the corresponding series is divergent.

(iii) if $l = 1$ the test is inconclusive. Indeed, the series with $a_n = 1$ for all n has limit $l = 1$ and it is divergent. However the series

$$\sum_{n=0}^{\infty} \frac{1}{(n+2)(n+1)} = \sum_{n=1}^{\infty} \frac{1}{(n+1)n}$$

is convergent and

$$\lim_{n\to\infty} \left(\frac{1}{(n+2)(n+1)} \right)^{\frac{1}{n}} = \lim_{n\to\infty} \mathrm{e}^{-\frac{1}{n} \ln((n+2)(n+1))}.$$

Since

$$\frac{1}{n} \ln((n+2)(n+1)) = \frac{\ln(n+2)}{n} + \frac{\ln(n+1)}{n} \to 0, \quad n \to \infty$$

we have that

$$\lim_{n\to\infty} \left(\frac{1}{(n+2)(n+1)} \right)^{\frac{1}{n}} = 1.$$

□

6.2 Application of the root test to power series

In this section we will apply the root test to the power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

We begin by introducing the following definition.

Definition 6.2.1. *Let* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$. *We say that* $0 \leq R \leq +\infty$ *is the radius of convergence of this power series if* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ *converges absolutely for* $|x-x_0| < R$ *and diverges for* $|x-x_0| > R$.

Proposition 6.2.2. *Every power series has a radius of convergence.*

Proof. The power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ definitely converges at $x = x_0$ so in the worst case the radius of convergence is $R = 0$. If there exists $x_1 \neq x_0$ such that $\sum_{n=0}^{\infty} a_n(x_1-x_0)^n$ is convergent then the series converges absolutely for $|x-x_0| < |x_1-x_0|$ and $R \geq |x_1-x_0|$. Indeed,

$$\sum_{n=0}^{\infty} |a_n||(x-x_0)^n| \leq \sum_{n=0}^{\infty} |a_n| \frac{|(x-x_0)^n|}{|x_1-x_0|^n} |x_1-x_0|^n.$$

Since the series $\sum_{n=0}^{\infty} a_n(x_1-x_0)^n$ is convergent then $a_n(x_1-x_0)^n \to 0$ so it is bounded as a sequence. Therefore, there exists $c > 0$ such that

$$\sum_{n=0}^{\infty} |a_n||(x-x_0)^n| \leq \sum_{n=0}^{\infty} |a_n| \frac{|(x-x_0)^n|}{|x_1-x_0|^n} |x_1-x_0|^n \leq c \sum_{n=0}^{\infty} \left| \frac{x-x_0}{x_1-x_0} \right|^n.$$

The last series is a convergent geometric series when $|x-x_0| < |x_1-x_0|$ so by the comparison test the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is absolutely convergent and its radius of convergence is greater equal than $|x_1-x_0|$. $\square$

We conclude this section by showing how the root test can be used to find the radius of convergence of a power series.

Proposition 6.2.3. *Suppose that* $a_n \neq 0$ *for* n *sufficiently large and that the limit*

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{R}.$$

Then the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ *has radius of convergence* R.

Proof. By the root test we have that the series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

is absolutely convergent if

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} |x-x_0| = \frac{|x-x_0|}{R} < 1.$$

Hence, the power series is absolutely convergent for all $x \in \mathbb{R}$ with $|x-x_0| < R$. The series is divergent if

$$\frac{|x-x_0|}{R} > 1.$$

$\square$

Remark 6.2.4. Note that to determine the radius of convergence of the power series you can also use a different test, for instance the ratio test applied to the series

$$\sum_{n=0}^{\infty} |a_n||x - x_0|^n.$$

Also, in this case

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{R}.$$

When $|x - x_0|$ is equal to the radius of convergence, i.e., $x - x_0 = \pm R$, we do not know if the series is convergent there or divergent. We need to evaluate case by case. For instance the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

has radius 1. When $x = 1$ we get the harmonic series which is divergent. When $x = -1$ we get

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which is convergent by the alternating series test. So, the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ is convergent for $x \in [-1, 1)$ and absolutely convergent for $x \in (-1, 1)$.

We conclude this chapter with some important examples of power series that you will encounter again in Analysis 2.

Example 6.2.5.

1. The series
$$\sum_{n=0}^{\infty} \frac{x^n}{n!},$$
where $0! = 1$ defines the exponential function e^x. It has radius of convergent ∞ so it is absolutely convergent on the whole of $\mathbb{R}$.

2. The series
$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$
defines $\sin(x)$ and has radius of convergence ∞.

3. The series
$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}$$
has radius of convergence ∞ and defines $\cos(x)$.

All the series above are also called Maclaurin series of e^x, $\sin(x)$ and $\cos(x)$, respectively.

Chapter 7

Continuous functions in one real variable

This chapter is dedicated to the notion of functions in one real variable. During your studies in mathematics you will work with functions all the time. Why? Because they are fundamental in modelling any physical object we might think of. For instance, we will often have functions of time t, or functions of space x, or functions of both time and space. In Analysis 1 we will study functions of one real variable x. In Analysis 2 and 3 you will study functions of more than one variable. Let us start with some basic definition.

Definition 7.0.1. *A* function f *is a relation between two sets* X *and* Y *which links every element* $x \in X$ *to one and only one element* $y \in Y$*. We use the notations* $f : X \to Y$ *and* $y = f(x)$*.* X *is the domain of* f *and* $f(X) = \{y = f(x) : \quad x \in X\} \subseteq Y$ *is the image set of* f*. When* $X \subseteq \mathbb{R}$ *and* $Y = \mathbb{R}$ *then* $f : X \to Y$ *is a* function of one real variable x with values in $\mathbb{R}$.

Often X will be an interval of $\mathbb{R}$, for instance, (a, b) or $[a, b]$.

Definition 7.0.2. *Let* $f : X \subseteq \mathbb{R} \to \mathbb{R} : x \to f(x)$*. The curve* $y = f(x)$ *in the plane* $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ *is the graph of* f*. The plane* $\mathbb{R}^2$ *is equipped with cartesian* x*-axis (horizontal) and* y*-axis (vertical), so the point of the graph of* f *has coordinates* $(x, f(x))$*.*

Example 7.0.3.

(i) Let $a \in \mathbb{R}$. The constant function

$$f : \mathbb{R} \to \mathbb{R} : x \to a$$

has image set $f(\mathbb{R}) = \{a\}$. Its graph is a line parallel to the x-axis through the point $(0, a)$.

(ii) The identity function

$$f : \mathbb{R} \to \mathbb{R} : x \to x$$

has a straight line has a graph as well. It is the diagonal line $y = x$.

(iii) The function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

jumps endlessly between 1 and 0. Its image set is given by $\{0, 1\}$. We will come back to this function later in the chapter.

Definition 7.0.4.

(i) A function $f : X \to Y$ is injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

(ii) A function $f : X \to Y$ is surjective if for all $y \in Y$ there exists $x \in X$ such that $y = f(x)$.

(iii) A $f : X \to Y$ is bijective if its is injective and surjective.

When f is bijective then for all $y \in Y$ there exists a unique $x \in X$ such that $f(x) = y$. This means that we can define the inverse of f as the function

$$f^{-1} : Y \to X : y \to x,$$

where $f(x) = y$. We say that f is invertible on X. Note that any two functions $f : X \to Y$ and $g : Y \to Z$ can be composed creating a new function:

$$g \circ f : X \to Z : x \to g(f(x)).$$

So, when f is a bijective we have that

$$f \circ f^{-1} : Y \to Y : y \to y$$

and

$$f^{-1} \circ f : X \to X : x \to x.$$

If $f : X \to Y$ is injective but not surjective then it is a bijection as $f : X \to f(X)$. A function could be not injective on the whole of X but injective and therefore invertible on a subset of X. For instance,

$$f : \mathbb{R} \to \mathbb{R} : x \to \sin(x),$$

is not injective on $\mathbb{R}$ (it is periodic) but it is injective on $[-\pi/2, \pi/2]$. So,

$$f : [-\pi/2, \pi/2] \to [-1, 1] : x \to \sin(x)$$

is invertible and has inverse

$$f^{-1} : [-1, 1] \to [-\pi/2, \pi/2] : x \to \arcsin(x).$$

Many other examples can be found in Mathematical Methods.

