Queen Mary University of London

MTH5104: Convergence and Continuity 2023–2024

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Contents

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Introduction and Motivation

MTH5104 and MTH5105 together form an introduction to analysis, the branch of mathematics devoted to the precise study of sequences, series, differentiation and integration. These courses bring rigour to calculus.

Some examples, questions and paradoxes

1. While already ancient Greek mathematicians (such as Archimedes) from the school of Pythagoras anticipated the integral calculus, they were very reluctant to use any kind of infinite process. This is best shown by considering a famous paradox of Zeno ($\sim$ 460 BC), *Achilles and the tortoise*: If Achilles starts at A and the tortoise starts at B then Achilles can never catch the tortoise since by the time Achilles reaches B, the tortoise will be at some further point C, and by the time Achilles reaches C, the tortoise will be further ahead at D and so on *ad infinitum*. So the tortoise will always be ahead!

   Let us model this argument with some numbers. (We don't try to actually model realistic speeds or distances!) Let us assume the tortoise is at distance 1 from Achilles and moves with speed 1, while Achilles runs with speed 4. When will he catch the tortoise?

   The time needed for Achilles to reach B is $1/4$. Hence the tortoise moved $1/4$, i.e. C is $1/4$ away from B. So the time for Achilles to reach C is $1/4 + 1/16$. The tortoise has now moved to D which is $1/16$ away from C, so the time for Achilles to reach D is $1/4 + 1/16 + 1/64$, etc. So Achilles runs for time

   $$\sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots$$

   behind the tortoise. But on the other hand, we can also solve this with an equation: Achilles runs for time t with speed 4, so he will be $4t$ away from A, while the tortoise at the same time is $1 + t$ away from A. So Achilles catches the tortoise when

   $$4t = 1 + t.$$

   Obviously, the solution of this equation is $t = 1/3$. The two results contradict each other if the above infinite sum becomes larger than $1/3$. *Could it be, that although we keep on adding positive numbers to the sum, the value of the total sum is still finite (and actually less than or equal to* $1/3$*)?*

   Somewhat surprisingly, the answer is *yes*, in fact

   $$\sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots = \frac{1}{3}. \tag{0.1}$$

2. Let us now look at the series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

As before, we infinitely often add a positive number, and the numbers added become smaller and smaller. *Will the sum therefore also converge to a finite value, as in the example above?*

This time, the answer is *no*! We can see this as follows.

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots \\ &\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} + \underbrace{\frac{1}{16} + \ldots} + \cdots \\ &= 1 + \frac{1}{2} \quad + \frac{1}{2} \quad\quad + \frac{1}{2} \quad\quad + \frac{1}{2} \quad + \cdots \\ &= \infty \end{aligned}$$

What is different in example 2 from example 1? In this lecture course, we will develop general tools which allow us to investigate whether or not an infinite series converges.

3. Of course, (0.1) is nothing else than a *geometric series*, and we know from calculus that

$$\sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k = \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k - 1 = \frac{1}{1-\frac{1}{4}} - 1 = \frac{1}{3},$$

using the rule $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$ (which agrees with what we claimed above).

Using this rule, we have for example

$$\frac{x}{1-x} = x\Big(\frac{1}{1-x}\Big) = x(1 + x + x^2 + \ldots) = x + x^2 + x^3 + \cdots \tag{0.2}$$

or also

$$\frac{x}{x-1} = \Big(\frac{1}{1-\frac{1}{x}}\Big) = 1 + x^{-1} + x^{-2} + \cdots \tag{0.3}$$

Adding (0.2) and (0.3) thus yields

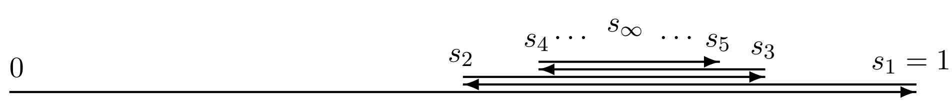
$$0 = \ldots + x^{-2} + x^{-1} + 1 + x + x^2 + \cdots$$

If $x > 0$, this means that adding up infinitely many positive numbers gives 0. *What is wrong with this argument?* The answer is that we have to consider for which values of x each series converges. The series on the right hand side of (0.2) only converges for $|x| < 1$, the series on the right hand side of (0.3) only converges for $|x| > 1$. It becomes clear that we need to define *convergence* very carefully and worry about the set of values for which a series converges.

4. The last series we consider in this section is the alternating harmonic series. It is given by

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

As we will see later in this course, this series converges. Intuitively, this is clear if we draw the first few elements, denoting $s_n = \sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k}$, we have the following picture.



(One can prove that the value of the infinite series is $\ln 2$, the natural logarithm of 2.) Now, we will re-order the sequence, taking always a positive element and then two negative ones, so that we obtain

$$S := 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \dots$$

Note that we still have all the elements from the original series, we just changed the order of summation! As addition is commutative, we might expect that the value should be the same and the order of summation should not matter. However, we get

$$\begin{aligned} S &= (1 - \frac{1}{2}) - \frac{1}{4} + (\frac{1}{3} - \frac{1}{6}) - \frac{1}{8} + (\frac{1}{5} - \frac{1}{10}) - \frac{1}{12} + \cdots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots \\ &= \frac{1}{2}\Big(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\Big) \end{aligned}$$

So now, the value of the sum is *half* of what it was before! *How did this happen? Did we do something wrong?* One of the goals of this course is to answer these questions.

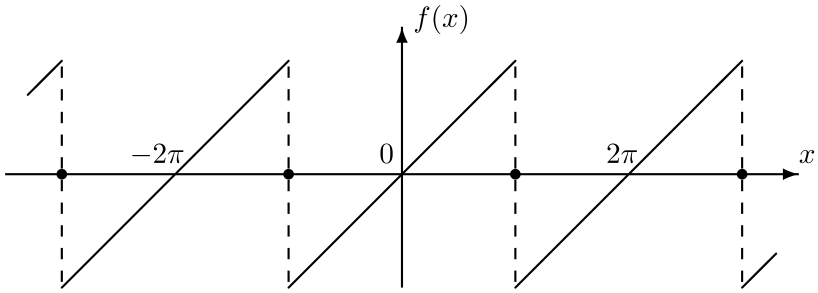
5. Finally, let us move on to some questions related to functions.

   - *Is the sum of two continuous functions continuous?* Yes!

- *Is the sum of infinitely many continuous functions continuous?* No! For example the function

$$f(x) = \frac{\sin(x)}{1} - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \frac{\sin(4x)}{4} + \cdots,$$

  has the following graph:



- *For any sequence of continuous functions f_n which converges to $f = \lim_{n\to\infty} f_n$, is f continuous?* No!

  At least not in general. We will see that this depends on the *type* of convergence – we will define different notions of convergence for functions.

- *Similarly, one could ask whether for a sequence of differentiable functions f_n converging to $f = \lim_{n\to\infty} f_n$, the limit f is also differentiable. Or one could ask whether for any sequence of integrable functions f_n, is it true that*

$$\int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n.$$

  Again, the answer is *no* in general. The theory needed to understand (most of) these questions will be covered in the course MTH5105.

Conclusion

We will have to make precise definitions and carefully restudy topics from calculus such as limits, convergence, continuity, etc. In fact, in order to do this precisely, we will have to begin in an elementary manner, namely with the properties of real numbers.

However, before we start with this, we will introduce a method which makes it easier to read, understand and prove statements like

$$\forall x \in \mathbb{R}\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in \mathbb{R}, |x - y| < \delta\ :\ |x^2 - y^2| < \varepsilon. \tag{0.4}$$

As we will later see, this complicated looking expression simply states that $f(x) = x^2$ is continuous. Don't be afraid: At the end of this lecture course, you will not only be able to understand such statements, but also to write and prove them yourselves!

“But I already know all this!”

You have likely encountered the main players in this course already: sequences, series, and functions. You may think that you understand them very well. However, as the above examples demonstrate, when a theory is not given a rigorous basis, confusion and paradox are always lurking.

This course will rebuild the edifice of calculus from the ground up. It will require you to take a step back, to re-examine your preconceptions, and to give careful meaning to notions which you previously took for granted.

Why go to all the trouble? There are at least two compelling reasons:

1. **Practical.** It’s well-known that employers like to hire mathematicians. Why? After all, physicists and engineers also know calculus, and are much more geared towards real-world problems. The answer is: employers want you, not primarily because of what you know, but because of *how you think*. Mathematicians are unique in our ability to reason with absolute precision, allowing us to solve problems that nobody else can. This isn’t a skill you can gain by watching a YouTube video or even reading a book: you have to *earn it*, through countless hours of hard work, at a desk, solving exercises.

2. **Spiritual.** Mathematics is beautiful. When founded in rigour, an elegant proof is both convincing and inspiring. The satisfaction of understanding an argument — truly understanding it — is unparalleled. We each have a finite amount of time on this world, and our goal should be to fill it with as much beauty (and love) as we can.

1 Sets and logic

Most of this should be familiar from MTH4213. We will provide a quick review, but *please* make sure you are on top of this. Ask for help if you need it. There are no shortcuts here: you will quickly become lost if you do not internalise these ideas.

1.A Symbols

We begin by collecting the most common mathematical symbols that you will encounter. It is *crucial* to use these! Doing so will force you to think in a precise way.

Set-theoretic symbols

- $\in$ is an element of
- $\subseteq$ is a subset of
- $\cup$ union
- $\cap$ intersection
- $\emptyset$ the empty set.

We can also negate these symbols. For instance, the negation of $\in$ is obtained by striking out the symbol: $\notin$.

Logical symbols

- $\Rightarrow$ implies
- $\Leftrightarrow$ if and only if (iff)
- $\forall$ for all
- $\exists$ there exists

1.B Set theory

Definition 1.1
A *set* is a collection of objects. The objects of a set S are referred to as the *elements* of S. We write

$$x \in S$$

to indicate that x is an element of S.

We can define a set by listing its elements inside curly braces. For example:

$$S = \{1, 2, 3\}. \tag{1.1}$$

We have $1 \in S$ but $4 \notin S$. In a set we do not allow repeated elements, so for example:

$$\{1, 1, 2, 3, 3, 3\} = \{1, 2, 3\}.$$

We can also define a set by stating a property that determines its elements. For example, we can define the set of all positive real numbers as follows:

$$T = \{x \in \mathbb{R} : x > 0\} \qquad \text{or} \qquad T = \{x \in \mathbb{R} | \, x > 0\}.$$

Here both ":" and "|" should be read as "such that".

Definition 1.2
Given two sets S and T, we say that S is a *subset* of T, written

$$S \subseteq T$$

if and only if every element of S is an element of T, i.e:

$$x \in S \Rightarrow x \in T.$$

Taking $S = \{1, 2, 3\}$ as in (1.1), we have

$$\{1, 2\} \subseteq \{1, 2, 3\}, \qquad \{1\} \subseteq \{1, 2, 3\}, \qquad 1 \not\subseteq \{1, 2, 3\}, \qquad \{1, 4\} \not\subseteq \{1, 2, 3\}.$$

Definition 1.3
The *empty set* $\emptyset$ is the set with no elements. It is a subset of every set.

Exercise 1.4
Write down all the subsets of $\{1, 2\}$. Then do the same for $\{1, 2, 3\}$, and for $\{1, 2, 3, 4\}$. Based on this, make a guess for how many subsets $\{1, 2, \ldots, n\}$ has, and prove that your guess is correct.

Definition 1.5
Let A and B be two sets. The *union* of A and B is the set of elements belonging to either A or B (or both):

$$A \cup B := \{x : x \in A \text{ or } x \in B\}.$$

The *intersection* of A and B is the set of elements belonging to both:

$$A \cap B := \{x : x \in A \text{ and } x \in B\}.$$

The set of elements of A which do not belong to B is denoted:

$$A \setminus B := \{x : x \in A \text{ and } x \notin B\}.$$

Remark 1.6
The intersection $A \cap B$ can be empty even if A and B are nonempty. We have inclusions:

$$\begin{aligned} A \cap B &\subseteq A, \\ A \cap B &\subseteq B, \\ A &\subseteq A \cup B, \\ B &\subseteq A \cup B, \\ A \cap B &\subseteq A \cup B, \\ A \setminus B &\subseteq A. \end{aligned}$$

Example 1.7
Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 5\}$. Compute $A \cup B$, $A \cap B$ and $A \setminus B$. What is the relation between $(A \cup B) \setminus (A \cap B)$ and $A \setminus B$?

You are already familiar with the following sets of numbers:

$$\begin{aligned} \mathbb{N} &= \{1, 2, 3, 4, \cdots\} && \text{natural numbers} \\ \mathbb{N}_0 &= \{0, 1, 2, 3, \cdots\} && \text{natural numbers with 0 included} \\ \mathbb{Z} &= \{0, \pm 1, \pm 2, \pm 3, \cdots\} && \text{integers} \\ \mathbb{Q} &= \{a/b : a \in \mathbb{Z}, b \in \mathbb{N}\} && \text{rational numbers} \end{aligned}$$

By definition of these sets we have immediately

$$\mathbb{N} \subseteq \mathbb{N}_0 \subseteq \mathbb{Z} \subseteq \mathbb{Q}.$$

There exists a bigger set which contains all of these sets of numbers: it is the set $\mathbb{R}$ of real numbers. Intuitively it is the set of all points on a straight line extending indefinitely in both directions. We will give a rigorous definition later on but at the moment we can already write the following chain of inclusions:

$$\mathbb{N} \subseteq \mathbb{N}_0 \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

1.C Logic: implication and equivalence

Recall that "$\Rightarrow$" means "implies" or "therefore". Here is an example of a true statement:

I live in London $\Rightarrow$ I live in England.

The order is crucial! The following statement is false, as any Mancunian will proudly attest:

I live in England $\Rightarrow$ I live in London.

It is a very common mistake to mix up $A \Rightarrow B$ and $B \Rightarrow A$. You are *much* less likely to make this mistake if you use the mathematical symbol "$\Rightarrow$" when writing your arguments. This is better than using terms like "therefore" or the dreaded

$$\therefore$$

which causes endless logical headaches for students.

Occasionally we have two statements which each imply the other. For instance:

It is sunny $\Leftrightarrow$ It is daytime and there are no clouds blocking the sun.

In this case we say that the statements are *equivalent*. We write

$$A \Leftrightarrow B$$

which means that A is true *if and only if* B is true. Here are some mathematical examples:

1. $x > 0 \Leftrightarrow x + 1 > 1$
2. If n is an integer, then $n > \frac{1}{2} \Leftrightarrow n \geq 1$
3. If m is an integer, then m is even $\Leftrightarrow m^2$ is even.

1.D Logic: quantifiers

We use the quantifiers $\forall$ ("for all") and $\exists$ ("there exists"). Here are some examples translating statements from English into mathematical symbols:

English	**Mathematics**
For all real numbers x, we have $x^2 \geq 0$	$\forall x \in \mathbb{R}\colon x^2 \geq 0$
For all integers a and b, we have $a + b = b + a$	$\forall a, b \in \mathbb{Z}\colon a + b = b + a$
For every real number r there exists an integer n with $n > r$	$\forall r \in \mathbb{R}\ \exists n \in \mathbb{Z}\colon n > r$

Notice that here the colon ":" is read sometimes as "we have" and sometimes as "such that" (this is really a defect of the English language; from the logical perspective there is no difference).

1.E Demon Games

Thought experiments are mental exercises. These are very common in science. There are a couple of thought experiments that involve Demons, in particular in physics and mathematics. Famous examples are Laplace's Demon or Maxwell's Demon.

In this lecture course, we will sometimes translate statements like

$$\forall n \in \mathbb{N}\ \exists m \in \mathbb{N} \ : \ m > n \tag{1.2}$$

(read "for all natural numbers n there exists a natural number m such that $m > n$") into a Demon Game, a thought experiment in which we play against an imaginary Demon. This will provide a convenient framework for formulating mathematical proofs.

The rules for the game are as follows:

1. We read the statement from left to right.
2. Whenever there is a "for all" quantifier $\forall$, the Demon gets to choose. If there are restrictions, the Demon has to fulfil them.
3. Whenever there is a "there exists" quantifier $\exists$, then we get to choose. If there are restrictions, we have to fulfil them.
4. If the last statement is true, then we win. Otherwise the Demon wins.

Example 1.8
The expression (1.2) corresponds to the Demon Game

- First the Demon picks $n \in \mathbb{N}$.
- Then we pick $m \in \mathbb{N}$.
- We win if $m > n$.

Example 1.9
The expression (0.4) corresponds to the Demon Game

- First the Demon picks $x \in \mathbb{R}$.
- Then the Demon picks $\varepsilon \in \mathbb{R}$ satisfying the restriction $\varepsilon > 0$.

- Then we pick $\delta \in \mathbb{R}$ satisfying the restriction $\delta > 0$.
- Then the Demon picks $y \in \mathbb{R}$ *satisfying the restriction* $|x - y| < \delta$.
- We win if $|x^2 - y^2| < \varepsilon$.

Be careful with the order. In Example 1.8, the Demon picks n first and we pick m afterwards *knowing what the Demon picked*. This is different from the statement $\exists m \in \mathbb{N}\ \forall n \in \mathbb{N}\ :\ m > n$ which translates into a Demon Game in which we have to choose first and the Demon picks after us *knowing what we picked*. Of course, neither we nor the Demon can look into the future, so in the Example 1.9 above, we can pick a δ that depends on x and ε which the Demon has already chosen, but our δ cannot depend on y, as y is picked by the Demon *afterwards*.

Note that we abbreviated the expression $\forall \varepsilon \in \mathbb{R}, \varepsilon > 0$ by $\forall \varepsilon > 0$ and we have also abbreviated $\exists \delta \in \mathbb{R}, \delta > 0$ by $\exists \delta > 0$. In this lecture course, ε and δ are *always* real numbers, so these abbreviations should cause no confusion.

Trial games: These are just examples of possible games that could have been played, including the statement of who would have won the game if these moves would have been played. It is best to write them down in a table.

Example 1.10
For the game from Example 1.8, we have:

	Trial Game 1	Trial Game 2
Demon picks $n =$	44	18
We pick $m =$	27	55
Who wins?	Demon (as $27 \not> 44$)	Us (as $55 > 18$)

Example 1.11
For the game from Example 1.9, we have:

	Trial Game 1	Trial Game 2
Demon picks $x =$	4	4
Demon picks $\varepsilon =$	1	1
We pick $\delta =$	1.5	0.2
Demon picks $y =$	5	4.1
Who wins?	Demon (as $\lvert x^2 - y^2\rvert = 9 \not< 1 = \varepsilon$)	Us (as $\lvert x^2 - y^2\rvert = 0.81 < 1 = \varepsilon$)

Winning strategy: A winning strategy is a strategy with which we *always* win the game, no matter what the Demon plays (in particular, the strategy must be such that we win even if the Demon plays particularly cleverly or stupidly).

Example 1.12
For example in Example 1.8 above, a winning strategy is the following.

- The Demon picks $n \in \mathbb{N}$.
- We pick $m := n + 1 \in \mathbb{N}$.
- Then $m > n$, so we win.

Note that we could have also chosen $m = n + 37$ to get another winning strategy, or $m = n^3 + 12$ to obtain yet another one. On the other hand, setting $m = \frac{3n+7}{2}$ is *not* a winning strategy – it satisfies $m > n$, but it is not an allowed move because in general this m is not a natural number.

Make sure that all moves are legal (i.e., possible restrictions are fulfilled). Remember that the Demon has a free choice of legal moves. Also remember that we can't see into the future, so for example in the game from Example 1.9, when we pick δ, we can choose it depending on x and ε (which the Demon already picked), but *not* on y, which the Demon will pick later!

Formal proofs: If we found a winning strategy for the Demon Game, then the original mathematical expression is true. A winning strategy can then be translated into a formal proof, using the following simple rules.

1. Whenever the Demon picks x, we write "Given x".
2. Whenever we pick y, we write "Let y be ...", or "Choose y such that ...".
3. At the end, replace the "we win" by an *end of proof box*, i.e. $\square$.

Example 1.13
The winning strategy from Example 1.12 translates into the following formal proof.

Theorem. $\forall n \in \mathbb{N}\ \exists m \in \mathbb{N} : m > n$.

Proof. Given $n \in \mathbb{N}$, choose $m := n + 1 \in \mathbb{N}$. Then $m > n$. □

Negation: Now we know how we can prove that a statement is true: we have to find a winning strategy for the corresponding Demon Game. But how do we prove that a statement is false?

We produce the *negation* $(\neg A)$ of a mathematical statement A in a mechanical way as follows. We change all $\exists$ into $\forall$ and vice-versa, and we negate the final expression after the colon. Whenever we want to prove that some mathematical expression is false, we will negate it and prove this negation.

Example 1.14
Consider the statement

$$\exists n \in \mathbb{N}\ \forall m \in \mathbb{N} : m \leq n.$$

We want to prove that this statement is false. We follow the rules above to produce the negation:

$$\forall n \in \mathbb{N}\ \exists m \in \mathbb{N} : m > n.$$

We now have to prove that the negation is true, by finding a winning strategy for the corresponding Demon Game. This is precisely what we did in Example 1.13 above.

2 Real Numbers

Recall the following systems of numbers:

$\mathbb{N}$ — the natural numbers: $1, 2, 3, 4, \ldots$
$\mathbb{N}_0$ — the natural numbers including zero: $0, 1, 2, 3, 4, \ldots$
$\mathbb{Z}$ — the integers: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
$\mathbb{Q}$ — the rationals: $\{p/q \ : \ p, q \in \mathbb{Z}, q \neq 0\}$.

We assume all the standard properties of these number systems:

- in $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{Q}$, we can do addition and multiplication.
- in $\mathbb{Z}$ and $\mathbb{Q}$, we can also do subtraction.
- in $\mathbb{Q}$, we can also do division (except by 0).

We also assume the standard rules:

- Commutativity of addition and multiplication:
  $a + b = b + a, \quad ab = ba.$
- Associativity of addition and multiplication:
  $(a + b) + c = a + (b + c), \quad (ab)c = a(bc).$
- Distributivity of addition over multiplication:
  $a(b + c) = ab + ac.$

Moreover, $\mathbb{Q}$ is *a field*, which means that there is an additive neutral element 0 and a multiplicative neutral element 1 as well as inverses for addition and multiplication (except for a multiplicative inverse of 0). Equivalently, this means that the following equations always have a (unique) solution for x: $a + x = b$ and $ax = b$ (if $a \neq 0$).

Finally, we assume the usual properties of order ($>$), i.e.

- every number a satisfies either $a > 0$, $a = 0$, or $-a > 0$.
- if $a > 0$ and $b > 0$, then $a + b > 0$ and $ab > 0$.
- if $a \geq b$ and $b \geq a$ then $a = b$.

The real numbers contain $\mathbb{Q}$ (and hence $\mathbb{Z}$ and $\mathbb{N}$). There are various ways of constructing the real numbers, e.g. as decimals, as "limits of Cauchy sequences", or as "Dedekind cuts", etc. But we shall characterise the real numbers by their properties rather than defining them using a particular model. They should have all the properties of $\mathbb{Q}$ (and in fact contain $\mathbb{Q}$), but we ask for more than that!

In particular, we consider the question whether or not there are any "gaps" in the real numbers, like the "gap" in $\mathbb{Q}$ between $\{z \in \mathbb{Q} \ : \ z^2 \leq 2\}$ and $\{z \in \mathbb{Q} \ : \ z^2 \geq 2\}$.