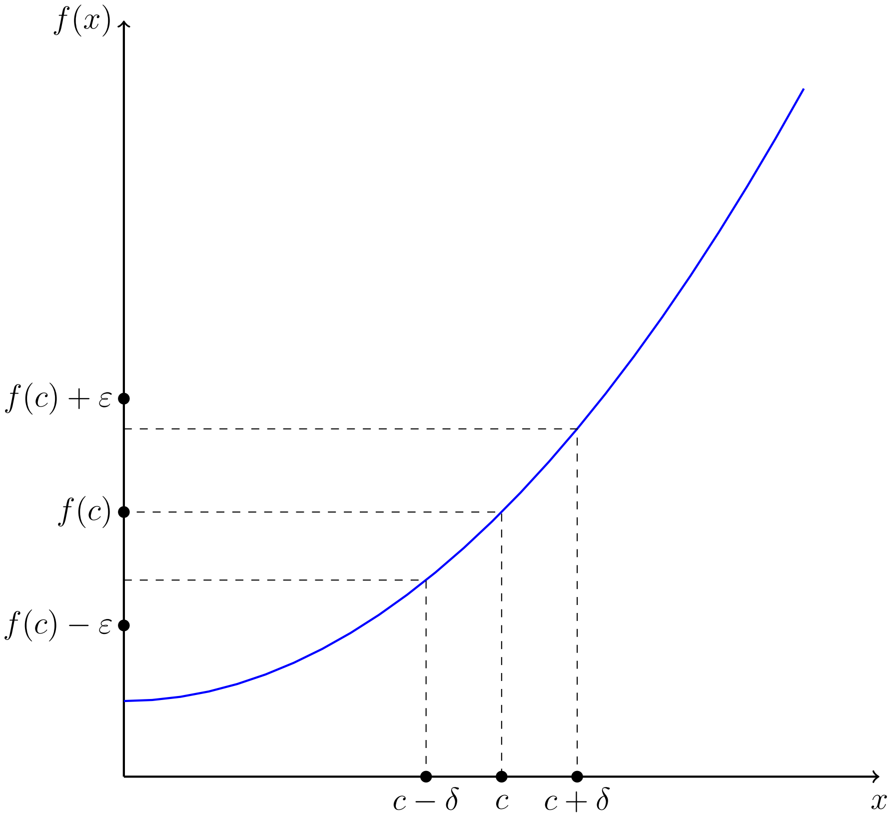

Figure 7.1: This function is continuous at c: for any $\varepsilon > 0$ that we choose, we may find $\delta > 0$ such that, if x is between $c-\delta$ and $c+\delta$, then $f(x)$ is between $f(c)-\varepsilon$ and $f(c)+\varepsilon$.

7.1 Continuity

In the rest of the chapter we will focus on functions which have an additional property: when x is close to c then (x) is close to $f(x)$. This is the basic idea of continuity. The following notes are partially taken from Alan Thompson's Analysis 2 Lecture Notes (thank you Alan!).

Definition 7.1.1. *A function $f\colon X \to \mathbb{R}$ is* continuous at $c \in X$ *if and only if*

$$\forall \varepsilon > 0\ \exists \delta > 0 \textit{ such that } \forall x \in X,\ \textit{ if } |x-c| < \delta,\ \textit{ then } |f(x) - f(c)| < \varepsilon.$$

A function $f\colon X \to \mathbb{R}$ is said to be continuous *if and only if it is continuous at all points $c \in X$.*

In plain English this roughly says that, for any $\varepsilon > 0$, we can find a $\delta > 0$ so that if x is no more than distance δ from c, then $f(x)$ is no more than distance ε from $f(c)$; this idea is illustrated in Figure 7.1.

Remark 7.1.2. The value of δ that you find will normally depend on both ε and c.

Definition 7.1.3. *A function $f\colon X \to \mathbb{R}$ is called* discontinuous at $c \in X$ *if and only if it is not continuous at c, i.e.*

$$\exists \varepsilon > 0 \text{ such that } \forall \delta > 0,\ \exists x \in X \text{ with } |x-c| < \delta \text{ and } |f(x) - f(c)| \geq \varepsilon.$$

Let's do some examples to see how this definition works.

Example 7.1.4. We begin with an easy one: let's look at the identity function again.

$$\begin{aligned} f&\colon \mathbb{R} \to \mathbb{R} \\ f(x) &= x. \end{aligned}$$

Choose $c \in \mathbb{R}$ and let $\varepsilon > 0$. To prove that f is continuous at c, we want to find $\delta > 0$ such that, whenever $|x - c| < \delta$, we have $|f(x) - f(c)| < \varepsilon$.

Take $\delta = \varepsilon$. Then if $|x - c| < \delta$, we also have $|x - c| < \varepsilon$, so it follows that $|f(x) - f(c)| = |x - c| < \varepsilon$. This shows that f is continuous at any $c \in \mathbb{R}$, so it is continuous.

Example 7.1.5. For a harder example, consider the square function

$$\begin{aligned} f&\colon \mathbb{R} \to \mathbb{R} \\ f(x) &= x^2. \end{aligned}$$

Choose $c \in \mathbb{R}$ and let $\varepsilon > 0$. To prove that f is continuous at c, we want to find $\delta > 0$ (which may depend on ε and c) such that, whenever $|x-c| < \delta$, we have $|f(x)-f(c)| < \varepsilon$.

To find such δ, note that $|f(x)-f(c)| = |x^2-c^2| = |x-c||x+c|$. Moreover, if $|x-c| < \delta$, then $|x - c| + 2|c| < \delta + 2|c|$ and hence $|x + c| < \delta + 2|c|$, by the triangle inequality. So if $|x - c| < \delta$, we have

$$|f(x) - f(c)| = |x^2 - c^2| = |x - c||x + c| < \delta(\delta + 2|c|).$$

Therefore, we should choose δ so that $\delta(\delta + 2|c|) \leq \varepsilon$. Solving the quadratic equation $\delta(\delta + 2|c|) = \varepsilon$, we find that $\delta = -|c| + \sqrt{|c|^2 + \varepsilon}$ will do.

Let's check it works. Choose $\delta = -|c| + \sqrt{|c|^2 + \varepsilon} > 0$. Then if $|x - c| < \delta$, we have

$$\begin{aligned} |f(x) - f(c)| &= |x - c||x + c| \\ &< \delta(\delta + 2|c|) \\ &= (-|c| + \sqrt{|c|^2 + \varepsilon})(|c| + \sqrt{|c|^2 + \varepsilon}) \\ &= |c|^2 + \varepsilon - |c|^2 \\ &= \varepsilon, \end{aligned}$$

as required. This shows that f is continuous at c. Since c was an arbitrary point in $\mathbb{R}$, we see that f is continuous.

Exercise: Show that the absolute value function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$ is continuous.

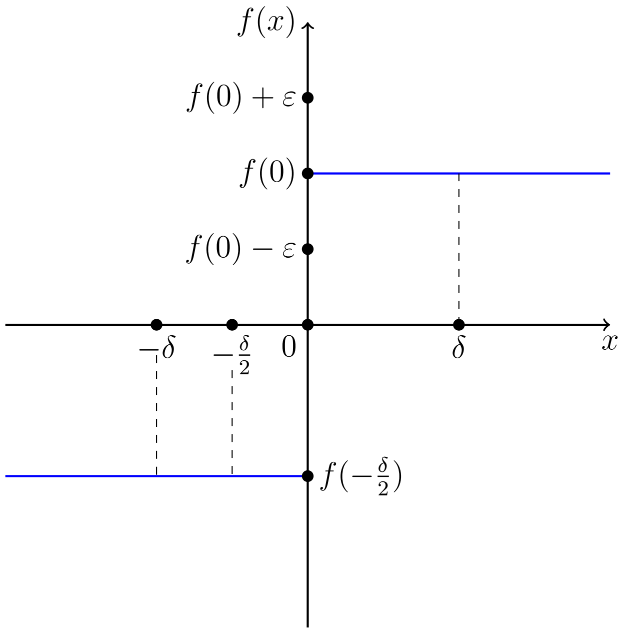

Figure 7.2: The function from Example 7.1.6 is discontinuous at 0: if $\varepsilon = \frac{1}{2}$, then no matter which $\delta > 0$ we choose, $f(-\frac{\delta}{2}) = -1$ will never lie between $f(0) - \varepsilon = \frac{1}{2}$ and $f(0) + \varepsilon = \frac{3}{2}$.

Example 7.1.6. Now let's try an example that we expect to be discontinuous. Consider

$$f\colon \mathbb{R} \to \mathbb{R}$$
$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

We expect this function to be discontinuous at $c = 0$. To show this, we need to find an $\varepsilon > 0$ such that, for all $\delta > 0$, there is an $x \in \mathbb{R}$ (which may depend on δ) with $|x - c| = |x| < \delta$ but $|f(x) - f(c)| = |f(x) - 1| \geq \varepsilon$.

Set $\varepsilon = \frac{1}{2}$ and choose any $\delta > 0$. Then let $x = -\frac{\delta}{2}$; note that $|x - c| = |x| = \frac{\delta}{2} < \delta$. Moreover since $x < 0$, we have $f(x) = -1$ and so

$$|f(x) - f(c)| = |-1-1| = 2 \geq \tfrac{1}{2} = \varepsilon.$$

Thus f is discontinuous at $c = 0$. This argument is illustrated by Figure 7.2.

7.2 Sequential continuity

The notion of continuity can also be stated in the language of sequences. In the sequel we will define what it means for a function to be sequentially continuous and we will prove that continuity is equivalent to sequential continuity in our context.

Definition 7.2.1. *A function* $f: X \to \mathbb{R}$ *is* sequentially continuous at $c \in X$ *if and only if, for any sequence* $(x_n) \subset X$ *that converges to* c*, the sequence* $(f(x_n))$ *converges to* $f(c)$*.*

$f: X \to \mathbb{R}$ *is said to be* sequentially continuous *if and only if it is sequentially continuous at every* $c \in X$*.*

The negation of Definition 7.2.1 is also pretty useful.

Definition 7.2.2. *A function* $f: X \to \mathbb{R}$ *is* sequentially discontinuous *at* $c \in X$ *if and only if there exists a sequence* $(x_n) \subset X$ *that converges to* c*, such that the sequence* $(f(x_n))$ *does not converge to* $f(c)$*.*

Let's check that these definitions deliver sane results in some easy examples.

Example 7.2.3. First, let's try the easiest function of all: a constant function. Let $a \in \mathbb{R}$ be any real number and define

$$\begin{aligned} f&: \mathbb{R} \to \mathbb{R} \\ f(x) &= a. \end{aligned}$$

Choose $c \in \mathbb{R}$ and let (x_n) be any sequence of real numbers converging to c. Then we have $f(x_n) = a$ for all $n \in \mathbb{N}$, so the sequence $(f(x_n))$ is just the constant sequence $(a, a, a, \ldots)$. Thus $(f(x_n)) \to a$ and $a = f(c)$, so f is sequentially continuous at c. But this holds for all $c \in \mathbb{R}$, so f is sequentially continuous.

Example 7.2.4. Now let's look at a marginally harder function: the identity function.

$$\begin{aligned} f&: \mathbb{R} \to \mathbb{R} \\ f(x) &= x. \end{aligned}$$

Choose $c \in \mathbb{R}$ and let (x_n) be any sequence of real numbers converging to c. Then we have $f(x_n) = x_n$ for all $n \in \mathbb{N}$, so the sequence

$$(f(x_n)) = (x_n) \to c = f(c).$$

Thus $(f(x_n)) \to f(c)$ and f is sequentially continuous at c. As this holds for all $c \in \mathbb{R}$, we see that f is sequentially continuous.

Example 7.2.5. Now consider the function

$$\begin{aligned} f&: \mathbb{R} \to \mathbb{R} \\ f(x) &= \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \end{aligned}$$

First consider the sequence $(x_n) = (-\frac{1}{n})$. Note that $(x_n) \to 0$ and $f(0) = 1$. However, $x_n < 0$ for all $n \in \mathbb{N}$, so $f(x_n) = -1$ for all $n \in \mathbb{N}$. Therefore $(f(x_n)) \to -1$ and we find that $(f(x_n))$ does not converge to $f(0)$. We see that f is sequentially discontinuous at 0.

Next, let $c \in \mathbb{R}$ be any nonzero real number and let $(x_n) \to c$ be any sequence. Set $\varepsilon = |c| > 0$. Then, by the definition of convergence of a sequence, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - c| < \varepsilon = |c|$. Consider first the case where $c > 0$, so $f(c) = 1$. Then for $n > N$ we have $-c < x_n - c < c$, which gives $x_n > 0$. So $f(x_n) = 1$ for all $n > N$ and thus $(f(x_n)) \to 1$. So we see that $(f(x_n)) \to f(c)$.

A similar argument works for $c < 0$. Together, this shows that f is sequentially continuous at all $c \in \mathbb{R} \setminus \{0\}$.

So far so good: this definition seems to correspond fairly well with our intuitive idea of what a continuous function should be. However, as the following examples show, we should be wary about trusting our intuition too much. Hold on to your hats folks, because things are about to get weird.

Example 7.2.6. Our intuitive ideas about continuity may lead us to think that any function $f\colon \mathbb{R} \to \mathbb{R}$ should be continuous at "most" points $c \in \mathbb{R}$, except for maybe a few discontinuous points here and there where the function jumps. However, let's consider the function

$$\begin{aligned} &f\colon \mathbb{R} \to \mathbb{R} \\ &f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases} \end{aligned}$$

I claim that f is sequentially discontinuous at every point $c \in \mathbb{R}$!

Indeed, let $c \in \mathbb{Q}$, so that $f(c) = 1$. Define a sequence by $(x_n) = \left(c + \frac{\sqrt{2}}{n}\right)$. Then $x_n \to c$ and $x_n \in \mathbb{R} \setminus \mathbb{Q}$ for all $n \in \mathbb{N}$. So $f(x_n) = 0$ for all $n \in \mathbb{N}$, which gives $(f(x_n)) \to 0$. But $0 \neq f(c) = 1$. So f is sequentially discontinuous at any $c \in \mathbb{Q}$.

Now let $c \in \mathbb{R} \setminus \mathbb{Q}$. We can find a sequence (x_n) of rational numbers such that $(x_n) \to c$ (just let x_n be the rational number given by the first n digits in the decimal expansion of c). So $f(x_n) = 1$ for all $n \in \mathbb{N}$, which gives $(f(x_n)) \to 1$. But $1 \neq f(c) = 0$. So f is also sequentially discontinuous at any $c \in \mathbb{R} \setminus \mathbb{Q}$, and hence is sequentially discontinuous at every $c \in \mathbb{R}$.

Example 7.2.7. OK, so functions can be discontinuous everywhere. But what about functions $f\colon \mathbb{R} \to \mathbb{R}$ that are continuous at at least one point? Surely a function that is continuous at some $c \in \mathbb{R}$ must also be continuous near c? Again, our intuition fails us here: consider the function

$$\begin{aligned} &f\colon \mathbb{R} \to \mathbb{R} \\ &f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases} \end{aligned}$$

I claim that f is sequentially continuous at 0 and sequentially discontinuous everywhere else!

The same argument that we used in Example 7.2.6 shows that f is sequentially discontinuous at all points $c \neq 0$. But what about at $c = 0$? Let (x_n) be any sequence with $(x_n) \to 0$. Note that $0 \leq |f(x)| \leq |x|$ for all $x \in \mathbb{R}$. So $0 \leq |f(x_n)| \leq |x_n|$ for all $n \in \mathbb{R}$ and $(|x_n|) \to 0$, which gives $(|f(x_n)|) \to 0$ by the Sandwich Theorem. But this implies that $(f(x_n)) \to 0 = f(0)$ as well. Thus we find that f is sequentially continuous at 0.