Definition 2.1
Let $a, b \in \mathbb{R}$ be real numbers with $a < b$. We then define the *intervals*:

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}.$$
$$(a, b) := \{x \in \mathbb{R} : a < x < b\}.$$
$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}.$$
$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}.$$

2.A Maximum and minimum of sets

Definition 2.2 (Maximum and minimum)
Suppose $A \subseteq \mathbb{R}$. Then a real number x is called a *maximum* of A iff

1. $x \in A$,
2. $\forall y \in A \ : \ y \leq x$.

A real number x is called a *minimum* of A iff

1. $x \in A$,
2. $\forall y \in A \ : \ x \leq y$.

Examples 2.3 (i) $A = [0, 1] = \{x \in \mathbb{R} \ : \ 0 \leq x \leq 1\}$.
$x = 1$ is a maximum of the set A, because

1. $1 \in A$,
2. $\forall y \in A \ : \ y \leq 1$.

(ii) $A = \{z \in \mathbb{Z} \ : \ z^2 < 101\}$.
A has a maximum of 10 and a minimum of -10. (Check!)

(iii) $A = (0, 1) = \{x \in \mathbb{R} \ : \ 0 < x < 1\}$.
$x = 1$ is *not* a maximum of A, since $x \notin A$.
$x = 0.9$ is *not* a maximum of A, since $y = 0.99 \in A$ and $y > x$.

In fact, this set A does not have a maximum. Let us prove this! The statement that A has a maximum is:

$$\exists x \in A \ \forall y \in A : y \leq x.$$

We want to show that this is false – or equivalently, that the negation is true. The negation (which we want to prove) is

$$\forall x \in A \ \exists y \in A : y > x. \tag{2.1}$$

Proof. Given $x \in A$, choose $y = \frac{x+1}{2}$. We must show that $y \in A$ and $y > x$. Since $x < 1$ we have $x + 1 < 2$ and so $y = \frac{x+1}{2} < 1$. We conclude that $y \in A$. Moreover we have

$$y = \frac{x+1}{2} = x + \frac{1-x}{2}$$

and since $x < 1$ we have $\frac{1-x}{2} > 0$ so that $y > x$. □

(iv) $A = \mathbb{N}$. Does this set have a maximum? No! We actually already proved this last week, when giving a proof (respectively a winning strategy for the corresponding Demon Game) of Example 1.14.

Could a set A have two different maxima?

Lemma 2.4 (Maximum is unique)
Let $A \subseteq \mathbb{R}$. If A has a maximum, then this maximum is unique (and we denote it by $\max(A)$).

Proof. Suppose we are given x and x' are two maxima of A. Then since x is a maximum of A and $x' \in A$, we have $x' \leq x$. Similarly, since x' is a maximum of A and $x \in A$, we have $x \leq x'$. From $x' \leq x$ and $x \leq x'$ we deduce $x = x'$. So the maximum is unique (if it exists). □

Of course, the minimum (if it exists) is unique as well and we denote it by $\min(A)$.

2.B Upper and lower bounds of sets

Definition 2.5 (Upper and lower bounds)
Let A be a subset of $\mathbb{R}$. Then we say that $x \in \mathbb{R}$ is an *upper bound* for A iff

$$\forall y \in A : y \leq x.$$

Similarly, $x \in \mathbb{R}$ is a *lower bound* for A iff

$$\forall y \in A : x \leq y.$$

Obviously, if A has a maximum then that maximum is an upper bound for A. But a set A can have an upper bound even if it does not have a maximum, and it can have many different upper bounds.

Examples 2.6 (i) $A = [0, 1]$. We have seen that $x = 1$ is the maximum of A, so this is an upper bound. But any real number bigger than 1 is also an upper bound for $[0, 1]$, for example $x = 738$ or $x = 10^{12}$.

(ii) $A = (0, 1)$. We have seen that $(0, 1)$ does not have a maximum. However, $x = 1$ is an upper bound for A, as any element $y \in A$ satisfies $y \leq 1$. Again, every real number greater than 1 is an upper bound as well.

(iii) $A = \mathbb{N}$. We have seen that A does not have a maximum, but does it have an upper bound? No! (We will prove this soon!)

(iv) $A = \{z \in \mathbb{Q} \ : \ z^2 \leq 2\}$. This set has an upper bound, for example $x = 3$. To prove this, we must show that *every* $y \in A$ satisfies $y \leq 3$, or equivalently *no* element of A is strictly greater than 3.

But given $y \in \mathbb{Q}$ with $y > 3$, we have $y^2 > 9$, so in particular $y^2 > 2$ and thus $y \notin A$.

Note that the set $A = \{z \in \mathbb{Q} \ : \ z^2 \leq 2\}$ does not have a maximum (we will prove this formally later), but it does have many upper bounds (e.g. any real number ≥ 3). In fact, any real number $x > 0$ satisfying $x^2 \geq 2$ is an upper bound for A.

Definition 2.7 (Bounded sets)
We use the following terminology. A subset A of $\mathbb{R}$ is said to be *bounded above* if it has an upper bound, and *bounded below* if it has a lower bound. We say that A is *bounded*, if it is both bounded above and bounded below.

2.C Supremum and infimum of sets

Definition 2.8 (Supremum and infimum)
Suppose $A \subseteq \mathbb{R}$. Then a real number x is called the *least upper bound* of A iff

1. x is an upper bound for A,
2. every upper bound z for A satisfies $x \leq z$.

The least upper bound of A (if it exists) is also called the *supremum* of A and we denote it by $\sup(A)$.

A real number x is called the *greatest lower bound* of A iff

1. x is a lower bound for A,

2. every lower bound z for A satisfies $z \leq x$.

The greatest lower bound of A (if it exists) is also called the *infimum* of A and we denote it by $\inf(A)$.

How do we prove that a particular real number x is the supremum of a particular given set A of real numbers? We must prove

1. x is an upper bound for A, so
   $\forall y \in A \; : \; y \leq x$,
2. There is no upper bound for A which is smaller than x, i.e.
   $\forall z \in \mathbb{R}$ with $z < x$, $\exists y \in A \; : \; z < y$.

Example 2.9
$A = (0, 1)$. To show that $\sup(A) = 1$, we have to prove

1. $\forall y \in A \; : \; y \leq 1$

   This is obvious from the definition of A.

2. $\forall z \in \mathbb{R}$ with $z < 1$, $\exists y \in A = (0, 1) \; : \; z < y$.

   We are given $z \in \mathbb{R}$ with $z < 1$. If $z \in (0, 1)$, let $y = \frac{1+z}{2}$. If $z \leq 0$, let $y = \frac{1}{2}$. In both cases, $y \in A$ and $z < y$.

Remark 2.10
We have talked about "the" supremum of A (if A has a supremum), so we should check that A cannot have two different suprema. Given a subset $A \subseteq \mathbb{R}$, consider the set:

$$U(A) := \{x \in \mathbb{R} : x \text{ is an upper bound of } A\}.$$

Then we have

$$\sup(A) = \min(U(A)).$$

We know by Lemma 2.4 that the minimum is unique. Hence if A has a supremum, that supremum is unique. Similarly, the infimum (if it exists) is unique, since it may be expressed as

$$\inf(A) = \max\{x \in \mathbb{R} : x \text{ is a lower bound of } A\}.$$

Lemma 2.11 (Maximum versus supremum)
Consider a set $A \subseteq \mathbb{R}$.

(i) Suppose A has a maximum. Then A has a supremum, and $\sup(A) = \max(A)$.

(ii) Suppose A has a supremum. Then A has a maximum if and only if $\sup A \in A$. In this case, we have $\max(A) = \sup(A)$.

There are many sets A which have a supremum but do not have a maximum, because $\sup(A) \notin A$. For example, $A = (0, 1)$.

Proof. (i) Let $x = \max(A)$. We have to check that x is an upper bound for A and that there is no $z \in \mathbb{R}$ with $z < x$ which is also an upper bound for A.

By the definition of a maximum, we know that $\forall y \in A \ : y \leq x$, so x is an upper bound for A.

If $z \in \mathbb{R}$ with $z < x$, then there is an element of A which is bigger than z, namely x itself. Hence, z is not an upper bound for A.

(ii) Now let $x = \sup(A)$. To show that x is a maximum for A, we have to check that $x \in A$ and that x is an upper bound for A. But this is automatic: we are given $x \in A$, and by definition the supremum is an upper bound. ☐

Of course, the same result holds replacing "maximum" by "minimum" and "supremum" by "infimum".

2.D The properties of the real numbers

We think of the real numbers as decimal numbers with an infinite number of decimal places after the decimal point. More formally, the real numbers can be defined using "Dedekind cuts" or "Cauchy sequences", but we will not do this here. We shall just characterise them by a collection of properties and not worry about exactly how they are constructed. When dealing with the real numbers, the following properties are the only facts we will ever need to use.

Definition 2.12 (The real numbers)
The *real numbers* are a set $\mathbb{R}$ with the following properties.

1. $\mathbb{Q} \subseteq \mathbb{R}$, i.e. every rational number is a real number.
2. $\mathbb{R}$ is a field, i.e. there are operations $+, \cdot$ (with inverse operations $-, \div$, except for division by zero) satisfying the usual rules mentioned before, i.e. commutativity, associativity and distributivity, as well as existence of inverses (except for a multiplicative inverse of zero).
3. $\mathbb{R}$ is totally ordered, i.e. $\forall x, y \in \mathbb{R}$ exactly one of the following is true: $x > y$, $x = y$, $x < y$.
4. *The Completeness Axiom:* Every non-empty set of real numbers which is

bounded above has a least upper bound in $\mathbb{R}$.

Remark 2.13
$\mathbb{Q}$ satisfies (1)–(3), but not (4). $\mathbb{C}$ satisfies (1) and (2), but not (3). It is the Completeness Axiom which ensures that there are no "gaps" in $\mathbb{R}$ like those in $\mathbb{Q}$.

Theorem 2.14 (Archimedean property)
For every real number x, there exists a natural number $n > x$.

Remark 2.15
Although this seems "obvious" if we think of the real numbers as decimals, we cannot be sure that the Completeness Axiom has not allowed "∞" to be a real number. So we need to carefully prove this!

Proof. Suppose there exists a real number $x \in \mathbb{R}$ such that for all $n \in \mathbb{N} : x \geq n$. We aim for a contradiction. If such an x exists, it is an upper bound for $\mathbb{N}$. But because $\mathbb{N}$ is a non-empty subset of $\mathbb{R}$ (e.g. $1 \in \mathbb{N}$), the Completeness Axiom tells us that there exists a supremum, say $y \in \mathbb{R}$.

Then $y-1$ is *not* an upper bound for $\mathbb{N}$. So $\exists n \in \mathbb{N}$ with $n > y-1$. But as $n+1 \in \mathbb{N}$ and $n+1 > y$, y can *not* be an upper bound for $\mathbb{N}$, which contradicts how y was chosen (as the least upper bound for $\mathbb{N}$). ↯ □

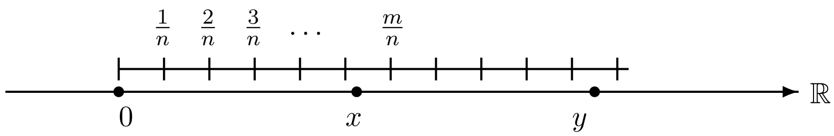
Corollary 2.16
For all $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

Proof. We have $\frac{1}{\varepsilon} \in \mathbb{R}$ (because $\mathbb{R}$ is a field). By the Archimedean property (Theorem 2.14), $\exists n \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < n$. But then by the usual rules of order, $\frac{1}{\varepsilon} < n \Leftrightarrow 1 < n\varepsilon \Leftrightarrow \frac{1}{n} < \varepsilon$. □

Next, we will see that $\mathbb{Q}$ lies "dense" in $\mathbb{R}$.

Theorem 2.17 (The rational numbers are dense in $\mathbb{R}$.)
Suppose $x, y \in \mathbb{R}$ with $x < y$. Then $\exists z \in \mathbb{Q}$ with $x < z < y$.

Proof. First, assume $0 \leq x < y$.



In the picture above, we first want to make the size $(\frac{1}{n})$ of the "grid" $\{\frac{k}{n} : k \in \mathbb{N}\}$ small enough, to make sure there is at least one element between x and y. We then take the first of these which is strictly larger than x.

We now turn this intuition into a mathematical proof. By Corollary 2.16 we can choose an $n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$. We now wish to find m.

By the Archimedean property (Theorem 2.14), there exists $s \in \mathbb{N}$ such that $s > nx$ and hence $x < \frac{s}{n}$. Consider the set:

$$A = \{s \in \mathbb{Z} : x < \tfrac{s}{n}\}.$$

We have just shown that A is nonempty. Since $x \geq 0$ and $n > 0$ we see that A is bounded below, e.g. by 0. From the Completeness Axiom formulated for infimum (Theorem 2.22), we conclude that A has an infimum, say $m = \inf A$. Clearly $m > 0$.

We claim that $m \in \mathbb{Z}$. Suppose for a contradiction that $m \notin \mathbb{Z}$. Then there exists an $\varepsilon > 0$ such that $[m - \varepsilon, m] \cap \mathbb{Z} = \emptyset$ which implies $[m - \varepsilon, m] \cap A = \emptyset$. It follows that $m - \varepsilon$ is a lower bound for A, contradicting the fact that m is the greatest lower bound.

So we have $m \in \mathbb{Z}$ with $m < s$ for all $s \in A$. It follows that $m \in A$: if not, then $m + 1$ would still be a lower bound for A, contradicting that m is the greatest lower bound. We conclude that $m \in A$ is a minimum for A (Lemma 2.11), and hence is the least integer such that $x < \frac{m}{n}$. It remains to show that $\frac{m}{n} < y$.

But we know that $\frac{m-1}{n} \leq x$ (since $m = \min A$). Moreover, we also know that $\frac{1}{n} < y - x$. Adding these two inequalities, we have

$$\frac{m}{n} = \frac{m-1}{n} + \frac{1}{n} < x + (y - x) = y.$$

So we are done in this case. But we still need to consider the cases $x < 0 < y$ and $x < y \leq 0$.

In the case $x < 0 < y$, we observe that there is an obvious rational number between x and y, namely 0. So we don't have to do anything!

In the case $x < y \leq 0$, we observe that $0 \leq -y < -x$. So from our first case, we know that there exists $\frac{m}{n}$ with $-y < \frac{m}{n} < -x$ and hence $x < -\frac{m}{n} < y$. □

Remark 2.18

In fact, we can always choose $z \neq 0$ by modifying the second case: If $x < 0 < y$ we set $x' = 0$ and search for $z \in \mathbb{Q}$ with $x' < z < y$. Such a z can be found according to the first case and it will then obviously also satisfy $x < z < y$ as well as $z \neq 0$. We will make use of this remark in Corollary 2.21 below.

Next, we show, using only our axioms for $\mathbb{R}$, that there is a real number $\sqrt{2}$.

Theorem 2.19
There exists $x \in \mathbb{R}$ such that $x^2 = 2$.

Proof. Let $A = \{z \in \mathbb{R} \ : \ z^2 \leq 2\}$. A is non-empty (for example $1 \in A$) and A is bounded above (for example 3 is an upper bound for A, since for any $z \in \mathbb{R}$ with $z > 3$, we have $z^2 > 9$, so $z \notin A$).

So by the Completeness Axiom, A has a least upper bound $x = \sup(A)$ in $\mathbb{R}$. Since $\mathbb{R}$ is totally ordered, there are three possibilities:

$$x^2 > 2, \quad x^2 = 2, \quad x^2 < 2.$$

If we can rule out the first and last one, then we can conclude that $x^2 = 2$.

<u>If $x^2 < 2$</u>
Our idea is to show that there is some $n \in \mathbb{N}$ such that $(x + \frac{1}{n})^2 < 2$, which will give $x + \frac{1}{n} \in A$ which contradicts x being an upper bound for A.

> (Rough working.) How do we find such an $n \in \mathbb{N}$? Write $\varepsilon := 2 - x^2 > 0$. Then
> $$\left(x + \frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} \leq (2 - \varepsilon) + \frac{6}{n} + \frac{1}{n},$$
> where we used that $x \leq 3$ (because 3 is an upper bound for A) and $\frac{1}{n^2} \leq \frac{1}{n}$ for all natural numbers. We want the last expression to be smaller than 2, so we need
> $$\frac{6}{n} + \frac{1}{n} < \varepsilon,$$
> or equivalently $\frac{1}{n} < \frac{\varepsilon}{7}$. But such an n exists by Corollary 2.16.

We are now ready to continue the proof. If $x^2 < 2$, let $\varepsilon = 2 - x^2 > 0$. By Corollary 2.16, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\varepsilon}{7}$ and hence $\frac{7}{n} < \varepsilon$. Now

$$\left(x + \frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} \leq (2 - \varepsilon) + \frac{6}{n} + \frac{1}{n} \leq 2 - \varepsilon + \frac{7}{n} < 2,$$

so $x + \frac{1}{n} \in A$. But this contradicts the fact that x is an upper bound for A. ↯

<u>If $x^2 > 2$</u>
Our idea is to show that there is some $n \in \mathbb{N}$ such that $(x - \frac{1}{n})^2 > 2$. It will follow that $x - \frac{1}{n}$ is an upper bound for A, contradicting the fact that x is the *least* upper bound for A.

(Rough working.) How do we find such an $n \in \mathbb{N}$? We set $\varepsilon = x^2 - 2 > 0$ and compute

$$\left(x - \frac{1}{n}\right)^2 = x^2 - \frac{2x}{n} + \frac{1}{n^2} \geq 2 + \varepsilon - \frac{6}{n},$$

using as above that $x \leq 3$ (and this time using $\frac{1}{n^2} \geq 0$). We want the last expression to be bigger than 2, so we need $\frac{6}{n} < \varepsilon$, or equivalently $\frac{1}{n} < \frac{\varepsilon}{6}$, which again is doable by Corollary 2.16.

Now we can complete the proof. If $x^2 > 2$, let $\varepsilon = x^2 - 2 > 0$. By Corollary 2.16, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\varepsilon}{6}$, and hence $\frac{6}{n} < \varepsilon$. Now

$$\left(x - \frac{1}{n}\right)^2 = x^2 - \frac{2x}{n} + \frac{1}{n^2} \geq 2 + \varepsilon - \frac{6}{n} > 2,$$

since $x \leq 3$. This shows that $x - \frac{1}{n}$ is an upper bound for A. (Recall that $x \geq 1$. Suppose $z > x - \frac{1}{n}$. Since $z > x - \frac{1}{n} \geq 0$ we have $z^2 > (x - \frac{1}{n})^2 > 2$, and hence $z \notin A$.) But this contradicts the fact that x was chosen as the *least* upper bound for A. ↯ □

In general (i.e., except for training purposes) we don't have to reveal our rough working!

Remark 2.20
Earlier in your life, you will have seen the proof that $\sqrt{2}$ is irrational (it's not difficult; go ahead and look it up if you've forgotten how it goes). Strictly speaking, that proof only shows that $\sqrt{2}$ is irrational *if it exists!* We have now proven that it exists, using the axioms for the real numbers (Definition 2.12).

The existence of irrational numbers was a fraught topic for a very long time. According to legend, the Pythagorean philosopher Hippasus was murdered due to his insistence that they exist.

We have proved $\sqrt{2} \in \mathbb{R}$, but we know that $\sqrt{2} \notin \mathbb{Q}$, i.e. $\sqrt{2}$ is *irrational*. But what about $\frac{m}{n}\sqrt{2}$ (with $\frac{m}{n} \in \mathbb{Q}$)? Obviously, $\frac{m}{n}\sqrt{2}$ is rational if $m = 0$ (since then $\frac{m}{n}\sqrt{2} = 0$). Suppose that $\frac{m}{n}\sqrt{2}$ is rational for some $m, n \in \mathbb{Z}$ with $m \neq 0$. Then $\frac{m}{n}\sqrt{2} = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ with $q \neq 0$. So $\sqrt{2} = \frac{pn}{qm}$, contradicting the fact that $\sqrt{2}$ is irrational. Hence $\frac{m}{n}\sqrt{2}$ is irrational, provided $m \neq 0$.

We can use this to prove that not only the rational numbers are dense in $\mathbb{R}$ (see Theorem 2.17), but also the irrational numbers.

Corollary 2.21 (The irrational numbers are dense in $\mathbb{R}$)

For any $x, y \in \mathbb{R}$ with $x < y$ there exists an irrational number z with $x < z < y$.

Proof. It will be enough to show that there exists $r \in \mathbb{Q}$, $r \neq 0$, with $x < r\sqrt{2} < y$. This means, it is enough to show that $\exists r \in \mathbb{Q}$, $r \neq 0$, with $\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$.

But in the case that $0 \leq x < y$ or $x < y \leq 0$, Theorem 2.17 shows that there is such an r. If $x < 0 < y$, such an r can be found according to Remark 2.18. □

On the last few pages, we have been concerned with *suprema* of sets in $\mathbb{R}$. We finish this chapter with a statement about *infima* in $\mathbb{R}$.

Theorem 2.22 (Completeness Axiom formulated for infima)
Suppose $A \subseteq \mathbb{R}$ is non-empty and bounded below, then A has a greatest lower bound.

Proof. We have $-A = \{-x : \; x \in A\}$ is non-empty (since A is non-empty) and $-A$ is bounded above (since A is bounded below). Hence, by the Completeness Axiom, there exists a supremum, $y = \sup(-A)$. We will show that $-y$ is the greatest lower bound of A, i.e., $-y = \inf(A)$.

1. $-y$ is a lower bound for A. Suppose $x \in A$. By definition of $-A$ we know that $-x \in -A$. Since y is an upper bound for $-A$ we know that $-x \leq y$ or, equivalently, $x \geq -y$. It follows that $-y$ is a lower bound for A.

2. $-y$ is the *greatest* lower bound for A. Suppose that $z > -y$ is a larger one. Since $-z < y$ and y is the least upper bound on $-A$, we know that $-z$ cannot be an upper bound on $-A$. Choose $x \in -A$ with $x > -z$. Now observe that $-x \in A$ and $-x < z$, contradicting the assumption that z is a lower bound for A.

(1) and (2) together show that $-y = \inf(A)$. □

Remark 2.23
We have used the Completeness Axiom (existence of the supremum for non-empty sets bounded above) to prove the existence of the infimum for non-empty sets bounded below. But using a very similar proof as above, one can easily deduce the Completeness Axiom from the existence of infima. In other words, Theorem 2.22 is *equivalent* to the Completeness Axiom.

3 Sequences

A sequence of real numbers is any infinite list of real numbers, for example

$$1, 2, 3, 4, \ldots \qquad \text{or} \qquad 1, 4, 9, 16, \ldots \qquad \text{or} \qquad 1, 1, 1, 1, \ldots$$

(there does *not* have to be a rule for how to find the next element). We write $(x_n)_{n=1}^{\infty}$ to denote the sequence $x_1, x_2, x_3, \ldots$. In this notation, the above three sequences are $(n)_{n=1}^{\infty}$, $(n^2)_{n=1}^{\infty}$, and $(1)_{n=1}^{\infty}$. We sometimes also simply write (x_n). Formally, a sequence is a *function* from $\mathbb{N}$ to $\mathbb{R}$, defined by $n \mapsto x_n$.