That's probably enough examples for now although, as we will see later, continuity still has a few pathological tricks up its sleeve that we have yet to encounter. For now, this is a pure maths course, so maybe it's about time we proved a theorem.

Proposition 7.2.8. *Let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be two functions defined on the same domain $X \subset \mathbb{R}$. Suppose that both f and g are sequentially continuous at some point $c \in X$. Then $f + g$, $f - g$, and $f \cdot g$ are all sequentially continuous at c. Moreover, if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is also sequentially continuous at c.*

Proof. We prove the statement for $f + g$, the others are analogous. Let $c \in X$ be a point where both f and g are sequentially continuous and let $(x_n) \subset X$ be a sequence that converges to c. Then $(f(x_n)) \to f(c)$ and $(g(x_n)) \to g(c)$, by sequential continuity. So, by the sum rule for sequences we see that $((f + g)(x_n)) = (f(x_n) + g(x_n)) \to f(c) + g(c) = (f + g)(c)$. Thus $f + g$ is also sequentially continuous at c. $\square$

This has the following useful corollary.

Corollary 7.2.9. *Any polynomial function*

$$\begin{aligned} &f\colon \mathbb{R} \to \mathbb{R} \\ &f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \end{aligned}$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$ are constants, is sequentially continuous.

Proof. Note that f can be constructed from sums and products of constant functions $x \mapsto a_i$ and the identity function $x \mapsto x$, and we saw that these functions are all sequentially continuous in Examples 7.2.3 and 7.2.4. So the result follows immediately from Proposition 7.2.8. $\square$

Continuity also behaves well under composition of functions.

Proposition 7.2.10. *Let $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$ be two functions defined on domains $X, Y \subset \mathbb{R}$, and suppose that $\operatorname{im}(f) \subset Y$. If f is sequentially continuous at $c \in X$ and g is sequentially continuous at $f(c) \in Y$, then the composition $g \circ f\colon X \to \mathbb{R}$ is sequentially continuous at $c \in X$.*

Proof. Let $(x_n) \subset X$ be any sequence that converges to c. As f is sequentially continuous at c and $\operatorname{im}(f) \subset Y$, we have that $(f(x_n)) \subset Y$ converges to $f(c)$. Now, as g is sequentially continuous at $f(c)$, we may apply g to the sequence $(f(x_n))$ to see that $(g \circ f(x_n)) \to g \circ f(c)$. This shows that $g \circ f$ is sequentially continuous at c. $\square$

The following theorem allows us to identify continuity with sequential continuity.

Theorem 7.2.11. *A function $f\colon X \to \mathbb{R}$ is continuous at $c \in X$ if and only if it is sequentially continuous at c.*

Proof. We prove the two implications in two separate steps.

Step 1. *If f is continuous at c, then f is sequentially continuous at c.* Suppose that f is continuous at c and let $(x_n) \subset X$ be a sequence converging to c. We want to show that $f(x_n)$ converges to $f(c)$, i.e.

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n > N, |f(x_n) - f(c)| < \varepsilon.$$

Choose $\varepsilon > 0$. Since f is continuous at c, by the definition of continuity we may find $\delta > 0$ such that

$$\forall x \in X, \ \text{if } |x - c| < \delta, \ \text{then } |f(x) - f(c)| < \varepsilon. \tag{7.2.1}$$

Moreover, since $(x_n) \to c$, by the definition of a convergent sequence we may find $N \in \mathbb{N}$ such that

$$\forall n > N, \ |x_n - c| < \delta \tag{7.2.2}$$

(and this is the same $\delta > 0$ that we just found above).

For this value of $N \in \mathbb{N}$, we therefore see that if $n > N$, then by (7.2.2) we have $|x_n - c| < \delta$ and so, noting that $(x_n) \subset X$, by (7.2.1) we get $|f(x_n) - f(c)| < \varepsilon$.

Thus $(f(x_n)) \to f(c)$ and f is sequentially continuous at c.

Step 2. *If f is sequentially continuous at c, then f is continuous at c.* We prove this by the contrapositive: we actually show that if f is discontinuous at c, then f is sequentially discontinuous at c.

So suppose f is discontinuous at c. Then, by the definition of discontinuity, there exists $\varepsilon > 0$ such that

$$\forall \delta > 0, \ \exists x \in X \text{ with } |x - c| < \delta \text{ and } |f(x) - f(c)| \geq \varepsilon. \tag{7.2.3}$$

Choose any such $\varepsilon > 0$. We will use (7.2.3) to construct a sequence $(x_n) \in X$ which converges to c, but such that $(f(x_n))$ does not converge to $f(c)$.

We define the sequence (x_n) by applying (7.2.3) with $\delta = \frac{1}{n} > 0$, for each $n \in \mathbb{N}$. Indeed, for each $n \in \mathbb{N}$, (7.2.3) says that we may find $x_n \in X$ with $|x_n - c| < \frac{1}{n}$ and $|f(x_n) - f(c)| \geq \varepsilon$.

For this choice of sequence (x_n), by construction we have $|x_n - c| < \frac{1}{n}$ for each n so, by the Sandwich Theorem, $(x_n) \to c$. However, we also have $|f(x_n) - f(c)| \geq \varepsilon$ for all $n \in \mathbb{N}$, so $(f(x_n))$ *does not* converge to $f(c)$. Thus f is sequentially discontinuous at c. □

Now we return to look at some more examples. The first two are designed to demonstrate that the domain is important.

Example 7.2.12. Consider the square root function

$$\begin{aligned} f\colon [0,\infty) &\to \mathbb{R} \\ f(x) &= \sqrt{x} \end{aligned}$$

(note that the square root is by definition positive). Is this function continuous at $c = 0$? Most people's intuition about continuity would say no. However it is, in fact, continuous. Let's go through it carefully using the definition.

Choose $\varepsilon > 0$ and let $c = 0$. We need to find a $\delta > 0$ such that, for all $x \geq 0$ with $|x - c| < \delta$, we have $|f(x) - f(c)| < \varepsilon$. Set $\delta = \varepsilon^2$. Then $x \geq 0$ and $|x - c| < \delta$ gives $0 \leq x < \varepsilon^2$, from which we obtain $0 \leq \sqrt{x} < \varepsilon$. This implies that $|f(x) - f(c)| = |\sqrt{x}| < \varepsilon$, so f is continuous at 0.

Example 7.2.13. Now for a somewhat more extreme example. Consider the function

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ f(x) &= \begin{cases} 1 & \text{if } x \in \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}. \end{cases} \end{aligned}$$

You can check for yourself that f is discontinuous at every integer $c \in \mathbb{Z}$.

But what happens if we shrink the domain of f? Define

$$\begin{aligned} g\colon \mathbb{Z} &\to \mathbb{R} \\ g(x) &= 1. \end{aligned}$$

This definitely *looks* discontinuous at every integer $c \in \mathbb{Z}$. But it is not!

Indeed, let $c \in \mathbb{Z}$ and choose $\varepsilon > 0$. Let $\delta = \frac{1}{2}$. Then if $x \in \mathbb{Z}$ and $|x - c| < \delta = \frac{1}{2}$, we must have $x = c$. So we obtain $|g(x) - g(c)| = |1 - 1| = 0 < \varepsilon$, and thus g is continuous at c.

The final example of this section shows how wild things can get if you're willing to let go of your intuition and follow the definitions to their ultimate conclusion. If you really understand this example, then continuity will hold no mysteries for you.

Example 7.2.14. This function is colloquially known as "the snowstorm"; if you try to draw the graph of it you'll understand why! Define a function $f\colon \mathbb{R} \to \mathbb{R}$ by:

- if $x \in \mathbb{R} \setminus \mathbb{Q}$ (i.e. if x is irrational), set $f(x) = 0$;
- if $x \in \mathbb{Q}$, write $x = \frac{p}{q}$ as a fraction in its lowest terms (i.e. $\frac{p}{q}$ satisfies $\gcd(p, q) = 1$ and $q > 0$), then define $f(x) = \frac{1}{q}$.

Begin by considering $c \in \mathbb{Q}$. Write $c = \frac{p}{q}$ as a fraction in its lowest terms. I claim that f is sequentially discontinuous at c (and so is discontinuous at c). To show this, I'll find a sequence $(x_n) \to c$ such that $(f(x_n))$ does not converge to $f(c)$. Indeed, consider $(x_n) = \left(c + \frac{\sqrt{2}}{n}\right)$. Then $(x_n) \subset \mathbb{R} \setminus \mathbb{Q}$, so $f(x_n) = 0$ for all $n \in \mathbb{N}$ and thus $\lim_{n\to\infty} f(x_n) = 0$ also. But $f(c) = \frac{1}{q} > 0$, so $(f(x_n))$ does not converge to $f(c)$.

Now things get interesting. Consider $c \in \mathbb{R} \setminus \mathbb{Q}$. *I claim that f is continuous at c!*

Let's prove it carefully. Let $c \in \mathbb{R} \setminus \mathbb{Q}$, choose $\varepsilon > 0$, and consider the following set:

$$S := \left\{ \frac{p}{q} \in \mathbb{Q} : 0 < q \leq \frac{1}{\varepsilon} \text{ and } \left| \frac{p}{q} - c \right| < 1 \right\}.$$

Crucially, this set is finite: there are only finitely many $q \in \mathbb{Z}$ satisfying $0 < q \leq \frac{1}{\varepsilon}$, and for each such q there are only finitely many $p \in \mathbb{Z}$ such that $|\frac{p}{q} - c| < 1$ (indeed, this is equivalent to $q(c-1) < p < q(c+1)$).

Now define $m \in \mathbb{R}$ as follows. If S is empty, set $m = 1$. If S is not empty, define $m := \min_{s \in S} |s - c|$; this is well-defined as S is finite. Note that we must have $m > 0$: if m were equal to zero, then we would have some $s \in S$ with $s = c$, contradicting $s \in S \subset \mathbb{Q}$ and $c \in \mathbb{R} \setminus \mathbb{Q}$.

Finally, define $\delta := \min\{1, m\}$; note that δ depends only on c and ε and that $0 < \delta \leq 1$. Let $x \in \mathbb{R}$ satisfy $|x - c| < \delta$ and suppose, for a contradiction, that $|f(x) - f(c)| \geq \varepsilon$.

Under this assumption, we have $|f(x) - f(c)| = |f(x) - 0| = f(x) \geq \varepsilon > 0$. We must therefore have $x \in \mathbb{Q}$ and, if we write $x = \frac{p}{q}$ as a fraction in its lowest terms, then $f(x) = \frac{1}{q} \geq \varepsilon$ and hence $0 < q \leq \frac{1}{\varepsilon}$. Moreover, as $|x - c| < \delta$ and $\delta \leq 1$, we also have $|\frac{p}{q} - c| < 1$. So $x = \frac{p}{q}$ is in the set S. But we also have $|x - c| < \delta \leq \min_{s \in S} |s - c|$, which shows that x cannot be in S. This is a contradiction, so we must have $|f(x) - f(c)| < \varepsilon$ and hence f is continuous at c.

Putting everything together, we have constructed a function that is continuous at every irrational number, and discontinuous at every rational number!

So far our repository of continuous functions is fairly small: we really only have polynomials, absolute values, and square roots, along with other functions that we can build from them using composition and the standard arithmetic operations (addition, subtraction, multiplication and division). How about more complicated functions? As the course proceeds, we will prove that functions like x^a (for $a \in \mathbb{Q}$), e^x and $\log(x)$ are all continuous, but we'll need to develop some more tools before we can do this. The standard trigonometric functions $\sin(x)$ and $\cos(x)$ are also continuous at all $x \in \mathbb{R}$, but proving this would require us to define these functions rigorously, which is a can of worms that we don't really have enough time to open here.

7.3 The Intermediate Value Theorem and its applications

Suppose now that we have a continuous function $f\colon [a,b] \to \mathbb{R}$, with $f(a) = \alpha$ and $f(b) = \beta$. Then our intuition tells us that f should also take all values between α and β. Thankfully this is a setting in which our intuition is correct, although the result is quite a bit harder to prove than one might initially expect. In fact, this humble property is even important enough to be elevated to the lofty status of a named theorem!

Theorem 7.3.1 (Intermediate Value Theorem). *Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous and that $f(a) < f(b)$. Then for all $v \in (f(a), f(b))$, there exists $c \in (a,b)$ such that $f(c) = v$.*

Proof. Let $v \in (f(a), f(b))$ and define the set $S := \{x \in [a,b] : f(x) \leq v\}$. Observe that $a \in S$ and therefore S is non-empty. Furthermore, S is bounded above (by b) and therefore, by completeness, $\sup(S)$ exists. Let $c = \sup(S)$. We will show that $f(c) = v$ by proving that, for every $\varepsilon > 0$, we have $v - \varepsilon < f(c)$ (Step 1) and $v + \varepsilon > f(c)$ (Step 2).

So let $\varepsilon > 0$. Since f is continuous at c, there exists $\delta > 0$ such that if $x \in [a,b]$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

Step 1. Note that if $c = b$, then $f(c) = f(b) > v > v - \varepsilon$. So assume that $c < b$; then we may find $x \in [a,b]$ such that $c < x < c + \delta$. Since c is an upper bound of S, we cannot have $x \in S$, so x must satisfy $f(x) > v$. Moreover, since $|x - c| < \delta$, we know that $|f(x) - f(c)| < \varepsilon$, so $f(c) > f(x) - \varepsilon > v - \varepsilon$.

Step 2. Since c is the least upper bound of S, there exists $x \in S$ with $c - \delta < x \leq c$. Therefore, since $|x - c| < \delta$, we know that $|f(x) - f(c)| < \varepsilon$, giving $f(c) < f(x) + \varepsilon$. But $f(x) \leq v$, as $x \in S$, so we obtain $f(c) < v + \varepsilon$.

Step 3. Now we put Steps 1 and 2 together. We have shown that for any $\varepsilon > 0$, we have $v - \varepsilon < f(c) < v + \varepsilon$. So the only possibility is $f(c) = v$. $\square$

Remark 7.3.2. If we have a continuous function $f\colon [a,b] \to \mathbb{R}$ with $f(a) > f(b)$, we may apply the Intermediate Value Theorem to $(-f)$ to prove that for all $v \in (f(b), f(a))$, there exists $c \in (a,b)$ such that $f(c) = v$.

In the remainder of this section, we will study some applications of the Intermediate Value Theorem. The most common application is to assert the existence of a solution to an equation. For example:

Example 7.3.3. Does there exist $x \in \mathbb{R}$ such that $x + e^{\sin(x)} = 42$?

Let $f(x) = x + e^{\sin(x)}$. Note that f is a composition of continuous functions, so is continuous; this means that we can use the Intermediate Value Theorem. To apply it to our question, we need to find two values a and b of x such that $f(a) < 42$ and $f(b) > 42$. Consider $a = 0$: then $f(a) = 0 + e^0 = 1$, so $f(a) < 42$. Next consider $b = 100\pi$: then $f(b) = 100\pi + e^0 = 100\pi + 1 > 42$. Thus, by the Intermediate Value Theorem, there exists $c \in (0, 100\pi)$ with $f(c) = 42$.