In this chapter, we are concerned with *convergence* of sequences. We start with the easiest situation, asking whether or not a sequence converges *to zero*.

3.A Convergence to zero

Definition 3.1 (Convergence to zero)
A sequence $(x_n)_{n=1}^{\infty}$ is said to *converge to* 0, denoted "$x_n \to 0$ as $n \to \infty$" or "$\lim_{n\to\infty} x_n = 0$" if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \in \mathbb{N}, n > N \; : \; |x_n| < \varepsilon. \tag{3.1}$$

Remark 3.2
We will *always* assume that $\varepsilon \in \mathbb{R}$ and $N, n \in \mathbb{N}$ in this course, even if we do not explicitly mention it. We will say that $(x_n)_{n=1}^{\infty}$ converges to $\ell \in \mathbb{R}$ if the sequence $(y_n)_{n=1}^{\infty}$, defined by $y_n = x_n - \ell$, converges to zero. So statements about convergence to other limits than zero can always be converted into statements about convergence to 0.

Demon Game The Demon Game corresponding to the definition that (x_n) converges to zero (i.e., to the statement (3.1)) is the following. (Of course, the sequence $(x_n)_{n=1}^{\infty}$ is given, before the game starts!)

- First the Demon picks $\varepsilon > 0$ ($\varepsilon \in \mathbb{R}$).
- Then we pick $N \in \mathbb{N}$.
- Then the Demon picks $n > N$ ($n \in \mathbb{N}$).
- We win if $|x_n| < \varepsilon$.

To *prove* that a given sequence (x_n) converges to 0, we have to find a winning strategy for this game. That is, we have to show that whatever (small) number

$\varepsilon > 0$ we are given, the $|x_n|$ are all smaller than ε for n sufficiently large. In other words, given $\varepsilon > 0$, we need a strategy to produce an N (which will depend on ε and the given sequence (x_n)) such that $|x_n| < \varepsilon$ for all $n > N$.

Examples 3.3 (i) $x_n = \frac{1}{n}$, so $x_1 = 1$, $x_2 = \frac{1}{2}$, $x_3 = \frac{1}{3}$, ... Let us first look at some trial games:

	Trial Game 1	Trial Game 2	Trial Game 3
Demon picks $\varepsilon =$	0.1	10^{-6}	10^{-6}
We pick $N =$	10	100	10^6
Demon picks $n =$	11	101	$10^6 + 1$
Who wins?	Us (as $\frac{1}{11} < 0.1$)	Demon (as $\frac{1}{101} \not< 10^{-6}$)	Us (as $\frac{1}{10^6+1} < 10^{-6}$)

The winning strategy is clear: if the Demon picks a particular value for ε, we choose N to be $\lceil \frac{1}{\varepsilon} \rceil$ (the least integer $\geq \frac{1}{\varepsilon}$). (Because of the Archimedean property (Theorem 2.14), we know that this N actually exists.) We can now translate this winning strategy into a formal proof.

Proof (that $x_n = \frac{1}{n} \to 0$ as $n \to \infty$). Given $\varepsilon > 0$, let $N = \lceil \frac{1}{\varepsilon} \rceil$, so $\frac{1}{N} \leq \varepsilon$ (with equality only if $\frac{1}{\varepsilon}$ is an integer). Now for all $n > N$, we have

$$|x_n| = \frac{1}{n} < \frac{1}{N} \leq \varepsilon.$$

So $|x_n| < \varepsilon$. □

(ii) $x_n = \frac{1}{n^2}$ converges to zero.

Strategy 1: Following the method of example 1, choose $N = \lceil \frac{1}{\sqrt{\varepsilon}} \rceil$, so $\frac{1}{N^2} \leq \varepsilon$. This gives the following formal proof:

Proof. Given $\varepsilon > 0$, let $N = \lceil \frac{1}{\sqrt{\varepsilon}} \rceil$, so $\frac{1}{N^2} \leq \varepsilon$. Then for all $n > N$, we have

$$|x_n| = \frac{1}{n^2} < \frac{1}{N^2} \leq \varepsilon,$$

so $|x_n| < \varepsilon$. □

Strategy 2: We can use the fact that $\frac{1}{n^2} \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. So the choice of N in Example (i) will still work! This gives the formal proof:

Proof. Given $\varepsilon > 0$, let $N = \lceil \frac{1}{\varepsilon} \rceil$, so $\frac{1}{N} \leq \varepsilon$. Now for all $n > N$, we have

$$|x_n| = \frac{1}{n^2} \leq \frac{1}{n} < \frac{1}{N} \leq \varepsilon.$$

So $|x_n| < \varepsilon$. □

(iii) $x_n = \frac{1}{n+n^2}$ converges to zero.

We note that $\frac{1}{n+n^2} < \frac{1}{n}$ (because $n^2 > 0$), so we can again use the strategy from Example (i). The formal proof is the following.

Proof. Given $\varepsilon > 0$, let $N = \lceil \frac{1}{\varepsilon} \rceil$, so $\frac{1}{N} \leq \varepsilon$. Now for all $n > N$, we have

$$|x_n| = \frac{1}{n+n^2} \leq \frac{1}{n} < \frac{1}{N} \leq \varepsilon.$$

So $|x_n| < \varepsilon$. □

(iv) $x_n = 1$ for all $n \in \mathbb{N}$. This is the sequence $1, 1, 1, 1, \ldots$ and it does *not* converge to 0. To prove that this sequence does not converge to 0, we must prove the negation of (3.1), namely

$$\exists \varepsilon > 0 \; \forall N \in \mathbb{N} \; \exists n > N \; : |x_n| \geq \varepsilon. \tag{3.2}$$

A formal proof that $x_n = 1$ does not converge to zero is the following.

Proof. Let $\varepsilon = \frac{1}{2}$. Then given any N, choose $n = N + 1$. Now, we have

$$|x_n| = 1 \geq \frac{1}{2} = \varepsilon.$$

So (3.2) is true and hence the sequence does not converge to zero. □

(v) Let (x_n) be the sequence defined by

$$x_n = \begin{cases} 1 & \text{if } n \text{ is prime,} \\ 0 & \text{if } n \text{ is not prime.} \end{cases}$$

Also this sequence does not converge to zero. A formal proof is the following

Proof. Let $\varepsilon = \frac{1}{2}$. Then given any N, choose n to be any prime number larger than N. Then $|x_n| = 1 \geq \frac{1}{2} = \varepsilon$. So (x_n) does not converge to zero. □

(vi) $x_n = \frac{2\sin n}{n}$ converges to zero.

We don't know the exact value of $\sin n$ as it depends on the value of n mod 2π, but we do know that $|\sin n| \leq 1$ for all $n \in \mathbb{N}$.

Proof. Given $\varepsilon > 0$, let $N = \lceil \frac{2}{\varepsilon} \rceil$, so $\frac{2}{N} \leq \varepsilon$. Now given any $n > N$, we have

$$|x_n| = \frac{|2\sin n|}{n} \leq \frac{2}{n} < \frac{2}{N} \leq \varepsilon.$$

so $|x_n| < \varepsilon$. □

(vii) Let (x_n) be the sequence given by

$$x_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is prime,} \\ 0 & \text{if } n \text{ is not prime.} \end{cases}$$

(This is the following sequence: $0, \frac{1}{2}, \frac{1}{3}, 0, \frac{1}{5}, 0, \frac{1}{7}, 0, 0, 0, \frac{1}{11}, 0, \ldots$)

Here, $|x_n| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$, so the *same* proof as in Example (i) proves that (x_n) converges to zero. More generally, this proof will work whenever we have a sequence $(x_n)_{n=1}^{\infty}$ with $|x_n| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$.

Motivated by the above examples, we are now going to prove how the convergence of some sequence implies the convergence of some other sequences.

Lemma 3.4 (Dominated convergence)
If $(x_n)_{n=1}^{\infty}$ is a sequence which converges to zero and $(y_n)_{n=1}^{\infty}$ is a sequence with $|y_n| \leq |x_n|$ for all $n \in \mathbb{N}$, then $(y_n)_{n=1}^{\infty}$ converges to zero.

Proof. Given any $\varepsilon > 0$, we must show that there exists N such that for all $n > N$ we have $|y_n| < \varepsilon$. But we know $\exists N \in \mathbb{N}$ such that for all $n > N$ $|x_n| < \varepsilon$ (since (x_n) converges to zero). So taking *this* value of N, we deduce that for all $n > N$ we have $|y_n| \leq |x_n| < \varepsilon$, so $|y_n| < \varepsilon$. This proves the lemma. □

Corollary 3.5
If $(x_n)_{n=1}^{\infty}$ is a sequence which does *not* converge to 0 and $(y_n)_{n=1}^{\infty}$ is a sequence with $|y_n| \geq |x_n|$ for all $n \in \mathbb{N}$, then $(y_n)_{n=1}^{\infty}$ does *not* converge to zero.

Proof. This is the contrapositive of Lemma 3.4. We give a proof by contradiction. If $(y_n)_{n=1}^{\infty}$ does converge to zero, then by Lemma 3.4 so does $(x_n)_{n=1}^{\infty}$, since $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$. ↯ □

Lemma 3.4 and Corollary 3.5 give us a lot of examples of sequences which converge to zero and sequences which do not converge to zero, but it will be useful to have some additional rules.

For example, if we have a sequence $(x_n)_{n=1}^{\infty}$ which converges to zero and a constant $c \in \mathbb{R}$, does the sequence $(y_n)_{n=1}^{\infty}$ given by $y_n = cx_n$ converge to zero? If $|c| \leq 1$ it does by Lemma 3.4, but what if $|c|$ is larger than 1?

Lemma 3.6
If $(x_n)_{n=1}^{\infty}$ is a sequence which converges to zero and $c \in \mathbb{R}$ is any constant, then the sequence $(y_n)_{n=1}^{\infty}$ defined by $y_n = cx_n$ for all $n \in \mathbb{N}$ also converges to zero.

How do we give a formal proof of this? Let us first consider a particular example, say $c = 100$. Suppose that we are given some $\varepsilon > 0$. We have to find N

such that $|y_n| < \varepsilon$ for all $n > N$,
i.e., such that $|100x_n| < \varepsilon$ for all $n > N$,
i.e., such that $|x_n| < \frac{\varepsilon}{100}$ for all $n > N$.

But such an N exists since we are given that $(x_n)_{n=1}^{\infty}$ converges to zero and we can just take the value of N to be that given by the "(x_n) Demon Game" for $\tilde{\varepsilon} = \frac{\varepsilon}{100}$. Formally, the proof is as follows:

Proof. If $c = 0$, there is nothing to do, since $(y_n)_{n=1}^{\infty}$ is then the constant sequence $0, 0, 0, \ldots$ which obviously converges to zero.

If $c \neq 0$, suppose we are given some $\varepsilon > 0$, set $\tilde{\varepsilon} = \frac{\varepsilon}{|c|} > 0$. Because $(x_n)_{n=1}^{\infty}$ converges to zero, we know that $\exists N \in \mathbb{N}$ such that for all $n > N$ we have $|x_n| < \tilde{\varepsilon} = \frac{\varepsilon}{|c|}$. Hence, choosing the *same* N, we have for all $n > N$ that

$$|y_n| = |cx_n| = |c| \cdot |x_n| < |c| \cdot \tilde{\varepsilon} = \varepsilon,$$

so $(y_n)_{n=1}^{\infty}$ converges to zero. □

Corollary 3.7
Let $(x_n)_{n=1}^{\infty}$ be a sequence which converges to zero and $(y_n)_{n=1}^{\infty}$ a sequence satisfying $|y_n| \leq c|x_n|$ for some constant $c \in \mathbb{R}$ with $c > 0$ and for all $n \in \mathbb{N}$. Then $(y_n)_{n=1}^{\infty}$ converges to zero.

Proof. By Lemma 3.6, $(cx_n)_{n=1}^{\infty}$ converges to zero. So by Lemma 3.4, $(y_n)_{n=1}^{\infty}$ converges to zero, since $|y_n| \leq c|x_n|$. □

Corollary 3.8
If $(x_n)_{n=1}^{\infty}$ is a sequence which does *not* converge to zero and $(y_n)_{n=1}^{\infty}$ is a sequence with $|y_n| \geq c|x_n|$ for some constant $c > 0$ and all $n \in \mathbb{N}$, then $(y_n)_{n=1}^{\infty}$ does *not* converge to zero.

Proof. By contradiction. Suppose $(y_n)_{n=1}^{\infty}$ converges to zero. Then by Lemma 3.6, the sequence $(\frac{y_n}{c})_{n=1}^{\infty}$ converges to zero. But $|x_n| \leq \frac{|y_n|}{c}$ for all $n \in \mathbb{N}$, so $(x_n)_{n=1}^{\infty}$ converges to zero by Lemma 3.4. $\lightning$ □

If $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are two sequences which both converge to zero, what can we say about the sequence $(z_n)_{n=1}^{\infty}$ where $z_n = x_n + y_n$ for each $n \in \mathbb{N}$?

We already know the answer in some examples, e.g. if $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n}$, so $z_n = \frac{2}{n}$. Because $(x_n)_{n=1}^{\infty}$ converges to zero and $z_n = 2x_n$, we know from Lemma 3.6 that $(z_n)_{n=1}^{\infty}$ also converges to zero. Moreover, if $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n^2}$, we know that $(z_n)_{n=1}^{\infty}$ converges to zero, since $\frac{1}{n} + \frac{1}{n^2} \leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$. What about the general case of adding sequences?

Theorem 3.9 (Summing sequences converging to zero)
If $(x_n)_{n=1}^{\infty}$ converges to zero and $(y_n)_{n=1}^{\infty}$ converges to zero then so does $(z_n)_{n=1}^{\infty}$ where $z_n = x_n + y_n$ for all $n \in \mathbb{N}$.

In order to win the Demon Game, given $\varepsilon > 0$, we must find N such that $|x_n + y_n| < \varepsilon$ for all $n > N$. We will need the following basic fact, which is used *all the time* in this module.

Theorem 3.10 (Triangle inequality)
$\forall x, y \in \mathbb{R} : |x + y| \leq |x| + |y|$.

Proof. Exercise. Use that $|x| = \max\{x, -x\}$. □

Because of the triangle inequality, we have $|x_n + y_n| \leq |x_n| + |y_n|$. To show $|x_n + y_n| < \varepsilon$ it is therefore enough to ensure that $|x_n| + |y_n| < \varepsilon$ (for $n > N$), but we can do this by ensuring that $|x_n| < \frac{\varepsilon}{2}$ and $|y_n| < \frac{\varepsilon}{2}$.

Proof of Theorem 3.9. Given $\varepsilon > 0$, we must show that there exists $N \in \mathbb{N}$ with the property that $\forall n > N$ we have $|x_n + y_n| < \varepsilon$. But since $(x_n)_{n=1}^{\infty}$ converges to zero, we know that $\exists N_x \in \mathbb{N}$ such that $|x_n| < \tilde{\varepsilon} = \frac{\varepsilon}{2}$ for all $n > N_x$. Moreover, since

$(y_n)_{n=1}^{\infty}$ converges to zero, we know that $\exists N_y \in \mathbb{N}$ such that $|y_n| < \tilde{\varepsilon} = \frac{\varepsilon}{2}$ for all $n > N_y$.

Set $N := \max\{N_x, N_y\}$. Then for all $n > N$, we have

$|x_n| < \frac{\varepsilon}{2}$ (since $n > N \geq N_x$)
and $|y_n| < \frac{\varepsilon}{2}$ (since $n > N \geq N_y$).

So using the triangle inequality, $|x_n + y_n| \leq |x_n| + |y_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. □

Using the results from 3.4–3.9, we can often prove that a given sequence converges to zero by breaking it up into simpler sequences that we already know converge to zero (similarly with sequences that do not converge to zero).

Examples 3.11 (i) Consider the sequence $x_n = \frac{3}{n^4} + \frac{4}{n+1}$.

We know that $(\frac{1}{n})$ converges to zero, so by Lemma 3.6, $(\frac{3}{n})$ and $(\frac{4}{n})$ converge to zero. But

$$|\tfrac{3}{n^4}| = \tfrac{3}{n^4} \leq \tfrac{3}{n} = |\tfrac{3}{n}|,$$

so $(\frac{3}{n^4})$ converges to zero by Lemma 3.4. Moreover,

$$|\tfrac{4}{n+1}| = \tfrac{4}{n+1} < \tfrac{4}{n} = |\tfrac{4}{n}|,$$

so $(\frac{4}{n+1})$ converges to zero by Lemma 3.4. We then deduce from Theorem 3.9 that $(\frac{3}{n^4} + \frac{4}{n+1})_{n=1}^{\infty}$ converges to zero.

We could have also proved this directly:

Proof. Given $\varepsilon > 0$, let $N = \lceil \frac{7}{\varepsilon} \rceil$. Then for given $n > N$, we have

$$|x_n| = \frac{3}{n^4} + \frac{4}{n+1} \leq \frac{3}{n} + \frac{4}{n} = \frac{7}{n} < \frac{7}{N} \leq \varepsilon.$$ □

(ii) Consider $x_n = \frac{100}{\sqrt{n}} - \frac{3\cos(\pi n)}{n^2+5n+7}$.

We first show that $\left(\frac{1}{\sqrt{n}}\right)$ converges to zero.

Indeed, given $\varepsilon > 0$, choose $N = \lceil \frac{1}{\varepsilon^2} \rceil$, so that $\frac{1}{N} \leq \varepsilon^2$ or equivalently $\frac{1}{\sqrt{N}} \leq \varepsilon$. Then for all $n > N$, we have

$$\left|\tfrac{1}{\sqrt{n}}\right| = \tfrac{1}{\sqrt{n}} < \tfrac{1}{\sqrt{N}} \leq \varepsilon.$$

Now that we know that $\left(\frac{1}{\sqrt{n}}\right)$ converges to zero, we also know that $\left(\frac{100}{\sqrt{n}}\right)$ converges to zero by Lemma 3.6. On the other hand, using

$$\left|-\tfrac{3\cos(\pi n)}{n^2+5n+7}\right| \leq 3\left|\tfrac{1}{n^2+5n+7}\right| \leq 3\left|\tfrac{1}{5n}\right| = \tfrac{3}{5}\left|\tfrac{1}{n}\right|,$$

we know that $\left(-\frac{3\cos(\pi n)}{n^2+5n+7}\right)$ converges to zero (by Corollary 3.7).

Hence, by Theorem 3.9, $(x_n)_{n=1}^{\infty}$ converges to zero.

Again, we could have also proved this directly:

Proof. Given $\varepsilon > 0$, let $N = \lceil (\frac{103}{\varepsilon})^2 \rceil$. Then for given $n > N$, we have

$$\begin{aligned} |x_n| &= \left|\frac{100}{\sqrt{n}} - \frac{3\cos(\pi n)}{n^2+5n+7}\right| \leq \left|\frac{100}{\sqrt{n}}\right| + \left|\frac{3\cos(\pi n)}{n^2+5n+7}\right| \leq \frac{100}{\sqrt{n}} + \frac{3}{n^2+5n+7} \\ &\leq \frac{100}{\sqrt{n}} + \frac{3}{n^2} \leq \frac{100}{\sqrt{n}} + \frac{3}{\sqrt{n}} = \frac{103}{\sqrt{n}} < \frac{103}{\sqrt{N}} \leq \varepsilon. \end{aligned}$$

□

(iii) Consider $z_n = \frac{n^2+1}{5n} - \frac{1}{n^2}$.

We write $z_n = x_n - y_n$ with $x_n = \frac{n^2+1}{5n}$ and $y_n = \frac{1}{n^2}$. Then, we see that $|x_n| > \frac{n^2}{5n} = \frac{1}{5}n$ for all $n \in \mathbb{N}$. So by Corollary 3.8, we see that (x_n) does *not* converge to zero.

On the other hand, we know (as seen before) that (y_n) converges to zero. Together, this implies that (z_n) does *not* converge to zero. [We can prove this by contradiction: Assume (z_n) converges to zero. Then, because (y_n) converges to zero, Theorem 3.9 shows that $x_n = z_n + y_n$ converges to zero, contradicting what we have just proved above. ↯]

We can also prove *directly* that (z_n) does not converge to zero, by noting that for $n \geq 5$, we have

$$|z_n| = \tfrac{n^3+n}{5n^2} - \tfrac{5}{5n^2} = \tfrac{n^3+n-5}{5n^2} \geq \tfrac{n^3}{5n^2} = \tfrac{1}{5}n. \tag{3.3}$$

We want to show that $\exists \varepsilon > 0\ \forall N \in \mathbb{N}\ \exists n > N\ : |z_n| \geq \varepsilon$. First we choose $\varepsilon = 1$. Then given any N, pick $n > \max\{N, 5\}$ (We want both $n > N$ – as this is the condition for how we have to chose n in the Demon Game – and also $n > 5$ – as this allows us to use the above estimate (3.3)). We then get $|z_n| \geq \frac{1}{5}n > 1 = \varepsilon$, proving that (z_n) does not converge to zero.

Example 3.12

Let $(x_n)_{n=1}^{\infty}$ be the sequence given by $x_n = \frac{1}{2^n}$, i.e. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$. We would like to compare this with the sequence $y_n = \frac{1}{n}$ (which we know converges to zero).

We claim that $\frac{1}{2^n} \leq \frac{1}{n}$, $\forall n \in \mathbb{N}$.

Proof. We prove this by induction. We know that the statement is true for $n = 1$ (where it says that $\frac{1}{2} \leq 1$). Now assume that the claim holds for *some* $n \in \mathbb{N}$. Then

$$\frac{1}{2^{n+1}} = \frac{1}{2}\frac{1}{2^n} \overset{(\star)}{\leq} \frac{1}{2}\frac{1}{n} = \frac{1}{n+n} \leq \frac{1}{n+1}.$$

So the fact that the claim holds for n [we used this in $(\star)$] implies that it also holds for $n+1$. Hence it holds for all $n \in \mathbb{N}$. □

The claim, together with Lemma 3.4, then implies that (x_n) converges to zero.

We want to generalise this and show that the sequence given by $x_n = c^n$ for some $c \in \mathbb{R}$ converges to zero if $|c| < 1$ (and does not converge to zero if $|c| \geq 1$). To prove this, it is useful to prove the following lemma first.

Lemma 3.13 (Bernoulli's inequality)
If $\alpha \in \mathbb{R}$, $\alpha \geq -1$, then $\forall n \in \mathbb{N} \; : \; (1+\alpha)^n \geq 1 + n\alpha$.

Proof. We prove this again by induction. Obviously the statement is true for $n = 1$ (where it just says $1+\alpha \geq 1+\alpha$). Assume now that it is true for *some* $n \in \mathbb{N}$. Then

$$(1+\alpha)^{n+1} = (1+\alpha)(1+\alpha)^n \overset{(\star)}{\geq} (1+\alpha)(1+n\alpha) = 1+\alpha+n\alpha+n\alpha^2 \geq 1+(n+1)\alpha,$$

where in $(\star)$ we used the induction hypothesis as well as $\alpha \geq -1$. Hence the claim being true for n implies that it is true for $n+1$ as well. Thus it holds for all $n \in \mathbb{N}$. □

Theorem 3.14 ("Geometric sequences")
If $c \in \mathbb{R}$ and the sequence $(x_n)_{n=1}^{\infty}$ is defined by $x_n = c^n$, then

(i) (x_n) converges to zero if $|c| < 1$.