Here is another application of the Intermediate Value Theorem. This result is a special case of an important theorem in topology, known as *Brouwer's Fixed Point Theorem.*

Theorem 7.3.4 (Fixed Point Theorem). *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function with* $\operatorname{im}(f) \subset [a,b]$*. Then there exists $c \in [a,b]$ such that $f(c) = c$. Such a c is called a* fixed point *of f.*

Proof. Define $g\colon [a,b] \to \mathbb{R}$ by $g(x) = x - f(x)$. Note that g is continuous, as a difference of continuous functions. Moreover, we have $g(a) = a - f(a) \leq 0$, as $f(a) \in [a,b]$, and $g(b) = b - f(b) \geq 0$, so $f(b) \in [a,b]$. If $g(a) = 0$ or $g(b) = 0$ then we have found our fixed point (it is a or b, respectively). If not, then by the Intermediate Value Theorem (7.3.1) there exists $c \in (a,b)$ such that $g(c) = 0$. But $g(c) = c - f(c)$, so we have $f(c) = c$ and c is a fixed point of f. □

The following lemma is another application of the Intermediate Value Theorem; it will come in handy when we come to prove the Inverse Function Theorem.

Lemma 7.3.5. *Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous. Then f is injective if and only if it is strictly monotone.*

Proof. Exercise! □

For us, however, the most important application of the Intermediate Value Theorem is the following result, which will allow us to infer the existence of continuous functions that are "inverse" to continuous functions that we already know. Like the Fixed Point Theorem, this theorem is also a special case of a more general result, known simply as the *Inverse Function Theorem*; you will see an enhanced version of the Inverse Function Theorem later in the course, and the full-fat version will appear in *Analysis 3.*

Theorem 7.3.6 (Inverse Function Theorem for Continuous Functions). *Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous and injective, with $f(a) < f(b)$. Then f is strictly increasing with image* $\operatorname{im}(f) = [f(a), f(b)]$*, and its inverse $f^{-1}\colon [f(a), f(b)] \to \mathbb{R}$ is continuous.*

Proof. **Step 1.** We being by showing that f is strictly increasing. Indeed, by Lemma 7.3.5 f is strictly monotone. As $f(a) < f(b)$, we must have f strictly increasing.

Step 2. Next we show that $\operatorname{im}(f) = [f(a), f(b)]$. Note that $f(a), f(b) \in \operatorname{im}(f)$. Moreover, for any $v \in (f(a), f(b))$, the Intermediate Value Theorem (7.3.1) tells us that there exists $c \in (a,b)$ with $f(c) = v$, so $v \in \operatorname{im}(f)$. Thus $[f(a), f(b)] \subset \operatorname{im}(f)$.

Next suppose that there exists $v \in \operatorname{im}(f)$ with $v < f(a)$. Then there exists $c \in [a,b]$ with $f(c) = v$, as c is in the image of f. But $c \geq a$ with $f(c) < f(a)$, contradicting the fact that f is strictly increasing, so there are no $v \in \operatorname{im}(f)$ with $v < f(a)$. A similar argument shows that there are also no $v \in \operatorname{im}(f)$ with $v > f(b)$. We conclude that $\operatorname{im}(f) = [f(a), f(b)]$.

Step 3. As f is a bijection on to its image, there exists an inverse function

$$f^{-1}\colon [f(a), f(b)] \to \mathbb{R}.$$

It just remains to show that f^{-1} is continuous.

Pick $v \in (f(a), f(b))$. Then there exists a unique $c = f^{-1}(v) \in (a, b)$ such that $f(c) = v$. Let $\varepsilon > 0$ satisfy $a < c - \varepsilon < c < c + \varepsilon < b$. Since f is strictly increasing, we have $f(c - \varepsilon) < v < f(c + \varepsilon)$. Define $\delta := \min\{v - f(c - \varepsilon), f(c + \varepsilon) - v\}$; note that $\delta > 0$. Now if $x \in [f(a), f(b)]$ satisfies $|x - v| < \delta$, then $v - \delta < x < v + \delta$, so it follows that $f(c - \varepsilon) < x < f(c + \varepsilon)$ and hence $c - \varepsilon < f^{-1}(x) < c + \varepsilon$, as f is strictly increasing. In other words, $|f^{-1}(x) - f^{-1}(v)| < \varepsilon$, and f^{-1} is continuous at v.

It remains to prove that f^{-1} is continuous at $f(a)$ and $f(b)$. Let us prove it for $f(a)$; the proof for $f(b)$ is similar. Let $\varepsilon > 0$ satisfy $a + \varepsilon < b$, and define $\delta := f(a + \varepsilon) - f(a)$. As f is strictly increasing, we have $\delta > 0$. Now if $x \in [f(a), f(b)]$ satisfies $|x - f(a)| < \delta$, then $f(a) \leq x < f(a) + \delta$, so it follows that $f(a) \leq x < f(a + \varepsilon)$ and hence $a < f^{-1}(x) < a + \varepsilon$, as f is strictly increasing. In other words, $|f^{-1}(x) - a| < \varepsilon$, and f^{-1} is continuous at $f(a)$. □

Remark 7.3.7. There is also a version of the Inverse Function Theorem for functions with $f(a) > f(b)$, proved in a very similar way. It states:
Suppose that $f\colon [a, b] \to \mathbb{R}$ is continuous and injective with $f(a) > f(b)$. Then f is strictly decreasing with image $\operatorname{im}(f) = [f(b), f(a)]$*, and its inverse $f^{-1}\colon [f(b), f(a)] \to \mathbb{R}$ is continuous.*

The Inverse Function Theorem is a powerful tool for constructing new examples of continuous functions. Consider the following example.

Example 7.3.8. Consider the function $f\colon [0, \infty) \to \mathbb{R}$ given by $f(x) = x^n$, for $n \in \mathbb{N}$ some natural number. This function is continuous by Corollary 7.2.9 and injective. Its inverse is the *nth root* function $f^{-1}\colon [0, \infty) \to \mathbb{R}$ given by $f^{-1}(x) = x^{\frac{1}{n}}$. By the Inverse Function Theorem, the nth root function is continuous on any closed interval $[0, b]$ for $b \in \mathbb{R}$; it is therefore continuous at every point in $c \in [0, \infty)$ (just choose b so that $c \in [0, b]$).

Exercise Show that the function $f\colon (0, \infty) \to \mathbb{R}$ given by $f(x) = x^a$, where $a \in \mathbb{Q}$ is any rational number, is continuous.

The final application of the Intermediate Value Theorem that we will discuss here will be useful when we come to think about calculus. Again, this is a special case of a more general result: the general version will appear in *Analysis 3*.

Theorem 7.3.9 (Extreme Value Theorem). *Suppose that $f\colon [a, b] \to \mathbb{R}$ is continuous. Then there exist $x_-, x_+ \in [a, b]$ such that $f(x_-) \leq f(x) \leq f(x_+)$ for all $x \in [a, b]$. In particular, f is bounded on $[a, b]$. We say that f has a* minimum *at x_- and a* maximum *at x_+.*

Remark 7.3.10. Before we prove this theorem, we will make a few comments on its hypothesis. We assume three things about f: it is continuous, its domain is closed, and its domain is bounded. Without any one of these, the theorem is false; I'll leave the finding of counterexamples in each case as an exercise.

Before we prove the Extreme Value Theorem in full generality, we first prove it for bounded functions.

Lemma 7.3.11. *Suppose that $g\colon [a,b] \to \mathbb{R}$ is continuous and bounded. Then there exist $x_-, x_+ \in [a,b]$ such that $g(x_-) \leq g(x) \leq g(x_+)$ for all $x \in [a,b]$.*

Proof. We prove the existence of x_+; the proof for x_- is very similar. Let $g\colon [a,b] \to \mathbb{R}$ be continuous and bounded. Then $m := \sup\{g(x) : x \in [a,b]\}$ is well-defined. As m is the least upper bound, for every $n \in \mathbb{N}$ we may find $x_n \in [a,b]$ such that $m - \frac{1}{n} < g(x_n) \leq m$. By the Sandwich Theorem, the sequence $(g(x_n)) \to m$. We want to show that $m = g(x_+)$ for some $x_+ \in [a,b]$.

As (x_n) is a sequence on the bounded set $[a,b]$, it has a convergent subsequence by the Bolzano-Weierstrass Theorem. Let (x_{n_k}) be such a convergent subsequence and define $x_+ := \lim_{k\to\infty} x_{n_k}$. Since $[a,b]$ is a closed interval, we must have $x_+ \in [a,b]$. Moreover, by continuity of g, we have $\lim_{k\to\infty} g(x_{n_k}) = g(x_+)$. But $(g(x_n)) \to m$, so we must have $g(x_+) = m$. □

Now we use this lemma to prove the full Extreme Value Theorem.

Proof of Theorem 7.3.9. Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous. Our strategy is to first prove that f is bounded, then simply apply Lemma 7.3.11 to get the result.

To show that f is bounded, first define a new function

$$g\colon [a,b] \to \mathbb{R} \text{ given by } g(x) = \frac{1}{1+|f(x)|}.$$

Note that g is continuous (as a composition of continuous functions) and $0 < g(x) \leq 1$ for all $x \in [a,b]$. So g is bounded. Thus, applying Lemma 7.3.11 to g, we see that there exists $x_- \in [a,b]$ such that $g(x) \geq g(x_-)$ for all $x \in [a,b]$. Let $g(x_-) = v$.

From this, we see that $\frac{1}{1+|f(x)|} \geq v$ for all $x \in [a,b]$. Rearranging, we obtain $|f(x)| \leq \frac{1}{v} - 1$ for all $x \in [a,b]$. This shows that f is bounded. We conclude the proof by applying Lemma 7.3.11 to f. □

7.4 Continuous limits

In this section, we will begin to lay the groundwork for differentiation that will be treated in Analysis 2. Suppose that we have a function $f\colon X \to \mathbb{R}$ which is defined on a set $X = (a,b) \setminus \{c\}$, for some $c \in (a,b)$ (i.e. we take the interval (a,b) and remove the point c). Is there a sensible way to assign a value to $f(c)$?

We could try something like "define $\lim_{x\to c} f(x)$ to be the limit of $(f(x_n))$ for any sequence $(x_n) \subset X$ converging to c". However, this has the same problem as sequential continuity, in that it requires us to check every possible sequence (x_n) converging to c, an almost impossible task. Instead, we will use a definition that looks more like the definition of continuity.

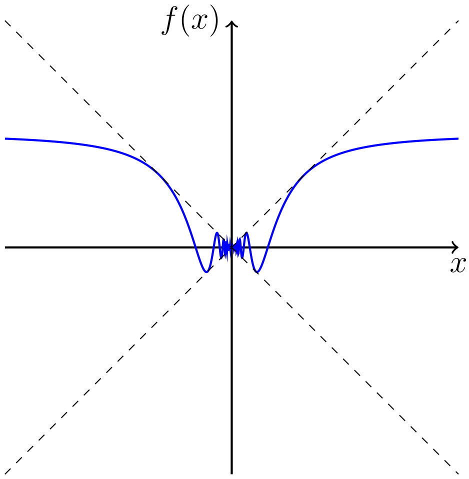

Figure 7.3: The function $f(x) = x\sin(\frac{1}{x})$ from Example 7.4.3. The dashed lines are the functions $f(x) = x$ and $f(x) = -x$, observe that $|x\sin(\frac{1}{x})| \leq |x|$.

Definition 7.4.1. *Let $f\colon X \to \mathbb{R}$ be a function, where $X = (a,b)\setminus\{c\}$ for some $c \in (a,b)$. We say that the* continuous limit $\lim_{x\to c} f(x)$ *exists and is equal to v if and only if*

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } \forall x \in X,\ \text{ if } |x-c| < \delta,\ \text{ then } |f(x) - v| < \varepsilon.$$

Remark 7.4.2. Note that this definition does not involve the value of f at the point c at all. In fact, as the next example will show, a function f can have a limit at a point c even if it is not well-defined there.

Example 7.4.3. Let $X = \mathbb{R}\setminus\{0\}$ and define $f\colon X \to \mathbb{R}$ by $f(x) = x\sin(\frac{1}{x})$. The graph of this function is shown in Figure 7.3. Does $\lim_{x\to 0} f(x)$ exist and, if so, what is it?

Note first that $f(0)$ does not make sense: the $\frac{1}{x}$ makes sure of that. However, we can still apply the definition. Let $\varepsilon > 0$ and choose $\delta = \varepsilon$. Then if $x \in X$ with $|x - 0| < \delta$, we have $|x\sin(\frac{1}{x})| \leq |x| < \varepsilon$ (as $|\sin(\frac{1}{x})| < 1$ for all $x \in X$), which gives $|x\sin(\frac{1}{x}) - 0| < \varepsilon$. Thus $\lim_{x\to 0} f(x)$ exists and is equal to 0.

Before we go any further, we should make sure that the definition we just wrote down makes sense. In particular, we should ensure that it is not possible for a function to have two different limits at the same point.

Proposition 7.4.4. *Let $f\colon X \to \mathbb{R}$ be a function, where $X = (a,b)\setminus\{c\}$ for some $c \in (a,b)$. If $\lim_{x\to c} f(x) = v$ and $\lim_{x\to c} f(x) = w$, then $v = w$.*

Proof. By definition, for any $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that if $|x-c| < \delta_1$, then $|f(x) - v| < \frac{\varepsilon}{2}$ and if $|x - c| < \delta_2$, then $|f(x) - w| < \frac{\varepsilon}{2}$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then if $|x - c| < \delta$, then $|x - c| < \delta_1$ and $|x - c| < \delta_2$, so both of the implications above hold. In particular,

$$\varepsilon > |f(x) - v| + |f(x) - w| = |f(x) - v| + |w - f(x)| \geq |v - w|$$

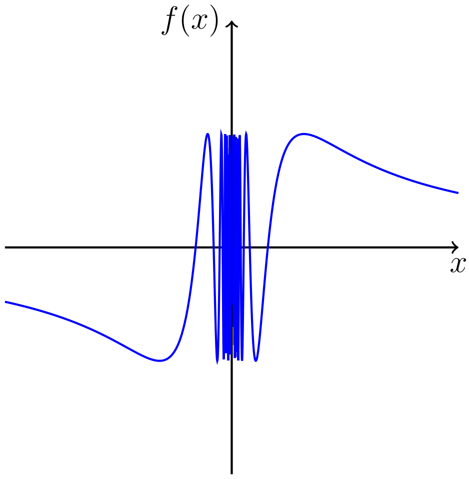

Figure 7.4: The function $f(x) = \sin(\frac{1}{x})$ from Example 7.4.7.

by the triangle inequality. But this is true for any $\varepsilon > 0$, so we must have $|v - w| = 0$, giving $v = w$. $\square$

Now we know that our definition of $\lim_{x\to c} f(x)$ is sensible, let's try to construct some tools that will allow us to compute it. The first result shows us that it is easy to compute limits on continuous functions.

Proposition 7.4.5. *A function $f\colon (a,b) \to \mathbb{R}$ is continuous at $c \in (a,b)$ if and only if $\lim_{x\to c} f(x)$ exists and is equal to $f(c)$.*

Proof. f is continuous at c if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in (a,b)$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. But this is precisely the definition of $\lim_{x\to c} f(x) = f(c)$. $\square$

The next proposition is more useful for proving that $\lim_{x\to c} f(x)$ does not exist.

Proposition 7.4.6. *Let (x_n) be a sequence in $X = (a,b) \setminus \{c\}$ which converges to c and let $f\colon X \to \mathbb{R}$ be a function. If $\lim_{x\to c} f(x) = v$ then $\lim_{n\to\infty} f(x_n) = v$.*

Proof. Choose $\varepsilon > 0$. Since $\lim_{x\to c} f(x) = v$, there exists $\delta > 0$ such that if $x \in X$ with $|x - c| < \delta$, then $|f(x) - v| < \varepsilon$. Moreover, since $(x_n) \to c$, there exists $N \in \mathbb{N}$ such that if $n > N$, then $|x_n - c| < \delta$. Combining these two statements, we see that if $n > N$, we have $|f(x_n) - v| < \varepsilon$, so $\lim_{n\to\infty} f(x_n) = v$. $\square$

The next example shows how we can use this result.