(ii) (x_n) does not converge to zero if $|c| \geq 1$.

Proof. If $c = 0$, then $x_n = 0$ for all n, and (x_n) converges to zero. So we may assume $c \neq 0$.

(i) If $|c| < 1$, then $|c| = \frac{1}{1+\alpha}$ for some $\alpha > 0$. So by Lemma 3.13, we have

$$|c^n| = |c|^n = \left(\frac{1}{1+\alpha}\right)^n \leq \frac{1}{1+n\alpha} < \frac{1}{n\alpha}.$$

But $(\frac{1}{n\alpha})$ converges to zero (by Lemma 3.6, since $\frac{1}{n}$ converges to zero). Hence the sequence (c^n) converges to zero by Lemma 3.4.

(ii) If $|c| \geq 1$ then $|c| = 1 + \alpha$ for some $\alpha \geq 0$. Hence, by Lemma 3.13

$$|c^n| = |c|^n \geq 1 + n\alpha \geq 1,$$

for all $n \in \mathbb{N}$. But the constant sequence $1, 1, 1, \ldots$ does not converge to zero and hence (c^n) does not converge to zero by Corollary 3.5. □

What about products and quotients of sequences? If $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ both converge to zero, what can we say about the sequences $(x_n y_n)_{n=1}^{\infty}$ and $\left(\frac{x_n}{y_n}\right)_{n=1}^{\infty}$?

We cannot say much about the quotient $\left(\frac{x_n}{y_n}\right)_{n=1}^{\infty}$. It will depend on whether the nominator or denominator go to zero *faster*. For example, let $(x_n)_{n=1}^{\infty}$ be given by $x_n = \frac{1}{n}$ and $(y_n)_{n=1}^{\infty}$ defined by $y_n = \frac{1}{n^2}$. We know that (x_n) and (y_n) both converge to zero. But we have:

- $\frac{x_n}{y_n} = n$, so $\left(\frac{x_n}{y_n}\right)$ does *not* converge to zero.
- $\frac{y_n}{x_n} = \frac{1}{n}$, so $\left(\frac{y_n}{x_n}\right)$ *does* converge to zero.

In contrast to this fact, we can *always* say something about the product $(x_n y_n)_{n=1}^{\infty}$.

Theorem 3.15 (Product of sequences converging to zero)
If $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ both converge to zero then $(x_n y_n)_{n=1}^{\infty}$ converges to zero.

Proof. We follow a similar strategy to the proof of Theorem 3.9, but instead of splitting ε into $\frac{\varepsilon}{2} + \frac{\varepsilon}{2}$, we split it into $\sqrt{\varepsilon}\sqrt{\varepsilon}$.

We must prove that

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \in \mathbb{N}, n > N \; : \; |x_n y_n| < \varepsilon.$$

To prove this, suppose we are given $\varepsilon > 0$. Let $\tilde{\varepsilon} = \sqrt{\varepsilon}$. Then since (x_n) converges to zero, we have

$$\exists N_x \in \mathbb{N} \; \forall n \in \mathbb{N}, n > N_x \; : \; |x_n| < \tilde{\varepsilon} = \sqrt{\varepsilon}$$

Similarly since (y_n) converges to zero, we have

$$\exists N_y \in \mathbb{N} \; \forall n \in \mathbb{N}, n > N_y \; : \; |y_n| < \tilde{\varepsilon} = \sqrt{\varepsilon}$$

Let $N = \max\{N_x, N_y\}$. Then for all $n > N$, we have

$$|x_n y_n| = |x_n||y_n| < \sqrt{\varepsilon}\sqrt{\varepsilon} = \varepsilon.$$

Hence $(x_n y_n)_{n=1}^{\infty}$ converges to zero. □

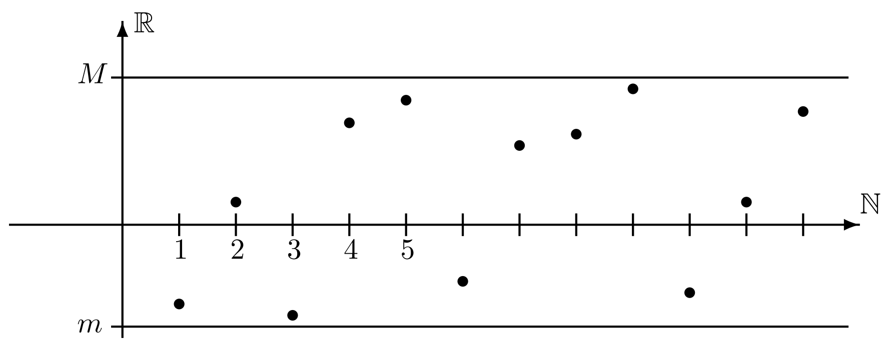
Example 3.16
Consider $x_n = \frac{2^n}{3^n(n+1)^2}$. By Lemma 3.4, $\left(\frac{1}{(n+1)^2}\right)$ converges to zero, as $\frac{1}{(n+1)^2} < \frac{1}{n}$. By Theorem 3.14, $\frac{2^n}{3^n} = (\frac{2}{3})^n$ converges to zero. Thus, by Theorem 3.15, (x_n) converges to zero.

Remark 3.17
Of course, we could have obtained this also by estimating $x_n < \frac{1}{n}$ and using only earlier results, but sometimes it is difficult to obtain such estimates.

The proof of Theorem 3.15 suggests that we might be able to prove more general results by splitting ε up in other ways, e.g. $\varepsilon = \frac{\varepsilon}{c} \cdot c$ for some real constant $c > 0$. This is our next goal.

Definition 3.18 (Bounded sequences)
We say that a sequence $(x_n)_{n=1}^{\infty}$ (of real numbers) is *bounded above* if $\exists M \in \mathbb{R}$ such that $x_n \leq M$ for all $n \in \mathbb{N}$. The sequence is *bounded below* if $\exists m \in \mathbb{R}$ such that $x_n \geq m$ for all $n \in \mathbb{N}$. We say that (x_n) is *bounded* if it is bounded above and below.

A sequence which satisfies the definition is illustrated here.

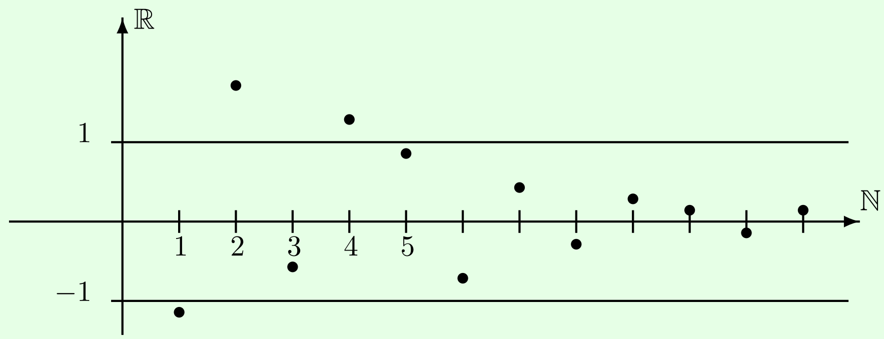


Remark 3.19
Clearly, $(x_n)_{n=1}^{\infty}$ is *bounded* if and only if $(|x_n|)_{n=1}^{\infty}$ is *bounded above*. We can write the condition for sequence $(x_n)_{n=1}^{\infty}$ to be bounded as follows:

$$\exists M \in \mathbb{R}\ \forall n \in \mathbb{N} : |x_n| \leq M.$$

Example 3.20
Every sequence $(x_n)_{n=1}^{\infty}$ which converges to zero is bounded.

Proof. If (x_n) converges to zero then taking $\varepsilon = 1$ in the definition of convergence to zero (Definition 3.1), there exists some $N \in \mathbb{N}$ such that for all $n > N$ we have $|x_n| < 1$. This is illustrated in the following figure.



Take M to be $\max\{|x_1|, |x_2|, \dots, |x_N|, 1\}$. Then for all $n \in \mathbb{N}$, we have $|x_n| \leq M$, i.e. $-M \leq x_n \leq M$ for all $n \in \mathbb{N}$.

This shows that M is an upper bound and $-M$ is a lower bound for the sequence (x_n), as defined in Definition 3.18. □

We are now ready to state and prove the more general version of Theorem 3.15.

Theorem 3.21 (Product of sequences, more general version)
If $(x_n)_{n=1}^{\infty}$ is *bounded* and $(y_n)_{n=1}^{\infty}$ converges to zero then $(x_n y_n)_{n=1}^{\infty}$ converges to zero.

Proof. As $(x_n)_{n=1}^{\infty}$ is bounded, there exists $M \in \mathbb{R}$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Hence

$$|x_n y_n| = |x_n||y_n| \leq M|y_n| = |My_n|, \qquad \forall n \in \mathbb{N}.$$

But (y_n) converges to zero and so (My_n) converges to zero as well (by Lemma 3.6), so $(x_n y_n)$ converges to zero by Lemma 3.4. □

Remark 3.22
Alternatively, we could prove this like we proved Theorem 3.15, splitting up $\varepsilon = M \cdot \frac{\varepsilon}{M}$. (As (y_n) converges to zero, we will get $|y_n| \leq \frac{\varepsilon}{M}$ for large enough n.) Try this!

3.B Convergence in general

Definition 3.23 (Convergence of a sequence to $x \in \mathbb{R}$)
A sequence $(x_n)_{n=1}^{\infty}$ converges to $x \in \mathbb{R}$ if and only if $(x_n - x)_{n=1}^{\infty}$ converges to zero. Equivalently, $(x_n)_{n=1}^{\infty}$ converges to $x \in \mathbb{R}$ if and only if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n > N \ : \ |x_n - x| < \varepsilon. \tag{3.4}$$

We use the notation $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

Theorem 3.24
Suppose $(x_n)_{n=1}^{\infty}$ converges to $x \in \mathbb{R}$ and $(y_n)_{n=1}^{\infty}$ converges to $y \in \mathbb{R}$. Then

(i) $(cx_n)_{n=1}^{\infty}$ converges to cx for any constant $c \in \mathbb{R}$.

(ii) $(x_n + y_n)_{n=1}^{\infty}$ converges to $x + y$.

(iii) $(x_n y_n)_{n=1}^{\infty}$ converges to xy.

(iv) if $y \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$, then $\left(\frac{x_n}{y_n}\right)_{n=1}^{\infty}$ converges to $\frac{x}{y}$.

Proof.

(i) By the definition of convergence of (x_n) to x, we know that the sequence $(x_n - x)$ converges to zero. So by Lemma 3.6, the sequence $c(x_n - x)$ converges to zero, i.e. $(cx_n - cx)$ converges to zero. But then by definition (cx_n) converges to cx.

(ii) As $x_n \to x$ and $y_n \to y$ as $n \to \infty$, we know that $(x_n - x)$ and $(y_n - y)$ are two sequences that converge to zero. Hence by Theorem 3.9, $((x_n - x) + (y_n - y))_{n=1}^{\infty}$ converges to zero. Rewriting this, we obtain that $((x_n + y_n) - (x + y))_{n=1}^{\infty}$ converges to zero. But then by definition $(x_n + y_n)$ converges to $x + y$.

(iii) We know that $(x_n - x)$ and $(y_n - y)$ converge to zero. We write

$$x_n y_n - xy = (x_n - x)(y_n - y) + x(y_n - y) + (x_n - x)y. \tag{3.5}$$

Then, we know that

- $((x_n - y)(y_n - y))$ converges to zero (by Theorem 3.15).
- $(x(y_n - y))$ converges to zero (by Lemma 3.6, as $(y_n - y)$ converges to zero).
- $((x_n - x)y)$ converges to zero (again by Lemma 3.6, as $(x_n - x)$ converges to zero).

So by Theorem 3.9, the right hand side of (3.5) converges to zero and hence $(x_n y_n - xy)$ converges to zero. But then by definition $(x_n y_n)$ converges to xy.

(iv) This is a question on one of the Problem Sheets. □

Examples 3.25 (i) Consider the sequence (x_n) given by $x_n = c$ for all $n \in \mathbb{N}$. This sequence converges to c.

Proof. We must prove that $(x_n - c)$ converges to zero. But $x_n - c = 0$ for all $n \in \mathbb{N}$ and we already know that the zero sequence converges to zero. □

(ii) $x_n = \frac{3n^2+5n+2}{2n^2+n+1}$. This sequence converges to $\frac{3}{2}$.

Proof. We write x_n in the form

$$x_n = \frac{3 + \frac{5}{n} + \frac{2}{n^2}}{2 + \frac{1}{n} + \frac{1}{n^2}}.$$

But $(3 + \frac{5}{n} + \frac{2}{n^2})$ converges to $3 + 0 + 0 = 3$. [This uses Theorem 3.24 (ii) and the facts that the constant sequence 3 converges to 3 while $(\frac{5}{n})$ and $(\frac{2}{n^2})$ converge to zero – the latter due to Lemma 3.4 and Lemma 3.6, as seen several times before.]

Similarly, $(2 + \frac{1}{n} + \frac{1}{n^2})$ converges to $2 + 0 + 0 = 2$, again by Theorem 3.24 (ii) [and the fact that $(\frac{1}{n})$ and $(\frac{1}{n^2})$ converge to zero].

In a last step, we then apply Theorem 3.24 (iv) to conclude that (x_n) converges to $\frac{3}{2}$. □

Lemma 3.26 (Limits are unique)
If $(x_n)_{n=1}^{\infty}$ converges to $x \in \mathbb{R}$ and also converges to $y \in \mathbb{R}$ then $x = y$. That is: if a limit exists, then it is unique.

Proof. Suppose $x \neq y$. Without loss of generality, assume $y > x$ and write α for the difference $y - x = |y - x|$.

Since (x_n) converges to x, we know that $\exists N_x \in \mathbb{N}$ such that $\forall n > N_x$, we have $|x_n - x| < \frac{\alpha}{2}$.

Similarly, since (x_n) converges to y, we know that $\exists N_y \in \mathbb{N}$ such that $\forall n > N_y$, we have $|x_n - y| < \frac{\alpha}{2}$.

Now consider some x_n with $n \geq \max\{N_x, N_y\}$. This x_n satisfies $|x_n - x| < \frac{\alpha}{2}$ *and*

$|x_n - y| < \frac{\alpha}{2}$. Hence, using the triangle inequality,

$$|y - x| \leq |y - x_n| + |x_n - x| = |x_n - y| + |x_n - x| < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.$$

So $|y - x| < \alpha$, contradicting our definition of α as $|x - y|$. ↯ □

Lemma 3.27
Suppose we have two sequences (x_n) and (y_n) with $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Suppose further that $x_n \leq y_n$ for all $n \in \mathbb{N}$. Then $x \leq y$.

Proof. Suppose for a contradiction that $x > y$. Take $\varepsilon = \frac{x-y}{2} > 0$. Since $x_n \to x$ there exists $N_x \in \mathbb{N}$ such that $\forall n > N_x : |x_n - x| < \varepsilon$. This in particular implies:

$$\forall n > N_x : x_n > x - \varepsilon = x - \frac{x-y}{2} = \frac{x+y}{2}.$$

Similarly since $y_n \to y$ there exists $N_y \in \mathbb{N}$ such that $\forall n > N_y : |y_n - y| < \varepsilon$. This in particular implies:

$$\forall n > N_y : y_n < y + \varepsilon = y + \frac{x-y}{2} = \frac{x+y}{2}.$$

Set $N = \max\{N_x, N_y\}$. Then for $n > N$ we have

$$\frac{x+y}{2} < x_n \leq y_n < \frac{x+y}{2}$$

which is a contradiction. ↯ □

Examples 3.28 (i) If $(x_n)_{n=1}^{\infty}$ converges to $x \in \mathbb{R}$ and there exists a constant $c \in \mathbb{R}$ such that $x_n \leq c$ for all $n \in \mathbb{N}$, then $x \leq c$.

Proof. We know that (y_n) defined by $y_n = c$ for all $n \in \mathbb{N}$ converges to c. We can then apply Lemma 3.27 to (x_n) and this choice of (y_n). □

Similarly, if $(x_n)_{n=1}^{\infty}$ converges to $x \in \mathbb{R}$ and there exists a constant $c \in \mathbb{R}$ such that $x_n \geq c$ for all $n \in \mathbb{N}$, then $x \geq c$.

(ii) If the inequality $x_n \leq y_n$ in Lemma 3.27 is replaced by a *strict* inequality $x_n < y_n$, we still can only conclude $x \leq y$ and *not* $x < y$. An example for this is $x_n = 0$ and $y_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Although we have $x_n < y_n$ for all $n \in \mathbb{N}$, both sequences converge to zero, so $x = \lim_{n\to\infty} x_n = 0$ and $y = \lim_{n\to\infty} y_n = 0$ and we have $x = y$.

Lemma 3.29 (The sandwich principle)
If (y_n) is *some sequence* and (x_n) and (z_n) are two converging sequences such that $x_n \leq y_n \leq z_n$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$, then (y_n) converges as well and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n$.

Proof. This is a question on the next Problem Sheet. □

3.C Monotonic sequences

Definition 3.30 (Monotonic sequences)
Let $(x_n)_{n=1}^{\infty}$ be a sequence.

- We say (x_n) is an *increasing* sequence if $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$.
- We say (x_n) is *strictly increasing* if $x_{n+1} > x_n$ for all $n \in \mathbb{N}$.
- We say (x_n) is a *decreasing* sequence if $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$.
- We say (x_n) is *strictly decreasing* if $x_{n+1} < x_n$ for all $n \in \mathbb{N}$.
- We say (x_n) is *monotonic* if it is either increasing or decreasing.

Examples 3.31 (i) $x_n = \frac{1}{n}$ is strictly decreasing since $\frac{1}{n+1} < \frac{1}{n}$ for all $n \in \mathbb{N}$. Similarly, $x_n = 3 + \frac{1}{n}$ is strictly decreasing as well.

(ii) $x_n = -\frac{1}{n}$ and $x_n = 2 - \frac{1}{n}$ are strictly increasing.

(iii) $x_n = 1$ for all $n \in \mathbb{N}$ is both increasing and decreasing (but not strictly).

(iv) $x_n = n$ is strictly increasing.

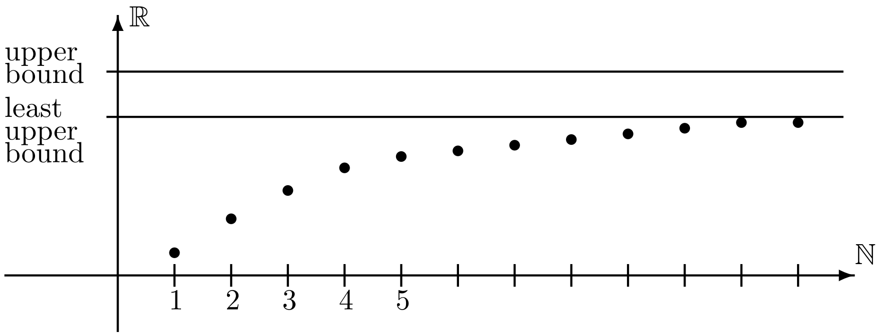
(v) $x_n = \frac{(-1)^n}{n}$ is neither increasing nor decreasing.

(vi) $x_n = (-1)^n$ is also neither increasing nor decreasing.

The sequences in the examples (i)–(iii) are monotonic and bounded, and all of them converge to a limit in $\mathbb{R}$. Example (iv) is monotonic but *not* bounded and it does not converge to a limit in $\mathbb{R}$. Examples (v) and (vi) are bounded but *not* monotonic; the first does converge to a limit in $\mathbb{R}$ and the second does not. The next theorem says that the fact that (i)–(iii) converge is not just a coincidence.

Theorem 3.32 (Bounded monotonic sequences converge)
If $(x_n)_{n=1}^{\infty}$ is a increasing sequence which is bounded above, then (x_n) converges

to some real number.



The set $A := \{x_n \ : \ n \in \mathbb{N}\}$ is a non-empty set of real numbers which is bounded above (by the hypothesis of the theorem). So by the Completeness Axiom for $\mathbb{R}$ it has a least upper bound, $x = \sup(A)$ (we will also denote this as $x = \sup_{n\in\mathbb{N}} x_n$). We aim to prove that (x_n) converges to this x.

Proof. Define $x = \sup_{n\in\mathbb{N}} x_n$. We must show that

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n > N \ : \ |x_n - x| < \varepsilon.$$

Suppose we are given $\varepsilon > 0$. We know that $x - \varepsilon$ is not an upper bound for the sequence (since x is the least upper bound). Hence there is some element of the sequence, say x_N which satisfies $x_N > x - \varepsilon$.

Because x_n is an increasing sequence, we now know that $\forall n > N$ we have $x_n \geq x_N > x - \varepsilon$. Moreover, x is an upper bound, so $\forall n > N$, we have $x_n \leq x < x + \varepsilon$. Hence $|x_n - x| < \varepsilon$. □

Remark 3.33
We don't really need to know the actual value of the limit in this theorem. We are proving that the limit *exists*.

Examples 3.34 (i) One way to specify the value of π is as the limit of the increasing sequence of rational numbers

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots$$

This is an increasing sequence which is bounded above (e.g. by 3.2), so by Theorem 3.32 there exists a real number to which this sequence converges. We call this real number π.

(ii) $x_n = \frac{n-1}{n}$. This is the sequence $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$. It is increasing and bounded above by 1 (for example). So by Theorem 3.32 it converges to a limit. In this case, we know what the limit is: Since $\frac{n-1}{n} = 1 - \frac{1}{n}$, we know that the limit is 1 (by Theorem 3.24 (ii)).

(iii) $x_n = \sum_{k=0}^{n} \frac{1}{k!}$ (where $0! = 1$). This is the sequence

$$\begin{aligned} x_1 &= 1 + \frac{1}{1!} = 2, \\ x_2 &= 1 + \frac{1}{1!} + \frac{1}{2!} = \frac{5}{2} = 2.5, \\ x_3 &= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} = \frac{5}{2} + \frac{1}{6} = \frac{8}{3} = 2.666\ldots. \end{aligned}$$

It is (strictly) increasing since $x_{n+1} - x_n = \frac{1}{(n+1)!} > 0$. It is also bounded above. To prove this, we observe that $\frac{1}{k!} = \frac{1}{k} \cdot \frac{1}{k-1} \cdot \cdots \cdot \frac{1}{2} \le \frac{1}{2^{k-1}}$ if $k \ge 2$. So

$$\begin{aligned} x_n &= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots \frac{1}{n!} \\ &\le 1 + 1 + \underbrace{\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n-1}}}_{=\frac{2^{n-1}-1}{2^{n-1}}<1}, \end{aligned}$$

so $x_n < 3$ for all $n \in \mathbb{N}$. Since (x_n) is increasing and bounded above, we can deduce from Theorem 3.32 that it converges to a real number. This real number is denoted "e".

It can be proved that e is not a rational number and in fact that it is not a root of any polynomial equation with integer coefficients (we say that e is "transcendental"). To define e, we need to express it as an *infinite sum* $\sum_{k=0}^{\infty} \frac{1}{k!}$. When we do this, we are using Theorem 3.32 (or a result equivalent to it).

Note that here we used

$$\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n-1}} = \sum_{k=1}^{n-1} \frac{1}{2^k} = \frac{2^{n-1}-1}{2^{n-1}}.$$

You might remember this formula from your Calculus class, but we will give a precise proof of it at a later stage (Lemma 4.10).

What about increasing sequences which are not bounded above?

Definition 3.35 (Tending to infinity)

We say that the sequence $(x_n)_{n=1}^{\infty}$ *tends to infinity* if

$$\forall K \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n > N\ :\ x_n > K. \tag{3.6}$$

In the literature, this is sometimes also called convergence to infinity. But note that "∞" is *not* a real number, so this is *not* the same as the definition of convergence to a real number! Also note that the "winning condition" in (3.6) is $x_n > K$ and *not* just $|x_n| > K$.

Examples 3.36 (i) $x_n = n$. To prove this tends to infinity, we must show that

$$\forall K \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n > N\ :\ n > K.$$

We have seen that this is true in the chapter on real numbers. Our winning strategy for the Demon Game corresponding to (3.6) is: Given $K \in \mathbb{R}$, let $N = \lceil |K| \rceil$. This number exists by the Archimedean property (Theorem 2.14). Now given any $n > N$, we have $n > N = \lceil |K| \rceil$, so $n > |K| \geq K$.

(ii) $x_n = \sqrt{n}$. To prove that this tends to infinity, we must show that

$$\forall K \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n > N\ :\ \sqrt{n} > K.$$

Proof. Given $K \in \mathbb{R}$, let $N = \lceil K^2 \rceil$. Now given any $n > N$, we have $n > N = \lceil K^2 \rceil$, so $n > K^2$, and thus $\sqrt{n} > |K| = K$. □

(iii) $x_n = c^n$, where $c > 1$ is a real constant. To prove that c^n tends to ∞, we must prove

$$\forall K \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n > N\ :\ c^n > K.$$

Proof. If $c > 1$ then $\frac{1}{c} < 1$ and we know from Theorem 3.14 that $(\frac{1}{c})^n$ converges to zero. So given any $K > 0$, we know that $\exists N$ such that $(\frac{1}{c})^n < \frac{1}{K}$ for all $n > N$. Thus $c^n > K$ for all $n > N$. If $K \leq 0$, then $c^n > K$ holds for all $n \in \mathbb{N}$. Hence, we proved that c^n tends to infinity. □

Comment: If $c < -1$ then it is not true that c^n tends to ∞. Although the even powers of c will tend to ∞, the odd powers will tend to $-\infty$.

Theorem 3.37 (Increasing sequences converge or tend to infinity)
Suppose $(x_n)_{n=1}^{\infty}$ is increasing. Then either (x_n) converges or it tends to infinity.

Proof. If the sequence is bounded above, then it converges (by Theorem 3.32). So suppose that (x_n) is *not* bounded above. We shall prove that then (x_n) tends to infinity, i.e. we shall prove that

$$\forall K \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n > N\ :\ x_n > K.$$

Given $K \in \mathbb{R}$, we know that K is not an upper bound for (x_n) (since the sequence is not bounded above, i.e. has no upper bounds). Therefore, there exists some x_N in the sequence with $x_N > K$. Now for all $n > N$ we have $x_n \geq x_N > K$ (using that (x_n) is increasing). □

Remark 3.38
Similarly, decreasing sequences either converge (if they are bounded below) or tend to $-\infty$. We leave it as an exercise to define precisely what it means that a sequence tends to $-\infty$ and to prove these results.

3.D Subsequences

Intuitively and informally, a subsequence of $(x_n)_{n=1}^{\infty} = x_1, x_2, x_3, x_4, \ldots$ is a sequence like $x_1, x_3, x_5, x_7, \ldots$ or $x_2, x_3, x_5, x_7, x_{11}, \ldots$. We take just *some* of the terms of the original sequence but we take infinitely many of them and we take them in the same order as the original sequence. In the examples above, we took the x_n where n is odd or the ones where n is a prime, but in general there does not have to be such a rule!

Definition 3.39 (Subsequence)
A *subsequence* of $(x_n)_{n=1}^{\infty}$ is a sequence $x_{r_1}, x_{r_2}, x_{r_3}, \ldots$ where $r_j \in \mathbb{N}$ for each $j \in \mathbb{N}$ and $r_{j+1} > r_j$ for each $j \in \mathbb{N}$. We also denote this as $(x_{r_j})_{j=1}^{\infty}$.

Examples 3.40 (i) $x_2, x_4, x_6, \ldots$ is a subsequence of $(x_n)_{n=1}^{\infty}$. Here, we took $r_1 = 2$, $r_2 = 4$, $r_3 = 6$, etc. So $r_j = 2j$ and the subsequence can be written as $(x_{2j})_{j=1}^{\infty}$.

(ii) $x_1, x_4, x_9, x_{16}, \ldots$ is another subsequence. Here we have chosen $r_j = j^2$, so the subsequence can be expressed as $(x_{j^2})_{j=1}^{\infty}$.

Remark 3.41
If we regard a sequence (x_n) as a function $\mathbb{N} \to \mathbb{R}$, $n \mapsto x_n$, then a subsequence of (x_n) is a composite function $\mathbb{N} \to \mathbb{N} \to \mathbb{R}$, $j \mapsto r_j \mapsto x_{r_j}$, where $j \mapsto r_j$ is *injective and order-preserving* (i.e., strictly increasing).

Very often a sequence which does not converge has subsequences which *do* converge! For example $x_n = (-1)^n$. Here $x_1, x_3, x_5, \ldots$ is a subsequence which converges to

-1 (in fact all the terms are -1) and $x_2, x_4, x_6, \ldots$ is a subsequence which converges to $+1$ (in fact all the terms are $+1$). The following theorem, which is of great importance later in the course, says that this is true in general.

Theorem 3.42 (Bolzano–Weierstrass, Version 1)
Every bounded sequence contains a convergent subsequence.

Proof. (The proof of this theorem is non-examinable.)

By Theorem 3.32 ("increasing bounded sequences converge") and its analogue for decreasing sequences, it will be enough to prove that every sequence of real numbers has a monotonic subsequence.

Suppose we are given a sequence $(x_n)_{n=1}^{\infty}$. There are two possibilities:

Case 1: For every $N \in \mathbb{N}$ the set $\{x_n \ : \ n \geq N\}$ has a maximum.
Case 2: For some $N \in \mathbb{N}$ the set $\{x_n \ : \ n \geq N\}$ does not have a maximum.

If we are in Case 1, we select a subsequence as follows:

1. Pick r_1 such that $x_{r_1} = \max\{x_n \ : \ n \geq 1\}$,
2. pick $r_2 > r_1$ such that $x_{r_2} = \max\{x_n \ : \ n \geq r_1 + 1\}$,
3. pick $r_3 > r_2$ such that $x_{r_3} = \max\{x_n \ : \ n \geq r_2 + 1\}$,

and so on. Now $x_{r_1} \geq x_{r_2} \geq x_{r_3} \geq \ldots$ since we are taking the maximum of smaller and smaller sets. That is, we found a *decreasing* subsequence.

If we are in Case 2, we let $r_1 = N$, so that $x_{r_1} = x_N$. Then as $\{x_n \ : \ n \geq N\}$ does *not* have a maximum, there exists some $r_2 > N = r_1$ with $x_{r_2} > x_{r_1}$. Moreover, the set $\{x_n \ : \ n \geq r_2\}$ cannot have a maximum. (If it did, say it had a maximum M, then $\max\{x_N, x_{N+1}, x_{N+2}, \ldots, x_{r_2-1}, M\}$ would be a maximum for $\{x_n \ : \ n \geq N\}$.) Thus there exists some $r_3 > r_2$ with $x_{r_3} > x_{r_2}$. Repeating this argument gives an *increasing* sequence $x_{r_1} < x_{r_2} < x_{r_3} < \ldots$. □

Definition 3.43 (Accumulation point)
A real number $x \in \mathbb{R}$ is called an *accumulation point* of a sequence (x_n) if for every $\varepsilon > 0$ there are infinitely many elements of the sequence which lie ε-close to x (i.e., satisfy $|x_n - x| < \varepsilon$). This means, x is an accumulation point of (x_n) if and only if

$$\forall \varepsilon > 0 \ \forall N \in \mathbb{N} \ \exists n > N \ : \ |x_n - x| < \varepsilon. \tag{3.7}$$

Lemma 3.44 (Limit is an accumulation point)
If (x_n) converges to $x \in \mathbb{R}$, then x is an accumulation point.

Of course, if (x_n) converges to x, then (by definition), given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that *all* x_n with $n > N$ lie ε-close to x (and thus in particular infinitely many). But let us formally prove this, using the precise mathematical definitions of convergence (3.4) and accumulation point (3.7).

Proof. We need to prove (3.7). So we are given $\varepsilon > 0$ and $N \in \mathbb{N}$ (picked by the Demon) and we need to find $n > N$ such that $|x_n - x| < \varepsilon$.

Since $x_n \to x$ as $n \to \infty$, we know by definition of convergence that for the given $\varepsilon > 0$, there exists some $\tilde{N}$ such that $\forall n > \tilde{N} \ : \ |x_n - x| < \varepsilon$. (Here we relabelled N to $\tilde{N}$ in the definition of convergence, because N is already used with a different meaning.)

Now, we can pick $n := \max\{N, \tilde{N}\} + 1$. This choice of n is allowed because $n > N$. From $n > \tilde{N}$, we conclude that $|x_n - x| < \varepsilon$, which is what we wanted to show. ☐

In general, a sequence can have an accumulation point even if it does not converge and it can have several accumulation points. (Later – on one of the Problem Sheets – we will actually find an example that has *infinitely many* accumulation points!)

Example 3.45
Let $x_n = (-1)^n$. Then we know that (x_n) does not converge. But both $x = +1$ and $x' = -1$ are accumulation points for (x_n).

Proof. For $x = +1$: Given any $\varepsilon > 0$ and any $N \in \mathbb{N}$, let $n := 2N$ so that $n > N$ and n even. Then $|x_n - 1| = |1 - 1| = 0 < \varepsilon$.

For $x' = -1$: Given any $\varepsilon > 0$ and any $N \in \mathbb{N}$, let $n := 2N + 1$ so that $n > N$ and n odd. Then $|x_n - (-1)| = |(-1) - (-1)| = 0 < \varepsilon$. ☐

We also know that the subsequence $(x_{2j})_{j=1}^{\infty}$ converges to $+1$ and the subsequence $(x_{2j+1})_{j=1}^{\infty}$ converges to -1. So both accumulation points are actually limits of subsequences. This is true in general, as the following lemma shows.

Lemma 3.46
The real number $x \in \mathbb{R}$ is an accumulation point of $(x_n)_{n=1}^{\infty}$ if and only if there is a subsequence $(x_{r_j})_{j=1}^{\infty}$ converging to x.

Proof. Suppose there is a subsequence $(x_{r_j})_{j=1}^{\infty}$ converging to x. Then for any $\varepsilon > 0$, there is $J \in \mathbb{N}$ such that all the elements of the subsequence (x_{r_j}) with $j > J$ satisfy $|x_{r_j} - x| < \varepsilon$. In particular, there are infinitely many. (One can write this more formally, as we have done in the proof of Lemma 3.44. I leave this as an exercise.)

Conversely, if x is an accumulation point, then we can construct a subsequence converging to x as follows: From (3.7), for $\varepsilon = 1$ and $N = 1$, we can find $r_1 > 1$ such that $|x_{r_1} - x| < \varepsilon = 1$. Then, using (3.7) again, this time with $\varepsilon = 1/2$ and $N = r_1$, we can find $r_2 > r_1$ with $|x_{r_2} - x| < \varepsilon = 1/2$. We then iterate this, i.e., if $x_{r_{j-1}}$ is already constructed, we use (3.7) with $\varepsilon = \frac{1}{j}$ and $N = r_{j-1}$ to find some $r_j > r_{j-1}$ with $|x_{r_j} - x| < \frac{1}{j}$. Clearly, the subsequence $(x_{r_j})_{j=1}^{\infty}$ then converges to x. □

We can now restate Theorem 3.42.

Theorem 3.47 (Bolzano–Weierstrass, Version 2)
Every bounded sequence of real numbers has an accumulation point.

Proof. This follows by combining the first version of the theorem with Lemma 3.46. □

The following lemma is often useful.

Lemma 3.48
Suppose that $(x_n)_{n=1}^{\infty}$ converges to $x \in \mathbb{R}$. Then every subsequence $(x_{r_j})_{j=1}^{\infty}$ also converges to x.

Proof. Suppose that $x_n \to x$ and fix $\epsilon > 0$. There exists an $N \in \mathbb{N}$ such that for $n > N$ we have $|x_n - x| < \epsilon$. Choose $J \in \mathbb{N}$ such that $r_J \geq N$. Then for $j > J$ we have $r_j > r_J \geq N$ and so $|x_{r_j} - x| < \epsilon$ as required. □

The preceding lemma provides an easy way to prove that a given sequence does not converge: identify two subsequences which converge to two different values (or, identify a single subsequence which does not converge).

3.E Cauchy sequences

If a bounded sequence is not monotonic, how can we prove that it is convergent if we don't have a candidate for the limit? We start with an easy result.

Lemma 3.49
Suppose that $(x_n)_{n=1}^{\infty}$ converges to some $x \in \mathbb{R}$. Then the sequence $(y_n)_{n=1}^{\infty}$ defined by $y_n = x_{n+1} - x_n$ converges to zero.

Proof. We first observe that $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x$. (This is "obvious", but we should still make sure we can formally prove it! See the next Problem Sheet.)

Then, using Theorem 3.24 (ii), we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} (x_{n+1} - x_n) = \lim_{n\to\infty} x_{n+1} - \lim_{n\to\infty} x_n = x - x = 0.$$

We can also give a direct proof. Given $\varepsilon > 0$ set $\tilde{\varepsilon} = \varepsilon/2$. Since $x_n \to x$ there is an $N \in N$ such that $\forall n > N : |x_n - x| < \tilde{\varepsilon}$. Then for $n > N$ we have

$$|y_n| = |x_{n+1} - x_n| = |(x_{n+1} - x) - (x_n - x)| \leq |x_{n+1} - x| + |x_n - x| < \tilde{\varepsilon} + \tilde{\varepsilon} = \varepsilon. \quad \square$$

Examples 3.50 (i) Let $x_n = \frac{n-1}{n}$. Then we know that $x_n \to 1$ as $n \to \infty$. So we obtain $y_n = x_{n+1} - x_n = \frac{n}{n+1} - \frac{n-1}{n} \to 0$ as $n \to \infty$. (Indeed, we can easily check that $y_n = \frac{n^2-(n-1)(n+1)}{n(n+1)} = \frac{1}{n(n+1)} \to 0$ as $n \to \infty$.)

(ii) Let $x_n = (-1)^n$. Then $|y_n| = |x_{n+1} - x_n| = 2$ for all $n \in \mathbb{N}$ and hence (y_n) does not converge to zero. By Lemma 3.49, (x_n) can thus not converge to any real number $x \in \mathbb{R}$ (as we have already seen before).

Note that the converse of Lemma 3.49 is false!

Examples 3.51 (i) Let $x_n = \sqrt{n}$. Then the sequence (x_n) does not converge to any real number. But the sequence $y_n = x_{n+1} - x_n = \sqrt{n+1} - \sqrt{n}$ does converge to zero (as we have seen in the Problem Sheet).

(ii) Let $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$. This sequence does *not* converge (we have actually already seen this in Chapter 0, but will see it again more precisely below). Nevertheless $y_n = x_{n+1} - x_n = \frac{1}{n+1}$ does converge to zero.

This means that for $(x_n)_{n=1}^{\infty}$ to converge to some $x \in \mathbb{R}$ it is *necessary* that the sequence $(x_{n+1} - x_n)_{n=1}^{\infty}$ converges to zero, but this condition is *not sufficient*. The goal of this section is to find a sufficient condition! For this reason, let us rephrase first the result from Lemma 3.49 using a quantifier statement.

Lemma 3.49 says that if (x_n) converges to some $x \in \mathbb{R}$, then

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n > N \; : \; |x_{n+1} - x_n| < \varepsilon.$$

(This is nothing else than the definition of $y_n = x_{n+1} - x_n \to 0$ for $n \to \infty$.) The following definition generalises this idea, testing not only the distance of x_n and x_{n+1} (for $n > N$) but actually *all* the distances of x_n to x_m for n and m both greater than N.

Definition 3.52 (Cauchy sequence)
A sequence $(x_n)_{n=1}^{\infty}$ is called a Cauchy sequence if and only if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n > N \; \forall m > N \; : \; |x_m - x_n| < \varepsilon. \tag{3.8}$$

Theorem 3.53 (Equivalence of Cauchy sequences and convergent sequences) Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. Then $(x_n)_{n=1}^{\infty}$ converges if and only if it is a Cauchy sequence.

Proof. (The proof of this theorem is non-examinable.)

We first show: Convergent sequences are Cauchy sequences.

Assume that there is some $x \in \mathbb{R}$ such that $x_n \to x$ as $n \to \infty$. We have to prove that (x_n) is a Cauchy sequence. So, we are given $\varepsilon > 0$ (by the Demon). Then by the definition of convergence, letting $\tilde{\varepsilon} = \frac{\varepsilon}{2}$, we can find some N such that for all $n > N$ we have $|x_n - x| < \tilde{\varepsilon} = \frac{\varepsilon}{2}$. We pick exactly this N. Then for all $n, m > N$, we have

$$|x_m - x_n| \leq |x_m - x| + |x - x_n| < \tilde{\varepsilon} + \tilde{\varepsilon} = \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Now we prove: Cauchy sequences do converge. We do this in several steps.

Step 1. Cauchy sequences are bounded: For $\varepsilon = 1$ (in the definition of a Cauchy sequence), we can find N_1 such that for all $n, m > N_1$ we have $|x_m - x_n| < 1$. In particular, letting $n = N_1 + 1$ be fixed, we find that for all $m > N_1$ we have

$$|x_m| = |(x_m - x_{N_1+1}) + x_{N_1+1}| \leq |x_m - x_{N_1+1}| + |x_{N_1+1}| < 1 + |x_{N_1+1}|.$$

Thus the sequence is bounded, since for all $m \in \mathbb{N}$ we have:

$$|x_m| \leq \max\{|x_1|, |x_2|, \dots, |x_{N_1}|, 1 + |x_{N_1+1}|\}.$$

Step 2. We can find a convergent subsequence: Indeed, this is just an application of the Theorem of Bolzano–Weierstrass (Theorem 3.42). So in particular, we have a subsequence $(x_{r_j})_{j=1}^{\infty}$ and some number $x \in \mathbb{R}$, such that $x_{r_j} \to x$ as $j \to \infty$.

Step 3. The whole sequence converges to x: As $x_{r_j} \to x$, by definition of convergence there is some N_2 such that $\forall r_j > N_2 \ : \ |x_{r_j} - x| < \tilde{\varepsilon} = \frac{\varepsilon}{2}$. Moreover, by the definition of a Cauchy sequence, there is some N_3 such that $\forall m, n > N_3 \ : \ |x_m - x_n| < \tilde{\varepsilon} = \frac{\varepsilon}{2}$. Thus for any $n > N := \max\{N_2, N_3\}$, we can pick some $r_\ell > N$ and estimate

$$|x_n - x| \leq |x_n - x_{r_\ell}| + |x_{r_\ell} - x| < \tilde{\varepsilon} + \tilde{\varepsilon} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $x_n \to x$ as $n \to \infty$. □

While this theorem seems hard at first look, Cauchy sequences actually have a lot of advantages:

- We have a condition that applies to all sequences (not just monotonic ones) and tells us whether the sequence converges or not.

- The statement is "if and only if", so the Cauchy criterion is necessary and sufficient.
- The theorem applies to sequences in $\mathbb{R}^n$ and $\mathbb{C}$ and $\mathbb{C}^n$, not just to $\mathbb{R}$. (In $\mathbb{R}^n$, by $|x_m - x_n|$ we mean the *distance* between x_m and x_n.)
- Finally, there is an easier proof of the Bolzano–Weierstrass Theorem using Cauchy sequences and it works in $\mathbb{R}^n$, not just in $\mathbb{R}$.

4 Series

4.A Definition and first examples

A *series* is a "sum of an infinite number of terms". Suppose $(x_n)_{n=1}^{\infty}$ is a sequence. We define the *partial sums* of the series $\sum_{k=1}^{\infty} x_k$ by $S_n = \sum_{k=1}^{n} x_k$. So

$$\begin{aligned} S_1 &= x_1, \\ S_2 &= x_1 + x_2, \\ S_3 &= x_1 + x_2 + x_3, \\ &\dots \\ S_n &= x_1 + x_2 + x_3 + \dots + x_n. \end{aligned}$$

We can ask whether the sequence $(S_n)_{n=1}^{\infty}$ of the n-th partial sums converges, or whether it tends to infinity, or whether it does neither.

Definition 4.1 (i) We say that $\sum_{k=1}^{\infty} x_k$ *exists* (or *converges*) if the sequence $(S_n)_{n=1}^{\infty}$ converges. If $S_n \to S$ for some real number S then we write:

$$\sum_{k=1}^{\infty} x_k = S.$$

(ii) We say that $\sum_{k=1}^{\infty} x_k$ *does not exist* (or *does not converge*) if the sequence $(S_n)_{n=1}^{\infty}$ does not converge.

(iii) We write $\sum_{k=1}^{\infty} x_k = \infty$ if $S_n \to \infty$. (If $\sum_{k=1}^{\infty} x_k = \infty$ then $\sum_{k=1}^{\infty} x_k$ "does not exist" since ∞ is not a number.)

Examples 4.2 (i) Let us look at $\sum_{k=1}^{\infty} x_k$ with $x_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Then

$$\begin{aligned} S_1 &= x_1 = 1 - \tfrac{1}{2} = \tfrac{1}{2}, \\ S_2 &= x_1 + x_2 = 1 - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{3} = 1 - \tfrac{1}{3}, \\ &\dots \\ S_n &= x_1 + x_2 + x_3 + \dots + x_n = 1 - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{3} - \dots - \tfrac{1}{n} + \tfrac{1}{n} - \tfrac{1}{n+1} \\ &= 1 - \tfrac{1}{n+1}. \end{aligned}$$

(This is called a "telescoping sum".) So $(S_n)_{n=1}^{\infty}$ converges to 1. Hence $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ exists and is equal to 1.

(ii) Let us look at $\sum_{k=0}^{\infty} \frac{1}{k!}$. Here $S_n = \sum_{k=0}^{n} x_k$ is

$$\begin{aligned} S_0 &= \tfrac{1}{0!} = 1, \\ S_1 &= \tfrac{1}{0!} + \tfrac{1}{1!} = 2, \\ &\dots \\ S_n &= \tfrac{1}{0!} + \tfrac{1}{1!} + \underbrace{\tfrac{1}{2!} + \tfrac{1}{3!} + \ldots + \tfrac{1}{n!}}_{\leq \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}} \leq 2 + \tfrac{2^{n-1}-1}{2^{n-1}} < 3, \end{aligned}$$

i.e., $S_n < 3$ for all $n \in \mathbb{N}$. Thus $(S_n)_{n=0}^{\infty}$ is increasing (since $S_{n+1} - S_n = \frac{1}{(n+1)!}$), bounded above (e.g., by 3) and thus converges to a real number in $\mathbb{R}$ which we denote by e. That is, $\sum_{k=0}^{\infty} \frac{1}{k!}$ exists and is equal to e.