Example 7.4.7. Let $X = \mathbb{R} \setminus \{0\}$ and define $f\colon X \to \mathbb{R}$ by $f(x) = \sin(\frac{1}{x})$. The graph of this function is shown in Figure 7.4. Does $\lim_{x\to 0} f(x)$ exist and, if so, what is it?

Consider the following two sequences:

$$(x_n) = \left(\frac{1}{2\pi n + \frac{\pi}{2}}\right),$$
$$(y_n) = \left(\frac{1}{2\pi n}\right).$$

We have $(x_n) \to 0$ and $(y_n) \to 0$, so if $\lim_{x\to 0} f(x)$ exists and is equal to v, then we should have $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) = v$, by Proposition 7.4.6. But $f(x_n) = \sin(2\pi n + \frac{\pi}{2}) = 1$ for all $n \in \mathbb{N}$, so $\lim_{n\to\infty} f(x_n) = 1$, and $f(y_n) = \sin(2\pi n) = 0$ for all $n \in \mathbb{N}$, so $\lim_{n\to\infty} f(y_n) = 0$. Thus we see that $\lim_{x\to 0} f(x)$ does not exist.

Continuous limits behave a lot like limits of sequences. Here are a couple of familiar results.

Proposition 7.4.8. *Let $f, g\colon X \to \mathbb{R}$ be two functions, where $X = (a, b) \setminus \{c\}$ for some $c \in (a, b)$. If $\lim_{x\to c} f(x) = v$ and $\lim_{x\to c} g(x) = w$, then*

$$\lim_{x\to c}(f+g)(x) = v + w$$
$$\lim_{x\to c}(f-g)(x) = v - w$$
$$\lim_{x\to c}(f\cdot g)(x) = vw$$

Moreover, if $g(x) \neq 0$ for all $x \in X$ and $w \neq 0$, then

$$\lim_{x\to c}(\tfrac{f}{g})(x) = \tfrac{v}{w}.$$

Proof. We prove the statement for $(f + g)$, the other cases are an exercise. Let $\varepsilon > 0$. Since $\lim_{x\to c} f(x) = v$, there exists $\delta_1 > 0$ such that if $x \in X$ with $|x - c| < \delta_1$, then $|f(x) - v| < \frac{\varepsilon}{2}$. Moreover, since $\lim_{x\to c} g(x) = w$, there exists $\delta_2 > 0$ such that if $x \in X$ with $|x - c| < \delta_2$, then $|g(x) - w| < \frac{\varepsilon}{2}$. Let $\delta = \min\{\delta_1, \delta_2\} > 0$. Then if $|x - c| < \delta$, both of the above statements hold, so $\varepsilon > |f(x) - v| + |g(x) - w| \geq |(f + g)(x) - (v + w)|$ by the triangle inequality. Thus $\lim_{x\to c}(f + g)(x) = v + w$, as required. $\square$

Theorem 7.4.9 (Sandwich Theorem for Continuous Limits)**.** *Consider three functions $f, g, h\colon X \to \mathbb{R}$, where $X = (a, b) \setminus \{c\}$ for some $c \in (a, b)$. Suppose that $f(x) \leq g(x) \leq h(x)$ for all $x \in X$. If $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = v$, then $\lim_{x\to c} g(x) = v$ also.*

Proof. Let $\varepsilon > 0$. Since $\lim_{x\to c} f(x) = v$, there exists $\delta_1 > 0$ such that if $x \in X$ with $|x - c| < \delta_1$, then $|f(x) - v| < \varepsilon$. Moreover, since $\lim_{x\to c} h(x) = v$, there exists $\delta_2 > 0$ such that if $x \in X$ with $|x - c| < \delta_2$, then $|g(x) - v| < \varepsilon$. Let $\delta = \min\{\delta_1, \delta_2\} > 0$. Then if $|x - c| < \delta$, both of the above statements hold so, noting that $f(x) \leq g(x) \leq h(x)$, we have

$$g(x) - v \leq h(x) - v \leq |h(x) - v| < \varepsilon,$$
$$v - g(x) \leq v - f(x) \leq |f(x) - v| < \varepsilon.$$

Putting these together, we obtain $-\varepsilon < g(x) - v < \varepsilon$, i.e. that $|g(x) - v| < \varepsilon$. Therefore $\lim_{x\to c} g(x) = v$, as required. $\square$

Example 7.4.10. The Sandwich Theorem gives us an easier way to do Example 7.4.3. Recall that, in that example, we had $X = \mathbb{R}\setminus\{0\}$ and $f\colon X \to \mathbb{R}$ given by $f(x) = x\sin(\frac{1}{x})$. The graph of this function is shown in Figure 7.3.

To compute $\lim_{x\to 0} f(x)$, we can use that fact that $|\sin(\frac{1}{x})| \leq 1$ for all $x \in X$, so $-|x| \leq f(x) \leq |x|$ for all $x \in X$. But $\lim_{x\to 0} |x| = \lim_{x\to 0}(-|x|) = 0$, so the Sandwich Theorem gives $\lim_{x\to 0}(f(x)) = 0$ also.

7.5 One-sided limits and limits at infinity

We conclude this chapter with a few variants on the theme of continuous limits. All of the results from 7.4.4 to 7.4.9 continue to hold in these settings, if suitably restated.

We begin with the concept of *one-sided limits.*

Definition 7.5.1. *Let $f\colon (c, b) \to \mathbb{R}$ be a function. We say that the* one-sided limit $\lim_{x\to c^+} f(x)$ *exists and is equal to v if and only if*

$$\forall\varepsilon > 0\ \exists\delta > 0 \text{ such that if } c < x < c + \delta, \text{ then } |f(x) - v| < \varepsilon.$$

Let $g\colon (a, c) \to \mathbb{R}$ be another function. We say that the one-sided limit $\lim_{x\to c^-} g(x)$ *exists and is equal to w if and only if*

$$\forall\varepsilon > 0\ \exists\delta > 0 \text{ such that if } c - \delta < x < c, \text{ then } |g(x) - w| < \varepsilon.$$

If we have a function $f\colon X \to \mathbb{R}$, where $X = (a, b) \setminus \{c\}$ for some $c \in (a, b)$, then we can talk about both of the one-sided limits $\lim_{x\to c^+}(f(x))$ and $\lim_{x\to c^-}(f(x))$. The following example shows that they do not have to be equal!

Example 7.5.2. Consider the function $f\colon X \to \mathbb{R}$, where $X = \mathbb{R} \setminus \{0\}$, defined by

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Then $\lim_{x\to 0^+}(f(x)) = 1$ and $\lim_{x\to 0^-}(f(x)) = -1$.

Exercise Consider a function $f\colon X \to \mathbb{R}$, where $X = (a, b) \setminus \{c\}$ for some $c \in (a, b)$. Show that $\lim_{x\to c} f(x)$ exists and is equal to v if and only if both of the one-sided limits $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ exist and are equal to v.

We can also talk about infinite limits and limits at infinity.

Definition 7.5.3. *Let $f\colon X \to \mathbb{R}$ be a function, where $X = (a, b)\setminus\{c\}$ for some $c \in (a, b)$. We say that $\lim_{x\to c} f(x) = +\infty$ if and only if*

$$\forall C > 0\ \exists\delta > 0 \text{ such that } \forall x \in X, \text{ if } |x - c| < \delta, \text{ then } f(x) > C.$$

Definition 7.5.4. *Let $f\colon (a,\infty) \to \mathbb{R}$ be a function. We say that the limit* $\lim_{x\to+\infty} f(x)$ *exists and is equal to v if and only if*

$$\forall \varepsilon > 0 \; \exists M \geq a \textit{ such that if } x > M, \textit{ then } |f(x) - v| < \varepsilon.$$

There are lots more variations of these ideas, for instance

$$\begin{aligned}
\lim_{x\to c} f(x) &= -\infty, \\
\lim_{x\to -\infty} f(x) &= c, \\
\lim_{x\to +\infty} f(x) &= +\infty, \\
\lim_{x\to c^+} f(x) &= +\infty,
\end{aligned}$$

and many more. Now that you have seen the definitions above, you should be able to write down whichever ones you may need.