Here, we have used the fact that $\frac{1}{k!} = \frac{1}{k} \cdot \frac{1}{k-1} \cdot \cdots \cdot \frac{1}{2} \leq \frac{1}{2^{k-1}}$ for $k \geq 2$ as well as

$$\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n-1}} = \sum_{k=1}^{n-1} \frac{1}{2^k} = \frac{2^{n-1} - 1}{2^{n-1}} < 1.$$

As mentioned before, this will be proved precisely in Lemma 4.10 below.

(iii) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$. We have

$$\begin{aligned} S_1 &= \tfrac{1}{\sqrt{1}} = 1, \\ S_2 &= \tfrac{1}{\sqrt{1}} + \tfrac{1}{\sqrt{2}}, \\ &\dots \\ S_n &= \tfrac{1}{\sqrt{1}} + \tfrac{1}{\sqrt{2}} + \ldots + \tfrac{1}{\sqrt{n}} \\ &\geq \tfrac{1}{\sqrt{n}} + \tfrac{1}{\sqrt{n}} + \ldots + \tfrac{1}{\sqrt{n}} = \tfrac{n}{\sqrt{n}} = \sqrt{n}. \end{aligned}$$

So the sequence $(S_n)_{n=1}^{\infty}$ does *not* converge to a real number. In fact, it tends to infinity. So we say that $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ does not exist and we write $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \infty$.

(iv) $x_k = (-1)^k$, i.e. we look at $\sum_{k=1}^{\infty} (-1)^k$. We have

$$\begin{aligned} S_1 &= -1, \\ S_2 &= -1 + 1 = 0, \\ S_3 &= -1 + 1 - 1 = -1, \\ S_4 &= -1 + 1 - 1 + 1 = 0, \\ S_n &= \begin{cases} 0 & \text{if } n \text{ is even}, \\ -1 & \text{if } n \text{ is odd}. \end{cases} \end{aligned}$$

So $(S_n)_{n=1}^\infty$ does not converge, and we say that $\sum_{k=1}^\infty (-1)^k$ does not exist.

4.B Some rules and criteria for convergence

There is an easy *necessary* condition for $\sum_{k=1}^\infty x_k$ to exist (respectively an easy *sufficient* condition for $\sum_{k=1}^\infty x_k$ to not exist).

Theorem 4.3
If $\sum_{k=1}^\infty x_k$ exists, then the corresponding sequence $(x_k)_{k=1}^\infty$ converges to zero. (Therefore, if $(x_k)_{k=1}^\infty$ does not converge to zero, then $\sum_{k=1}^\infty x_k$ does not exist.)

Proof. This is a direct application of Lemma 3.49. If $\sum_{k=1}^\infty x_k$ exists, then by definition the sequence of partial sums $(S_n)_{n=1}^\infty$ converges (to some real number). So by Lemma 3.49, $(S_{n+1} - S_n)_{n=1}^\infty$ converges to zero. But $S_{n+1} - S_n = x_{n+1}$, so the sequence $(x_{n+1})_{n=1}^\infty$ converges to zero and thus $(x_n)_{n=1}^\infty$ converges to zero. □

Note that the converse to Theorem 4.3 is false. For example in Example 3.2(iii) above the sequence $(x_k)_{k=1}^\infty$ converges to zero, but the sequence of partial sums $(S_n)_{n=1}^\infty$ does not converge. Nevertheless, this theorem can be very useful to prove non-convergence. For example in Example 3.2(iv) above, we know immediately that the series $\sum_{k=1}^\infty (-1)^k$ cannot exist, since the sequence $((-1)^k)_{k=1}^\infty$ does not converge to zero.

More generally, using the Cauchy criterion for convergence of sequences, we get a Cauchy criterion for the convergence of series.

Theorem 4.4 (Cauchy criterion)
The series $\sum_{k=1}^\infty x_k$ exists if and only if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall m > n > N \ : \ \left| \sum_{k=n+1}^{m} x_k \right| < \varepsilon.$$

Proof. By definition, $\sum_{k=1}^\infty x_k$ exists if and only if the sequence of partial sums (S_n) converges. By Theorem 3.53, (S_n) converges if and only if it is a Cauchy sequence, i.e. if and only if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall m > n > N \ : \ |S_m - S_n| < \varepsilon.$$

But $|S_m - S_n| = |\sum_{k=1}^m x_k - \sum_{k=1}^n x_k| = |\sum_{k=n+1}^m x_k|$. □

In the Problem Sheets, we have seen that convergent sequences build a ring and a vector space. Can we do the same for convergent series?

Lemma 4.5
If $\sum_{k=1}^{\infty} x_k = S$ and $\sum_{k=1}^{\infty} y_k = T$ (i.e. both exist) then $\sum_{k=1}^{\infty}(x_k + y_k) = S + T$.

Proof. Let $S_n = \sum_{k=1}^{n} x_k$ and $T_n = \sum_{k=1}^{n} y_k$. So we have $S_n \to S$ and $T_n \to T$ as $n \to \infty$. Moreover, let $U_n = \sum_{k=1}^{n}(x_k + y_k)$. We must show that $(U_n)_{n=1}^{\infty}$ converges to $S + T$. But

$$U_n = \sum_{k=1}^{n}(x_k + y_k) = \sum_{k=1}^{n} x_k + \sum_{k=1}^{n} y_k = S_n + T_n$$

by commutativity of addition in $\mathbb{R}$. So $(U_n)_{n=1}^{\infty}$ is the sequence with $U_n = S_n + T_n$ and hence by Theorem 3.24(ii) (U_n) converges to $S+T$. Hence $\sum_{k=1}^{\infty}(x_k+y_k)$ exists and is equal to $S + T$. □

Lemma 4.6
If $\sum_{k=1}^{\infty} x_k = S$ and $c \in \mathbb{R}$ then $\sum_{k=1}^{\infty} cx_k = cS$.

Proof. This is very similar to the proof of the lemma above. It is a question on the Problem Sheet. □

Remark 4.7
The previous two lemmas imply that the set of convergent series forms a vector space (details are left as an exercise). However, this set does not necessarily form a ring, as products of infinite series are not straightforward. It is very rarely true that

$$\sum_{k=1}^{\infty} x_k y_k = \Big(\sum_{k=1}^{\infty} x_k\Big)\Big(\sum_{k=1}^{\infty} y_k\Big)$$

since such an identity fails even for finite sums!

Our last easy convergence criterion is similar to the dominated convergence for sequences.

Theorem 4.8 (Comparison test)
Suppose $(x_k)_{k=1}^{\infty}$ and $(y_k)_{k=1}^{\infty}$ are sequences of real numbers such that $0 \le y_k \le x_k$ for all $k \in \mathbb{N}$. Then if $\sum_{k=1}^{\infty} x_k$ exists, it follows that $\sum_{k=1}^{\infty} y_k$ exists. Moreover, $\sum_{k=1}^{\infty} y_k \le \sum_{k=1}^{\infty} x_k$.

Remark 4.9
We require here that both sequences (x_k) and (y_k) consist only of *non-negative*

elements!

Proof. Write $S_n = \sum_{k=1}^n x_k$ and $T_n = \sum_{k=1}^n y_k$. We know that $(S_n)_{n=1}^\infty$ converges (to some real number $S \in \mathbb{R}$). We want to prove that $(T_n)_{n=1}^\infty$ also converges (to some real number $T \leq S$). But because $(T_n)_{n=1}^\infty$ is a *increasing sequence* (since $y_k \geq 0$ for all k), we know that it converges if we can show that it is bounded above (by Theorem 3.32 we know that bounded monotonic sequences converge).

But we have $T_n \leq S_n$ for all $n \in \mathbb{N}$ as $y_k \leq x_k$ for all $k \in \mathbb{N}$. Moreover, as $(S_n)_{n=1}^\infty$ is increasing and converges to S, we have $S_n \leq S$ for all $n \in \mathbb{N}$. Hence $T_n \leq S_n \leq S$ for all $n \in \mathbb{N}$, i.e. (T_n) is bounded above. So we see that (T_n) converges to some T. We want to finally show that $T \leq S$, but this follows directly from Lemma 3.27. □

4.C Specific series and more examples

We first have a look at some specific series: geometric series and the harmonic series. Then, we construct more examples using our old examples from before, these specific series and the rules from the last section.

Geometric series: We first prove a lemma for the partial sums of a geometric series.

Lemma 4.10
We have $S_n = \sum_{k=1}^n ar^{k-1} = \frac{a(r^n-1)}{r-1}$ if $r \neq 1$.

Proof. We have

$$S_n = \sum_{k=1}^n ar^{k-1} = a + ar + ar^2 + \ldots ar^{n-1}.$$

Multiplying by r yields

$$rS_n = r\sum_{k=1}^n ar^{k-1} = ar + ar^2 + \ldots ar^{n-1} + ar^n.$$

Subtracting the first equation from the second gives a telescoping sum:

$$(r-1)S_n = ar^n - a = a(r^n - 1),$$

and hence if $r \neq 1$, we have

$$S_n = \frac{a(r^n-1)}{r-1} = \frac{a(1-r^n)}{1-r}.$$

□

Theorem 4.11 (Geometric series)
Suppose $a \in \mathbb{R}$, $a \neq 0$ and $r \in \mathbb{R}$. Let $(x_k)_{k=1}^\infty$ be given by $x_k = ar^{k-1}$. Then

(i) if $|r| < 1$, then $\sum_{k=1}^{\infty} x_k$ exists and equals $\frac{a}{1-r}$.

(ii) if $|r| \geq 1$, then $\sum_{k=1}^{\infty} x_k$ does not exist.

Proof.

(i) For the partial sums, we have by Lemma 4.10 that $S_n = \sum_{k=1}^{n} x_k = \frac{a(1-r^n)}{1-r}$, so if $|r| < 1$ we deduce that $\lim_{n\to\infty} S_n$ exists and is equal to $\frac{a}{1-r}$ (since by Theorem 3.14, $r^n \to 0$ as $n \to \infty$ and thus by Theorem 3.24(ii) $S_n \to \frac{a(1-0)}{1-r} = \frac{a}{1-r}$). Since $\sum_{k=1}^{\infty} x_k$ is *defined* to be $\lim_{n\to\infty} S_n$, we have proved that $\sum_{k=1}^{\infty} x_k$ exists and equals $\frac{a}{1-r}$.

(ii) If $|r| \geq 1$, then $|x_k| = |ar^{k-1}| = |a||r|^{k-1} \geq |a|$. So (x_k) does not converge to zero as k goes to ∞ and therefore $\sum_{k=1}^{\infty} x_k$ does not exist (by Theorem 4.3). □

Harmonic series: As already seen in the introduction, the harmonic series does not converge.

Theorem 4.12 (Harmonic series)
We have $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$.

Proof. We estimate

$$S_{2^m} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>\frac{1}{2}} + \ldots + \frac{1}{2^m} \geq 1 + m \cdot \frac{1}{2}.$$

We want to prove that S_n tends to ∞, i.e., we want to show that

$$\forall K \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n > N : S_n > K \qquad (*)$$

For $K \leq 0$, $S_n > K$ is always true. So we assume that we are given $K > 0$ (by the Demon). Choose $m \in \mathbb{N}$ with $m \geq 2K - 2$ and set $N = 2^m$. Then for $n > N$ we have

$$S_n > S_N = S_{2^m} \geq 1 + \frac{m}{2} \geq K.$$

□

The series $\sum_{k=1}^{\infty} \frac{1}{k^2}$: This series exists!

Theorem 4.13
The series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ exists.

Proof. For $k \in \mathbb{N}$ we have $k \geq \frac{k+1}{2}$ and therefore $k^2 \geq k\frac{k+1}{2}$. Hence

$$0 \leq \frac{1}{k^2} \leq \frac{2}{k(k+1)}.$$

But $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ and we have seen in Example 3.2 that $\sum_{k=1}^{\infty}(\frac{1}{k} - \frac{1}{k+1})$ exists. Hence $\sum_{k=1}^{\infty} \frac{2}{k(k+1)}$ exists (by Lemma 4.6) and hence $\sum_{k=1}^{\infty} \frac{1}{k^2}$ exists by the comparison test, Theorem 4.8. □

Remark 4.14
This theorem tells us that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ exists, but it does *not* tell us the value of this infinite sum, only that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \leq \sum_{k=1}^{\infty} \frac{2}{k(k+1)} = 2.$$

In fact, one can show that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. This is best proved using complex analysis.

Remark 4.15
One can prove that $\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ exists if and only if $\alpha > 1$.

Examples 4.16 (i) Does the sum $\sum_{k=1}^{\infty} \frac{1}{2^k+k}$ exist?

We have $0 \leq \frac{1}{2^k+k} \leq \frac{1}{2^k}$ and we know that $\sum_{k=1}^{\infty} \frac{1}{2^k}$ exists (by Theorem 4.11). Hence by the comparison test, Theorem 4.8, $\sum_{k=1}^{\infty} \frac{1}{2^k+k}$ exists.

(ii) We have seen before that $\sum_{k=1}^{\infty} \frac{1}{k!}$ exists. We can give an easier proof of this, using that $0 \leq \frac{1}{k!} \leq \frac{1}{2^{k-1}}$ for all $k \in \mathbb{N}$. As $\sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$ exists (by Theorem 4.11), we obtain from the comparison test that $\sum_{k=1}^{\infty} \frac{1}{k!}$ exists.

(iii) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ does not exist according to Example 3.2. We can give a shorter proof of this fact: As $0 \leq \frac{1}{k} \leq \frac{1}{\sqrt{k}}$, if $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ would exist, then by the comparison test also $\sum_{k=1}^{\infty} \frac{1}{k}$ would exist. But Theorem 4.12 tells us that this is not the case, so $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ does not exist.

What about series where some of the terms are negative?

4.D Absolute convergence

Definition 4.17
We say that $\sum_{k=1}^{\infty} x_k$ *converges absolutely* if $\sum_{k=1}^{\infty} |x_k|$ exists.

Theorem 4.18 (Absolute convergence implies convergence)
If $\sum_{k=1}^{\infty} |x_k|$ exists then $\sum_{k=1}^{\infty} x_k$ exists.

Proof. Given a sequence $(x_k)_{k=1}^{\infty}$ in $\mathbb{R}$, define

$$x_k^+ := \begin{cases} x_k & \text{if } x_k \geq 0, \\ 0 & \text{if } x_k < 0, \end{cases}$$

and

$$x_k^- := \begin{cases} -x_k & \text{if } x_k \leq 0, \\ 0 & \text{if } x_k > 0. \end{cases}$$

We observe that

(i) $x_k = x_k^+ - x_k^-$ for all $k \in \mathbb{N}$,

(ii) $0 \leq x_k^+ \leq |x_k|$ for all $k \in \mathbb{N}$,

(iii) $0 \leq x_k^- \leq |x_k|$ for all $k \in \mathbb{N}$.

Therefore, by the comparison test, Theorem 4.8, $\sum_{k=1}^{\infty} x_k^+$ exists and $\sum_{k=1}^{\infty} x_k^-$ exists. Thus, by Lemma 4.5, $\sum_{k=1}^{\infty} x_k = \sum_{k=1}^{\infty} (x_k^+ - x_k^-)$ exists. □

Example 4.19
Let us look at the series

$$\sum_{k=1}^{\infty} \frac{\sin k}{2^k}.$$

Does it exist? We notice that this series converges absolutely, since $|\frac{\sin(k)}{2^k}| \leq \frac{1}{2^k}$ for all $k \in \mathbb{N}$ and we know that $\sum_{k=1}^{\infty} \frac{1}{2^k}$ exists, so $\sum_{k=1}^{\infty} |\frac{\sin k}{2^k}|$ exists by the comparison test. Theorem 4.18 then shows that $\sum_{k=1}^{\infty} \frac{\sin k}{2^k}$ exists.

One can "re-order" an absolutely convergent series.

Theorem 4.20 (Re-ordering an absolutely convergent series)
Let $\sum_{k=1}^{\infty} x_k$ be absolutely convergent and $\phi : \mathbb{N} \to \mathbb{N}$ a bijection. Then the

re-ordered series $\sum_{k=1}^{\infty} x_{\phi(k)}$ exists and

$$\sum_{k=1}^{\infty} x_k = \sum_{k=1}^{\infty} x_{\phi(k)}.$$

Proof. (The proof of this theorem is non-examinable.)

We aim to use the Cauchy criterion (Theorem 4.4) to show existence of $\sum_{k=1}^{\infty} x_{\phi(k)}$; that is, we want to show that

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall m > n > N \ : \ \Big| \sum_{k=n+1}^{m} x_{\phi(k)} \Big| < \varepsilon.$$

Let $S_n = \sum_{k=1}^{n} |x_k|$ be the partial sum. Let $\varepsilon > 0$ be given (by the Demon). Then as $(S_n)_{n=1}^{\infty}$ is increasing and converges to $S = \sum_{k=1}^{\infty} |x_k|$, we can find $N_1 \in \mathbb{N}$ such that $|S_{N_1} - S| = S - S_{N_1} < \varepsilon$. That is,

$$\sum_{k=N_1+1}^{\infty} |x_k| = \sum_{k=1}^{\infty} |x_k| - \sum_{k=1}^{N_1} |x_k| < \varepsilon. \tag{4.1}$$

Now set $N := \max\{\phi^{-1}(1), \ldots, \phi^{-1}(N_1)\}$. For any $k > N$ we have $k \notin \{\phi^{-1}(1), \ldots, \phi^{-1}(N_1)\}$ and hence $\phi(k) \notin \{1, \ldots, N_1\}$, where we have used the fact that ϕ is injective. In other words, $k > N$ implies $\phi(k) > N_1$. Therefore, with this choice of N, we find

$$\forall m > n \geq N \ : \ \Big| \sum_{k=n+1}^{m} x_{\phi(k)} \Big| \leq \sum_{k=n+1}^{m} \big| x_{\phi(k)} \big| \leq \sum_{k=N_1+1}^{\infty} |x_k| < \varepsilon.$$

Thus, by the Cauchy criterion, the re-ordered series exists. Notice that by setting $n = N$ above, we have that $\sum_{k=N+1}^{m} \big| x_{\phi(k)} \big| < \varepsilon$, for all $m > N$, and so

$$\sum_{k=N+1}^{\infty} \big| x_{\phi(k)} \big| \leq \varepsilon. \tag{4.2}$$

We now show that the rearranged series converges to the same value as the original series. We have:

$$\begin{aligned}
\Big| \sum_{k=1}^{\infty} x_k - \sum_{k=1}^{\infty} x_{\phi(k)} \Big| &= \Big| \sum_{k=1}^{N_1} x_k + \sum_{k=N_1+1}^{\infty} x_k - \sum_{k=1}^{N} x_{\phi(k)} - \sum_{k=N+1}^{\infty} x_{\phi(k)} \Big| \\
&\leq \Big| \sum_{k=1}^{N_1} x_k - \sum_{k=1}^{N} x_{\phi(k)} \Big| + \Big| \sum_{k=N_1+1}^{\infty} x_k \Big| + \Big| \sum_{k=N+1}^{\infty} x_{\phi(k)} \Big| \\
&\leq \Big| \sum_{k=1}^{N_1} x_k - \sum_{k=1}^{N} x_{\phi(k)} \Big| + \sum_{k=N_1+1}^{\infty} |x_k| + \sum_{k=N+1}^{\infty} \big| x_{\phi(k)} \big| \\
&\leq 3\varepsilon.
\end{aligned}$$

The steps in this estimate are justified as follows. In the second and third lines we have used the triangle inequality. In the final step we have used (4.1) and (4.2) to bound the second and third terms; also, the first term is a finite sum of terms taken from the series $\sum_{k=N_1+1}^{\infty} x_k$ and is hence bounded by ε.

Since the above estimate holds all $\varepsilon > 0$, we conclude that $\sum_{k=1}^{\infty} x_k = \sum_{k=1}^{\infty} x_{\phi(k)}$. □

What about series that do exist but do not converge absolutely?

Definition 4.21
A series $\sum_{k=1}^{\infty} x_k$ is *conditionally convergent* if it converges but does not converge absolutely.

Let us look at an example, namely the alternating harmonic series:

Theorem 4.22 (The alternating harmonic series exists)
The series

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

is conditionally convergent.

Proof. We have seen in Theorem 4.12 that the series does not converge absolutely. We thus simply need to show that the series converges. For $n, m \in \mathbb{N}$ with $m > n$, define

$$S_{n,m} = \sum_{k=n+1}^{m} (-1)^{k+1}\frac{1}{k}.$$

If we can show that $S_{n,m}$ is small when n, m are large, then the Cauchy Criterion will tell us that the infinite sum $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ exists.

If n is even then $S_{n,m} = \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \dots \pm \frac{1}{m}$. Grouping the terms in pairs,

$$S_{n,m} = \left(\tfrac{1}{n+1} - \tfrac{1}{n+2}\right) + \left(\tfrac{1}{n+3} - \tfrac{1}{n+4}\right) + \dots + \left(\tfrac{1}{m-1} - \tfrac{1}{m}\right),$$

assuming m is even. (If m is odd then there is an unpaired term $\frac{1}{m}$ at the end.) The terms are all positive, and hence $S_{n,m} \geq 0$. Grouping into pairs in the other possible way,

$$S_{n,m} = \tfrac{1}{n+1} - \left(\tfrac{1}{n+2} - \tfrac{1}{n+3}\right) - \dots - \left(\tfrac{1}{m-1} - \tfrac{1}{m}\right),$$

where we have assumed m is odd. (There is an unpaired term $-\frac{1}{m}$ at the end if m is even.) All terms other than the first are negative, and so $S_{n,m} \leq \frac{1}{n+1}$.

A similar argument for n odd gives $-\frac{1}{n+1} \leq S_{n,m} \leq 0$. Summarising, in all cases $|S_{n,m}| \leq \frac{1}{n+1}$.

Now apply Theorem 4.4 (the Cauchy Criterion). Given $\varepsilon > 0$, let $N = \lceil 1/\varepsilon \rceil$. Then, for all $m > n > N$ we have $|S_{n,m}| \leq \frac{1}{n+1} < 1/N \leq \varepsilon$. Hence the infinite sum $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ exists. □

Remark 4.23
One can prove that $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = \ln 2$, the natural logarithm of 2, but we will not do this in this course.

Now let us re-order the series with some bijective $\phi : \mathbb{N} \to \mathbb{N}$ which is chosen exactly in such a way that the sum $\sum_{k=1}^{\infty} x_{\phi(k)}$ is given by

$$\sum_{k=1}^{\infty} x_{\phi(k)} = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \dots.$$

(That is $\phi(1) = 1$, $\phi(2) = 2$, $\phi(3) = 4$, $\phi(4) = 3$, $\phi(5) = 6$, $\phi(6) = 8$, $\phi(7) = 5$, $\phi(8) = 10$, $\phi(9) = 12$, etc.) As seen in the introduction, we obtain

$$\begin{aligned}\sum_{k=1}^{\infty} x_{\phi(k)} &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \dots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \dots = \frac{1}{2}\ln 2.\end{aligned}$$

So the value of the re-ordered sum is half the value of the original alternating harmonic series! This might seem surprising, but we did *not* make a mistake! In fact, we have the following general theorem.

Theorem 4.24 (Riemann rearrangement theorem)
Let $\sum_{k=1}^{\infty} x_k$ be a conditionally convergent series. Then for any $L \in \mathbb{R}$ there exists a bijection $\phi_L : \mathbb{N} \to \mathbb{N}$ such that $\sum_{k=1}^{\infty} x_{\phi_L(k)}$ exists and has value L. Moreover, there also exists a bijection $\phi_\infty : \mathbb{N} \to \mathbb{N}$ such that $\sum_{k=1}^{\infty} x_{\phi_\infty(k)} = \infty$.

Proof. (The proof of this theorem is non-examinable.)

Step 1: We first prove that $\sum_{k=1}^{\infty} x_k^+ = \infty$ and $\sum_{k=1}^{\infty} x_k^- = \infty$. (Recall that x_k^+ denotes x_k when $x_k \geq 0$ and 0 otherwise, and that x_k^- denotes $-x_k$ when $x_k \leq 0$ and 0 otherwise.)

Since $\sum_{k=1}^{\infty} x_k^+$ and $\sum_{k=1}^{\infty} x_k^-$ are each sums of positive terms, each of these sums either converges (to an element of $\mathbb{R}$) or sums to ∞.

- It cannot be true that both $\sum_{k=1}^{\infty} x_k^+$ and $\sum_{k=1}^{\infty} x_k^-$ converge to elements of $\mathbb{R}$ because then $\sum_{k=1}^{\infty}|x_k| = \sum_{k=1}^{\infty}(x_k^+ + x_k^-)$ would converge to an element of $\mathbb{R}$.

- It cannot be true that $\sum_{k=1}^{\infty} x_k^+$ converges and $\sum_{k=1}^{\infty} x_k^- = \infty$, because this would imply that $\sum_{k=1}^{\infty} x_k = \sum_{k=1}^{\infty}(x_k^+ - x_k^-) = -\infty$.
- Similarly, it cannot be true that $\sum_{k=1}^{\infty} x_k^-$ converges and $\sum_{k=1}^{\infty} x_k^+ = \infty$, as then $\sum_{k=1}^{\infty} x_k = \infty$.

Hence both $\sum_{k=1}^{\infty} x_k^+ = \infty$ and $\sum_{k=1}^{\infty} x_k^- = \infty$.

Step 2: Assume we are given $L \in \mathbb{R}$, we construct ϕ_L.

It is easier to explain what to do if we first change the x_k^+, x_k^- notation to something more convenient. Suppose the x_k which are ≥ 0 are, in increasing order of their numbering, $y_1, y_2, y_3, \ldots$ and the x_k's which are < 0 are $z_1, z_2, z_3, \ldots$. We re-order the x_k's as follows. First we take just enough of the y_k's to make their sum $> L$ (we can do this since the sum of all of them is infinity by Step 1). So we take $y_1, y_2, \ldots, y_{i_1}$ say. Next we take just enough of the z_k's to bring the sum down below L, say $z_1, z_2, \ldots, z_{j_1}$. Then, we take just enough more of the y_k's to make the sum $> L$ again, starting with the first y_k we have not used yet, so we take $y_{i_1+1}, \ldots, y_{i_2}$ for some i_2. Then we take just enough more of the z_k's to make the sum $< L$ again, say $z_{j_1+1}, \ldots, z_{j_2}$. And so on. Each time we use up a finite number (≥ 1) more of the y_k's or z_k's, so they all get used eventually, so this process will really define a bijection $\phi_L : \mathbb{N} \to \mathbb{N}$.

Step 3: We show that the re-ordered series satisfies $\sum_{k=1}^{\infty} x_{\phi_L(k)} = L$.

Since at each stage we take just enough terms to move the sum to the other side of L, we know that after the first stage the partial sum is within $|y_{i_1}|$ of L, after the second stage it is within $|z_{j_1}|$ of L and so on. But since the sequence $(x_k)_{k=1}^{\infty}$ tends to zero (as $\sum_{k=1}^{\infty} x_k$ converges) so do its subsequences $(y_{i_m})_{m=1}^{\infty}$ and $(z_{j_m})_{m=1}^{\infty}$. Hence the partial sums of our re-ordered sequence converge to L.

Similarly, one can construct ϕ_∞ by taking just enough positive elements to make the partial sum larger than 1, then take one negative element. Then, continue with just enough positive elements to make the partial sum larger than 2 before taking again one negative term. Next, we take enough positive terms to make the partial sum larger than 3, etc. (The details are left as an exercise.) □

4.E Ratio test

The following test is extremely useful in practice.

Theorem 4.25 (Ratio test for positive series)
Suppose that $x_k > 0$ for all $k \in \mathbb{N}$. Consider the sequence $(x_{k+1}/x_k)_{k=1}^{\infty}$ and suppose that $(x_{k+1}/x_k) \to R$ for some $R \in \mathbb{R} \cup \{\infty\}$.

(i) If $R < 1$ then the series $\sum_{k=1}^{\infty} x_k$ exists.

(ii) if $R > 1$ or $R = \infty$ then $\sum_{k=1}^{\infty} x_k$ does not exist.

If $R = 1$ then we are out of luck, and cannot conclude anything.

Proof. Start with (i), assume that $R < 1$. We first show that $x_k \to 0$.

Choose an $\varepsilon_1 > 0$ such that $R + \varepsilon_1 < 1$. Since $x_{k+1}/x_k \to R$ there exists $N_1 \in \mathbb{N}$ such that for $k > N_1$ we have

$$|x_{k+1}/x_k - R| < \varepsilon_1.$$

In particular we have $x_{k+1}/x_k < R + \varepsilon_1$ and since $x_k > 0$ this implies

$$x_{k+1} < x_k(R + \varepsilon_1). \tag{4.3}$$

Inductively, we see that for $k \geq 2$ we have

$$|x_k| = x_k < x_1(R + \varepsilon_1)^{k-1}.$$

Since $R+\varepsilon_1 < 1$ we have $(R+\varepsilon_1)^{k-1} \to 0$ (Theorem 3.14) and hence from dominated convergence (Lemma 3.4) we see that $x_k \to 0$.

We now show that $\sum_{k=1}^{\infty} x_k$ exists, using the Cauchy criterion (Theorem 4.4). Fix $\varepsilon > 0$. For $m > n > N_1$ we have

$$\left|\sum_{k=n+1}^{m} x_k\right| = \sum_{k=n+1}^{m} x_k < \sum_{k=n+1}^{m} x_{n+1}(R + \varepsilon_1)^{k-(n+1)} = x_{n+1} \cdot \sum_{k=1}^{m-n} (R + \varepsilon_1)^{k-1}$$

where the inequality follows from (4.3). The latter sum is bounded above by the full geometric series

$$\sum_{k=1}^{m-n} (R + \varepsilon_1)^{k-1} \leq \sum_{k=1}^{\infty} (R + \varepsilon_1)^{k-1} =: S$$

and we note that $S > 0$. Since $x_k \to 0$ there exists $N_2 \in N$ such that for $k > N_2$:

$$|x_k| < \varepsilon/S.$$

Let $N = \max\{N_1, N_2\}$. Then for $m > n > N$ we have

$$\left|\sum_{k=n+1}^{m} x_k\right| < x_{n+1} \cdot \sum_{k=1}^{m-n} (R + \varepsilon_1)^{k-1} < \frac{\varepsilon}{S} \cdot S = \varepsilon.$$

We conclude from the Cauchy criterion that $\sum_{k=1}^{\infty} x_k$ exists. This completes the proof of (i).

Turning to (ii), assume that $R > 1$. Choose $\varepsilon_1 > 0$ such that $R - \varepsilon_1 > 1$. Since $x_{k+1}/x_k \to R$ there exists $N \in \mathbb{N}$ such that for $k > N$ we have

$$|x_{k+1}/x_k - R| < \varepsilon_1.$$

In particular we have $x_{k+1}/x_k > R - \varepsilon_1$ and since $x_k > 0$ this implies

$$x_{k+1} > x_k(R - \varepsilon_1).$$

Inductively, we see that for $k \geq 2$ we have

$$|x_k| = x_k > x_1(R - \varepsilon_1)^{k-1}. \tag{4.4}$$

Since $R - \varepsilon_1 > 1$ we have by Theorem 3.14 that the sequence $((R - \varepsilon_1)^{k-1})_{k=1}^{\infty}$ diverges. By Corollary 3.5 we conclude that $(x_k)_{k=1}^{\infty}$ diverges. It follows that $\sum_{k=1}^{\infty} x_k$ does not exist (Lemma 4.5).

The case $R = \infty$ is similar and left as an exercise. □

Theorem 4.26 (Ratio test for never-zero series)
Suppose that $x_k \neq 0$ for all $k \in \mathbb{N}$. Suppose that $|x_{k+1}|/|x_k| \to R$ for some $R \in \mathbb{R} \cup \{\infty\}$.

(i) If $R < 1$ then the series $\sum_{k=1}^{\infty} x_k$ is absolutely convergent.

(ii) If $R > 1$ or $R = \infty$ then the series $\sum_{k=1}^{\infty} x_k$ does not exist.

Proof. Part (i) follows directly from Theorem 4.25 applied to the series $\sum_{k=1}^{\infty} |x_k|$.

For part (ii), suppose for a contradiction that $\sum_{k=1}^{\infty} x_k$ exists. Then $x_k \to 0$ which implies $|x_k| \to 0$. But similar to (4.4) above, we have

$$|x_k| > |x_1|(R - \varepsilon_1)^{k-1}$$

with $R - \varepsilon_1 > 1$. This implies $|x_k| \to \infty$, a contradiction. □

Example 4.27
Consider the series

$$\sum_{k=1}^{\infty} \frac{2^k + 3^k}{3^k + 4^k}.$$

We will prove this converges using the ratio test. Letting $x_k = (2^k + 3^k)/(3^k + 4^k)$

we have

$$\begin{aligned}\frac{x_{k+1}}{x_k} &= \frac{2^{k+1}+3^{k+1}}{3^{k+1}+4^{k+1}} \cdot \frac{3^k+4^k}{2^k+3^k} \\ &= \frac{2^{k+1}+3^{k+1}}{2^k+3^k} \cdot \frac{3^k+4^k}{3^{k+1}+4^{k+1}} \\ &= \frac{(\frac{2}{3})^{k+1}+1}{\frac{1}{3}((\frac{2}{3})^k+1)} \cdot \frac{\frac{1}{4}((\frac{3}{4})^k+1)}{(\frac{3}{4})^k+1} \\ &\to \frac{1}{(1/3)} \cdot \frac{(1/4)}{1} = \frac{3}{4} < 1.\end{aligned}$$

We conclude from the ratio test that the series converges. Similar examples can be found on Problem Sheet 5.

4.F Power series

Consider the series

$$\sum_{k=0}^{\infty} x^k$$

for $x \in \mathbb{R}$ fixed. We know that this converges when $|x| < 1$ and that it does not converge when $|x| \geq 1$ (Theorem 4.11). So we define a function

$$\begin{aligned} f\colon (-1,+1) &\to \mathbb{R} \\ f(x) &= \sum_{k=0}^{\infty} x^k. \end{aligned}$$

Theorem 4.11 gives us an alternative formula for $f(x)$, namely $f(x) = \frac{1}{1-x}$, but note that while $\frac{1}{1-x}$ makes sense for all $x \in \mathbb{R}$ except $x = 1$, the infinite sum $\sum_{k=0}^{\infty} x^k$ only makes sense for $-1 < x < +1$.

Similarly, we can define the *exponential function* $\exp(x)$ by

$$\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

where this series exists. For what values $x \in \mathbb{R}$ does $\exp(x)$ exist? We know it exists for $x = 1$, because we proved earlier that $\sum_{k=0}^{\infty} \frac{1}{k!}$ exists (we called this sum "e").

So $\exp(x)$ exists for all $0 \leq x \leq 1$ by the comparison test, Theorem 4.8. It follows that $\exp(x)$ converges absolutely for all x with $|x| \leq 1$ and hence $\exp(x)$ exists for these values of x by Theorem 4.18 ("absolute convergence implies convergence").

Remark 4.28
One of the Problem Sheet questions shows that $\exp(x)$ exists for $x = 2$. By the above argument, it therefore exists for all x with $|x| \leq 2$.

We can do better, and prove that $\exp(x)$ exists for all $x \in \mathbb{R}$. It will suffice to prove this for all x with $x \geq 0$ (as absolute convergence implies convergence).

Theorem 4.29
$\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}$ exists for all $x \in \mathbb{R}$.

Proof. As mentioned above, we only need to consider $x \geq 0$. For M with $1 \leq M \leq k$ we have

$$k! = \underbrace{k \cdot (k-1) \cdot (k-2) \cdot \ldots \cdot (M+1)}_{k-M \text{ terms}} \cdot M \ldots 3 \cdot 2 \cdot 1 \geq M^{k-M}.$$

In fact, $k! \geq M^{k-M}$ for *all* $M \in \mathbb{N}$ (since if $M > k$ then $M^{k-M} < 1$).

Now given $x \in \mathbb{R}$ with $x \geq 0$, pick $M \in \mathbb{N}$ with $M \geq 2x$. Then

$$\frac{x^k}{k!} \leq \frac{x^k}{M^{k-M}} = M^M \left(\frac{x}{M}\right)^k \leq M^M \left(\frac{1}{2}\right)^k.$$

So $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges by the comparison test, Theorem 4.8 □

This shows that exp is a well-defined function from $\mathbb{R}$ to $\mathbb{R}$. Similarly, we can define

$$\sin(x) := \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$
$$\cos(x) := 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

for all values of $x \in \mathbb{R}$ for which these sums exist. For what values of $x \in \mathbb{R}$ are these defined?

The series for $\sin(x)$ converges absolutely for all $x \in \mathbb{R}$ by the comparison test. To see this, write the series for $\sin(x)$ as

$$\sin(x) = 0 + \frac{x}{1!} + 0x^2 - \frac{x^3}{3!} + 0x^4 + \frac{x^5}{5!} + 0x^6 - \frac{x^7}{7!} + \ldots$$

and note that every term in the series is less than or equal in absolute value to the corresponding term in the series for $\exp(x)$. But we have just proved that $\exp(x)$ exists for all x. Hence by Theorem 4.18 the series for $\sin(x)$ exists for all $x \in \mathbb{R}$. Similarly, the series for $\cos(x)$ exists for all $x \in \mathbb{R}$.

In general, we have the following result for a power series (that is a series of the form $\sum_{k=0}^{\infty} a_k x^k$).

Theorem 4.30 (Radius of convergence)
Given any power series $\sum_{k=0}^{\infty} a_k x^k$ (with coefficients $a_k \in \mathbb{R}$), exactly one of the following three statements is true:

(i) $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely for all $x \in \mathbb{R}$.

(ii) $\sum_{k=0}^{\infty} a_k x^k$ exists only for $x = 0$.

(iii) $\exists R > 0$ (called the *radius of convergence*) such that:

  (a) $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely for all x with $|x| < R$.
  (b) $\sum_{k=0}^{\infty} a_k x^k$ does not converge for any x with $|x| > R$.

  We cannot say what happens when $|x| = R$: depending on the value of x, the series may or may not converge.

Examples 4.31 (i) $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges absolutely for all $x \in \mathbb{R}$.

(ii) $\sum_{k=0}^{\infty}(k!)x^k$ only converges for $x = 0$.

(iii) $\sum_{k=0}^{\infty} x^k$ has radius of convergence $R = 1$.

(iv) More examples are given on the Problem Sheets.

Proof of Theorem 4.30. It is sufficient to prove the following statement

> $(*)$ If X is a real number such that the series converges for $x = X$, then the series converges absolutely for all $Y \in \mathbb{R}$ with $|Y| < |X|$.

Why is this sufficient? We set

$$S := \Big\{|X| \ : \ X \in \mathbb{R} \text{ and } \sum_{k=0}^{\infty} a_k x^k \text{ converges for } x = X\Big\}.$$

- If S has no upper bound, then given $Y \in \mathbb{R}$, we can find X such that $|X| \in S$ and $|X| > |Y|$. Then $(*)$ implies we are in case (i).
- If S has an upper bound, then it has a least upper bound R (by the completeness action). If $R = 0$ then $S = \{0\}$ and we are in case (ii).
- Finally if $R > 0$, given $Y \in \mathbb{R}$ with $|Y| < R$, there exists $X \in \mathbb{R}$ with $|X| \in S$ and $|X| > |Y|$. Then $(*)$ implies that we are in case (iii).

It remains to prove $(*)$: Assume $\sum_{k=0}^{\infty} a_k X^k$ converges and $|Y| < |X|$. Since $\sum_{k=0}^{\infty} a_k X^k$ converges, the terms $a_k X^k$ converge to zero and hence there exists $M \in \mathbb{R}$ such that $|a_k X^k| < M$ for all $k \in \mathbb{N}$ (by Example 3.20). Therefore

$$|a_k Y^k| = |a_k| \cdot |X|^k \cdot \left|\frac{Y}{X}\right|^k < M \cdot \left|\frac{Y}{X}\right|^k$$

and as $|\frac{Y}{X}| < 1$ the sum $\sum_{k=0}^{\infty} |\frac{Y}{X}|^k$ exists (Theorem 4.11). Hence by the comparison test (Theorem 4.8), $\sum_{k=0}^{\infty} |a_k Y^k|$ exists. This proves $(*)$ and hence the theorem. □

5 Continuity

A function from $\mathbb{R}$ to $\mathbb{R}$ assigns a real number $f(x)$ to each real number x. There doesn't have to be a single rule for this assignment. We think of f as a "black box" into which we input a real number x and we get a real number $f(x)$ as output. We only insist that if we input the same number on more than one occasion, we get the same answer. If so, we say we have a "well-defined function".

Possible examples include simple functions such as $f(x) = x$, $f(x) = 5$ $(\forall x \in \mathbb{R})$ or $f(x) = \sin(x^2)$, but also more complicated looking ones, for example

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q}, \\ 1 & \text{if } x \in \mathbb{Q}, \end{cases}$$

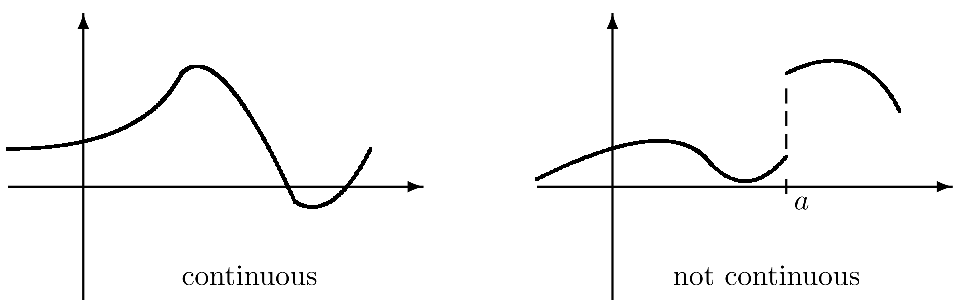
or

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \text{ or if } x = 0, \\ 1/q & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ in lowest terms, with } p > 0. \end{cases}$$

For the latter examples, it is hard to draw the graphs, as they look quite "wild". The goal of this section is to study continuity of functions and some of the properties of continuous functions.

5.A Definition and first examples

We shall formalize the idea that a function is continuous if one can draw its graph "without lifting the pen off the paper".

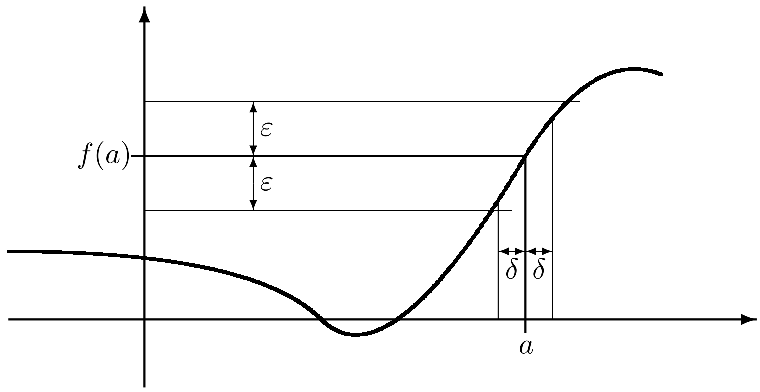


Having this picture in mind, we can now give a precise definition of continuity at some point $a \in \mathbb{R}$.

Definition 5.1 (Continuous functions)
Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function and $a \in \mathbb{R}$. We say that f is *continuous at a* if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall h \in \mathbb{R}, |h| < \delta \; : \; |f(a+h) - f(a)| < \varepsilon. \tag{5.1}$$

We say that f is *continuous* if it is continuous at a for all $a \in \mathbb{R}$.



Examples 5.2 (i) $f(x) = x$ is continuous at 0.

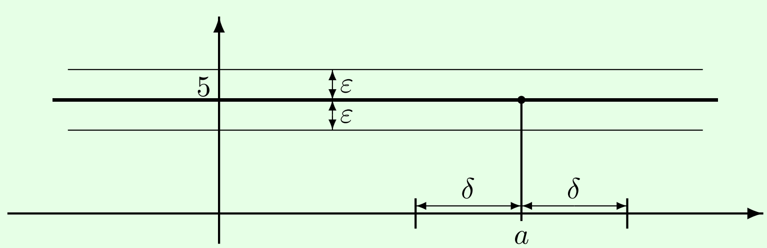
Winning strategy for the Demon Game. Suppose the Demon picks $\varepsilon > 0$. We then have to pick δ. Let us simply assume we have done this and continue with the game. The Demon then picks h with $|h| < \delta$. We win if $|f(0+h) - f(0)| < \varepsilon$. But $|f(0+h) - f(0)| = |h - 0| = |h| < \delta$, which suggests that we could have simply chosen $\delta = \varepsilon$ as our winning strategy! So we can turn this into a formal proof.

Proof. Given $\varepsilon > 0$, let $\delta = \varepsilon$. Then for all $h \in \mathbb{R}$ with $|h| < \delta$, we have

$$|f(0+h) - f(0)| = |h - 0| = |h| < \delta = \varepsilon,$$

so f is continuous at 0. □

(ii) The constant function $f(x) = 5$ is continuous at all points $a \in \mathbb{R}$.



Proof. Given $a \in \mathbb{R}$ and $\varepsilon > 0$, let δ be *any* positive real number, e.g. $\delta = 1$.

Then for all $h \in \mathbb{R}$ with $|h| < \delta$ we have

$$|f(a+h) - f(a)| = |5 - 5| = 0 < \varepsilon,$$

so f is continuous at a. □

(iii) $f(x) = 3x$ is continuous at all points $a \in \mathbb{R}$.

Proof. Given $a \in \mathbb{R}$ and $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{3}$. Then for any $h \in \mathbb{R}$ with $|h| < \delta$ we have

$$|f(a+h) - f(a)| = |3(a+h) - 3a| = |3h| < 3\delta = \varepsilon,$$

so f is continuous at a. □

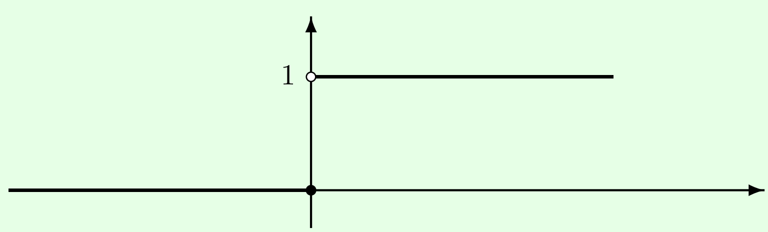
How do we prove that a function is *not* continuous at some $a \in \mathbb{R}$? We have to show the negation of (5.1), namely

$$\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists h \in \mathbb{R}, |h| < \delta \ : \ |f(a+h) - f(a)| \geq \varepsilon. \tag{5.2}$$

Example 5.3
Let $f(x)$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

We claim that $f(x)$ is not continuous at 0.



Proof. We pick $\varepsilon = \frac{1}{2}$. Now whatever $\delta > 0$ we are given (by the Demon), we can pick $h \in (0, \delta)$, for example pick $h = \frac{\delta}{2}$. Then

$$|f(0+h) - f(0)| = |f(h) - f(0)| = |1 - 0| = 1 \geq \frac{1}{2} = \varepsilon.$$

Thus we have shown that the negation of (5.1) is true and hence f is not continuous at 0. □

Recall that our definition (5.1) of "$f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$" was

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall h \in \mathbb{R}, |h| < \delta\ :\ |f(a+h) - f(a)| < \varepsilon.$$

So the Demon specifies $\varepsilon > 0$ which says how close $f(a+h)$ must be to $f(a)$. We then reply with $\delta > 0$ (possibly depending on a and on ε) which *guarantees* that if $|h| < \delta$ then $f(a+h)$ is indeed ε-close to $f(a)$.

Obviously, this is equivalent to the following statement:

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathbb{R}, |x-a| < \delta\ :\ |f(x) - f(a)| < \varepsilon. \tag{5.3}$$

From (5.1), we arrive at (5.3) by replacing h by $x - a$, and vice versa. The negation of (5.3) is

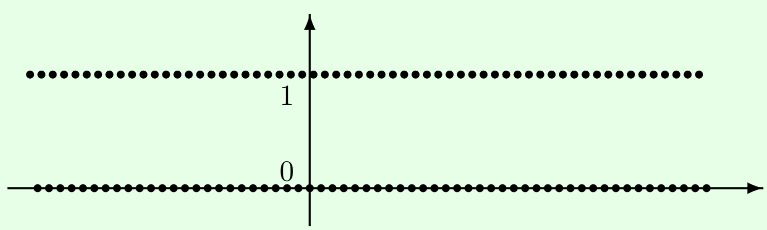
$$\exists \varepsilon > 0\ \forall \delta > 0\ \exists x \in \mathbb{R}, |x-a| < \delta\ :\ |f(x) - f(a)| \geq \varepsilon. \tag{5.4}$$

While sometimes it is easier to work with (5.1) and (5.2), there are other situations where it is more convenient to work with the equivalent statements (5.3) and (5.4).

Examples 5.4 (i) Let $f(x)$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Here is a schematic illustration of how $f(x)$ looks.



We claim that this function is not continuous anywhere.

Proof. We first show that if $a \in \mathbb{Q}$, then f is not continuous at a. We must show that

$$\exists \varepsilon > 0\ \forall \delta > 0\ \exists x \in \mathbb{R}, |x-a| < \delta\ :\ |f(x) - f(a)| \geq \varepsilon.$$

As our $a \in \mathbb{Q}$, we have $f(a) = 1$. We pick $\varepsilon = \frac{1}{2}$ (in fact any $\varepsilon \in (0, 1]$ would do). Now given any $\delta > 0$ there exists an irrational number x in the interval $(a - \delta, a + \delta)$ (by Corollary 2.21 "irrational numbers are dense in $\mathbb{R}$"). For such an x, $f(x) = 0$ and thus

$$|f(x) - f(a)| = |0 - 1| = 1 \geq \frac{1}{2} = \varepsilon.$$

Hence f is not continuous at a.

On the other hand, if $a \notin \mathbb{Q}$, then $f(a) = 0$. We pick again $\varepsilon = \frac{1}{2}$. Given any $\delta > 0$ (by the Demon), there exists a rational number x in the interval $(a - \delta, a + \delta)$ (by Theorem 2.17 "rational numbers are dense in $\mathbb{R}$"). For such x, $f(x) = 1$ and hence

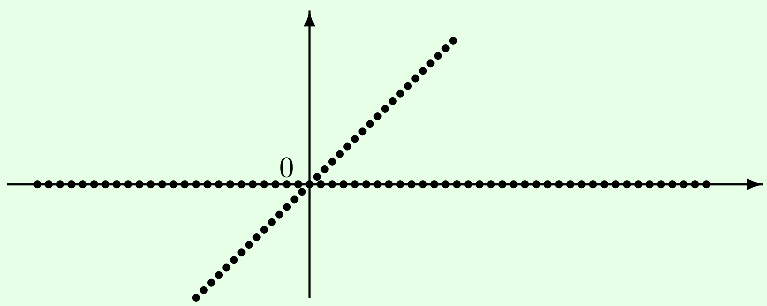
$$|f(x) - f(a)| = |1 - 0| = 1 \geq \frac{1}{2} = \varepsilon,$$

so f is not continuous at a. □

(ii) Let $f(x)$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

We claim that this function is continuous at $a = 0$ and not continuous at any other point $a \in \mathbb{R}$.



Proof. We first show that f is continuous at $a = 0$. We must show that

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in \mathbb{R}, |x| < \delta \ : \ |f(x)| < \varepsilon.$$

(This is just (5.3), where we plugged in $a = 0$ and $f(a) = 0$.)

Given any $\varepsilon > 0$ (by the Demon), we choose $\delta = \varepsilon$ (as this worked both for the linear function $f(x) = x$ and for the constant function $f(x) = 0$). Now, for all x with $|x| < \delta = \varepsilon$, $f(x)$ is equal to either x or 0. In either case, we have $|f(x)| < \varepsilon$. Thus f is continuous at $a = 0$.

Next, we show that if $a \in \mathbb{Q}, a \neq 0$, then f is not continuous at a. We must show that

$$\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x \in \mathbb{R}, |x - a| < \delta \ : \ |f(x) - f(a)| \geq \varepsilon.$$

As $a \in \mathbb{Q}$, we have $f(a) = a$ which motivates us to pick $\varepsilon = \frac{|a|}{2} > 0$ (in fact any $\varepsilon \in (0, |a|]$ would do). Now given any $\delta > 0$ there exists an irrational number x in the interval $(a - \delta, a + \delta)$ (by Corollary 2.21). For such an x, $f(x) = 0$ and thus

$$|f(x) - f(a)| = |0 - a| = |a| \geq \frac{|a|}{2} = \varepsilon.$$

Hence f is not continuous at a.

Similarly, if $a \notin \mathbb{Q}$, then $f(a) = 0$. We pick again $\varepsilon = \frac{|a|}{2}$. Given any $\delta > 0$ (by the Demon), we pick a rational number x such that

- $x \in (a, a + \delta)$ if $a > 0$,
- $x \in (a - \delta, a)$ if $a < 0$.

This ensures that $|x| > |a|$. We can find such an x by Theorem 2.17. As $f(x) = x$, we have

$$|f(x) - f(a)| = |x - 0| = |x| \geq |a| \geq \frac{|a|}{2} = \varepsilon,$$

so f is not continuous at a. □

Remark 5.5
It is possible to construct functions f for which the set of points a where f is continuous is a complicated set. For example, there exists functions that are continuous at every irrational $a \in \mathbb{R}$ but not continuous at any rational $a \in \mathbb{R}$. An example (see the Problem Sheet) is:

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \text{ or if } x = 0, \\ 1/q & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ in lowest terms, with } p > 0, \end{cases}$$

But let us go back to well-behaved functions.

Example 5.6
$f(x) = x^2$ is continuous at every $a \in \mathbb{R}$. To prove this, we have to show that for every $a \in \mathbb{R}$ the following statement holds:

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall h \in \mathbb{R}, |h| < \delta \; : \; |(a + h)^2 - a^2| < \varepsilon.$$

Let us start with an informal calculation: $(a + h)^2 - a^2 = a^2 + 2ah + h^2 - a^2 = 2ah + h^2 = (2a + h)h$. We want $|h|$ to be small enough that $|(2a + h)h| < \varepsilon$. What conditions do we need to place on $|h|$ so that the following sequence of inequalities hold

$$|(2a+h)h| \leq (2|a|+|h|)|h| < (2|a|+1)|h| < \varepsilon?$$

The first inequality always holds: it is the triangle inequality. The middle inequality will hold if $|h| < 1$. The final inequality will hold if $|h| < (2|a|+1)^{-1}\varepsilon$.

This leads us to the following formal proof.

Proof. Given any $a \in \mathbb{R}$ and $\varepsilon > 0$, choose $\delta = \min\left\{(2|a|+1)^{-1}\varepsilon, 1\right\}$. Then, for all h with $|h| < \delta$ we have

$$|(a+h)^2 - a^2| = |2ah + h^2| \leq (2|a|+|h|)|h| < (2|a|+\delta)\delta \leq (2|a|+1)\delta < \varepsilon.$$

So f is continuous at a. □

Remark 5.7
A similar proof works for all polynomial functions. You will see more examples of this type on the Problem Sheet!

What if the function f is only defined on a subset $D \subseteq \mathbb{R}$, not on the whole of $\mathbb{R}$?

Examples 5.8

(i) $f(x) = \frac{1}{x}$. Here f is a function $D \to \mathbb{R}$ where $D = \mathbb{R} \setminus \{0\}$.

(ii) $f(x) = \sum_{k=0}^{\infty} x^k$. Here $f : (-1,1) \to \mathbb{R}$.

(iii) $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k}$. Here $f : [-1,1) \to \mathbb{R}$.

Definition 5.9
We say that $f : D \to \mathbb{R}$ is continuous at $a \in D$ if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in D, |x-a| < \delta\ :\ |f(x) - f(a)| < \varepsilon. \tag{5.5}$$

We must specify $x \in D$ instead of $x \in \mathbb{R}$, as $f(x)$ only makes sense for $x \in D$.

Example 5.10
$f(x) = \sqrt{x}$. Here f is a function $f : [0,\infty) \to \mathbb{R}$. To prove that this function is continuous at $a = 0$, we must show that

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in [0,\infty), |x| < \delta\ :\ |\sqrt{x}\,| < \varepsilon.$$

But this is easy:

Proof. Given $\varepsilon > 0$, we take $\delta = \varepsilon^2$. Then if $x \in [0, \infty)$ and $x < \delta = \varepsilon^2$, we have $|\sqrt{x}\,| < \varepsilon$ as required. So f is continuous at 0. □

Remark 5.11
$f(x) = \sqrt{x}$ is not differentiable at $x = 0$, but it is continuous at $x = 0$.

Example 5.12
One can prove (and we will assume this in this course) that any power series

$$\sum_{k=0}^{\infty} a_k x^k$$

defines a continuous function from D to $\mathbb{R}$, where D is the *disc of convergence* of the series, i.e. $D = \{x \in \mathbb{R} \ : \ |x| < R\}$ where R is the radius of convergence of the series (see Theorem 4.30). In particular, $\exp(x)$, $\sin(x)$ and $\cos(x)$ are continuous functions on all of $\mathbb{R}$. Similarly, every polynomial function is continuous at every point $a \in \mathbb{R}$, as a polynomial is a power series with only finitely many non-vanishing coefficients a_k.

5.B Rules for continuous functions

Suppose that $f : D_1 \to \mathbb{R}$ and $g : D_2 \to \mathbb{R}$ are continuous. Let us construct more continuous functions from them:

- We define $cf : D_1 \to \mathbb{R}$ by $(cf)(x) = cf(x)$, for $c \in \mathbb{R}$.
- We define $f + g : D_1 \cap D_2 \to \mathbb{R}$ by $(f + g)(x) = f(x) + g(x)$.
- We define $f \cdot g : D_1 \cap D_2 \to \mathbb{R}$ by $(f \cdot g)(x) = f(x)g(x)$. (Do not confuse this *product* of f and g with the composition $f \circ g$.)
- We define $\frac{f}{g} : D_1 \cap D_2 \cap \{x \in \mathbb{R} \ : \ g(x) \neq 0\} \to \mathbb{R}$ by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$.

Examples 5.13

(i) $h(x) = \frac{1}{x}$ is the quotient of the constant function $f(x) = 1$ by the function $g(x) = x$, and it is defined whenever $g(x) \neq 0$, i.e. on $\mathbb{R} \setminus \{0\}$. Is this h continuous?

(ii) $h(x) = \frac{\sin(x)\cos(x)}{x^2+2}$ is the quotient of the function $f(x) = \sin(x)\cos(x)$ by the function $g(x) = x^2 + 2$ and it is defined whenever $g(x) \neq 0$, i.e. on all of $\mathbb{R}$. Moreover, $f(x)$ is the product of the continuous functions $\sin(x)$ and $\cos(x)$ and $g(x)$ is the sum of the continuous function x^2 and the constant function 2. We want to know whether this h is continuous.

Theorem 5.14 (Building continuous functions from other continuous functions) Suppose $f : D_1 \to \mathbb{R}$ and $g : D_2 \to \mathbb{R}$ are two functions, $c \in \mathbb{R}$ is a constant and $a \in D_1 \cap D_2$. Assume moreover that f and g are continuous at a. Then

(i) cf is continuous at a.

(ii) $f + g$ is continuous at a.

(iii) $f \cdot g$ is continuous at a.

(iv) If $g(a) \neq 0$, then $\frac{f}{g}$ is continuous at a.

Proof. Exercise (on the Problem Sheet). □

Example 5.15
Every polynomial function is continuous. This follows from the previous theorem and the fact that the identity function $f(x) = x$ and the constant function $f(x) = c$ (for $c \in \mathbb{R}$ fixed) are both continuous.

Now let us look at the composition of functions: Suppose $f : D_1 \to \mathbb{R}$ and $g : D_2 \to \mathbb{R}$ are two functions. Then we can define the composition $g \circ f$ by

$$(g \circ f)(x) = g(f(x))$$

at all points $x \in D_1$ such that $f(x) \in D_2$.

Example 5.16
$f(x) = 1 - \cos(x)$, so $f : \mathbb{R} \to [0, 2]$. $g(x) = \sqrt{x}$, so $g : [0, \infty) \to [0, \infty)$. Then $g \circ f$ is a function $\mathbb{R} \to [0, \infty)$

$$(g \circ f)(x) = \sqrt{1 - \cos(x)}.$$

In fact, $g \circ f$ is a function $\mathbb{R} \xrightarrow{f} [0, 2] \xrightarrow{g} [0, \sqrt{2}] \subseteq [0, \infty)$. Is it continuous?

Theorem 5.17 (Continuity of composition of functions)

Suppose $f : D_1 \to D_2$ and $g : D_2 \to \mathbb{R}$. Moreover, suppose that $a \in D_1$ and that f is continuous at a and g is continuous at $f(a)$. Then $(g \circ f)$ is continuous at a.

Informally: As f is continuous at a, we know that if x is close to a, then $f(x)$ is close to $f(a)$. As g is continuous at $f(a)$, we know that if $f(x)$ is close to $f(a)$, then $g(f(x))$ is close to $g(f(a))$. So the difficulty is just to convert the words "is close to" into a precise statement involving ε and δ.

Proof. We must prove that if we are given any $\varepsilon > 0$, then we can find $\delta > 0$ such that for all x with $|x - a| < \delta$ we have $|g(f(x)) - g(f(a))| < \varepsilon$. We know by the continuity of g at $f(a)$ that given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that for all $y \in D_2$ with $|y - f(a)| < \delta_1$, we have

$$|g(y) - g(f(a))| < \varepsilon.$$

We also know, by the continuity of f at a that given $\tilde{\varepsilon} = \delta_1$, there exists $\delta > 0$ such that for all $x \in D_1$ with $|x - a| < \delta$, we have

$$|f(x) - f(a)| < \tilde{\varepsilon} = \delta_1.$$

Thus, given $\varepsilon > 0$, we can choose δ_1 and then δ as above. We deduce that for $x \in D_1$ with $|x - a| < \delta$, we have $|f(x) - f(a)| < \delta_1$ and hence $|g(f(x)) - g(f(a))| < \varepsilon$. This proves that $(g \circ f)$ is continuous at a. □

Example 5.18
$f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(x)$, $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \exp(x)$.

f and g are continuous at all points $a \in \mathbb{R}$ (by our assumption that power series are continuous at all a inside their discs of convergence). Hence by the above theorem, $g \circ f$ is continuous at all $a \in \mathbb{R}$, i.e. the function $(g \circ f)(x) = \exp(\cos(x))$ is continuous everywhere.

5.C Continuous functions and limits of sequences

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $(x_n)_{n=1}^{\infty}$ is a convergent sequence. What can we say about the sequence $(f(x_n))_{n=1}^{\infty}$? The kind of question we are interested in is the following: we know that $x_n = \frac{1}{n}$ converges to 0. Does it follow that the sequence $\exp(\frac{1}{n})$ converges to $\exp(0)$?

Theorem 5.19
Let $f : D \to \mathbb{R}$ be a function which is continuous at $a \in D$. If $(x_n)_{n=1}^{\infty}$ is a sequence in D which converges to a, then the sequence $(f(x_n))_{n=1}^{\infty}$ converges to

$f(a)$.

This theorem can be upgraded to an "if and only if": see the Problem Sheet.

Proof. We must show that given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $|f(x_n) - f(a)| < \varepsilon$.

Since f is continuous at a, we know that given $\varepsilon > 0$, $\exists \delta > 0$ such that for all $x \in D$ with $|x - a| < \delta$, we have $|f(x) - f(a)| < \varepsilon$.

Since $(x_n)_{n=1}^{\infty}$ converges to a, we know that given $\delta > 0$, $\exists N \in \mathbb{N}$ such that for all $n > N$ we have $|x_n - a| < \delta$.

Combining these two facts, we see that for all $n > N$ we have $|x_n - a| < \delta$ and therefore $|f(x_n) - f(a)| < \varepsilon$. Hence $(f(x_n))_{n=1}^{\infty}$ converges to $f(a)$. □

Remark 5.20
The proof of Theorem 5.19 follows the same strategy as that of Theorem 5.17 ("continuity of the composition of continuous functions"). If we think of a convergent sequence $(x_n)_{n=1}^{\infty}$ as a function $\mathbb{N} \to \mathbb{R}$, $n \mapsto x_n$ and *extend* this to a function

$$\mathbb{N} \cup \{\infty\} \to \mathbb{R}$$

by sending ∞ to the limit a of $(x_n)_{n=1}^{\infty}$, then we can think of this function as being "continuous at ∞" (this is not rigorous, but there is a way to make it rigorous via the concept of a "topological space"). Hence, Theorem 5.19 might be interpreted as a special case of Theorem 5.17.

Examples 5.21 (i) $x_n = \cos\big((1 + \frac{1}{n})(1 + \frac{1}{n^2})\big)$. We know that $((1 + \frac{1}{n})(1 + \frac{1}{n^2}))_{n=1}^{\infty}$ converges to 1 by Theorem 3.24. Moreover, we know that $\cos(x)$ is continuous at $a = 1$ (since it is continuous everywhere). Hence Theorem 5.19 tells us that $(x_n)_{n=1}^{\infty}$ converges to $\cos(1)$.

(ii) We can use Theorem 5.19 to prove that $\big(\frac{1}{\sqrt{n}}\big)_{n=1}^{\infty}$ converges to 0 without going back to the definition of convergence. To do this, observe that

- $\frac{1}{n}$ converges to 0
- $x \overset{f}{\mapsto} \sqrt{x}$ is continuous at 0.

Hence by Theorem 5.19, the sequence $\frac{1}{\sqrt{n}} = f(\frac{1}{n})$ converges to $f(0) = \sqrt{0} = 0$.

(iii) We can use Theorem 5.19 to prove that a function is *not* continuous. For example let

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Now let $x_n = \frac{1}{n}$ so that $x_n \to 0$. Then $f(x_n) = 0$ for all n and so $f(x_n) \to 0$. But $f(0) = 1 \neq 0$, so f cannot be continuous at 0 by Theorem 5.19.

The Examples 5.4 can be tackled similarly.

Continuing the idea from Example (iii) above, we introduce the following:

Definition 5.22
We say that $\lim_{x\to a} f(x)$ exists and is equal to $\ell \in \mathbb{R}$ iff

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall h, 0 < |h| < \delta\ :\ |f(a+h) - \ell| < \varepsilon.$$

It is important that we require $|h| \neq 0$, because we do not want to consider $f(a)$ in the definition!

One can show (see the Problem Sheet) that $\lim_{x\to a} f(x) = \ell$ (according to the above definition) if and only if for *every* sequence $(x_n)_{n=1}^{\infty}$ which satisfies $x_n \neq a$ for all n as well as $x_n \to a$ for $n \to \infty$, we get $f(x_n) \to \ell$ as $n \to \infty$.

An equivalent definition to our earlier definition of "f is continuous at a" is to say that f is continuous at a if and only if $\lim_{x\to a} f(x)$ exists and is equal to $f(a)$. With this definition, it is obvious that the function

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases}$$

is *not* continuous at $x = 0$, since $\lim_{x\to 0} f(x) = 0$, but $f(0) = 1$.

5.D The Intermediate Value Theorem

Theorem 5.23 (Intermediate Value Theorem (IVT))
If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) \leq 0$ and $f(b) \geq 0$ then $\exists c \in [a, b]$ such that $f(c) = 0$.

As a first step to proving this theorem, we prove the following lemma.

Lemma 5.24
Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $c \in [a, b]$ such that $f(c) > 0$. Then $\exists \delta > 0$, such that $\forall x \in (c - \delta, c + \delta) \cap [a, b]\ :\ f(x) > 0$.

Proof. Since f is continuous at c, we know that $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall x \in (c-\delta, c+\delta) \cap [a,b]$ we have $|f(x) - f(c)| < \varepsilon$.

So picking $\varepsilon = \frac{f(c)}{2}$, we know that

$$\exists \delta > 0\ \forall x \in (c-\delta, c+\delta) \cap [a,b]\ :\ |f(x) - f(c)| < \frac{f(c)}{2}$$

and therefore

$$\exists \delta > 0\ \forall x \in (c-\delta, c+\delta) \cap [a,b]\ :\ \frac{f(c)}{2} < f(x) < \frac{3f(c)}{2},$$

so in particular $f(x) > 0$. □

Remark 5.25
The statement of the lemma includes the possibility that $c = a$ or $c = b$. There is an obvious analogous result for $f(c) < 0$.

Proof of Theorem 5.23. Let $A = \{x \in [a,b]\ :\ f(x) \leq 0\}$. This set is non-empty, since $a \in A$. It is also bounded above (by b). So by the Completeness Axiom, A has a least upper bound, say $c \in [a,b]$. We will prove that $f(c) = 0$.

If $f(c) < 0$ then $c \neq b$ and so $c < b$. By Lemma 5.24 $\exists \delta > 0$ such that for all $x \in (c-\delta, c+\delta) \cap [a,b]$ we have $f(x) < 0$.

Choose $x \in (c, c+\delta) \cap [a,b]$. This exists since $c < b$. Then $f(x) < 0$ and so $x \in A$. On the other hand $x > c$. This contradicts the fact that c is an upper bound for A.

If $f(c) > 0$ then $c \neq a$ and so $c > a$. By Lemma 5.24 $\exists \delta > 0$ such that for all $x \in (c-\delta, c+\delta) \cap [a,b]$ we have $f(x) > 0$.

Choose $x \in (c-\delta, c) \cap [a,b]$. This exists since $c > a$. Then $f(x) > 0$ and we see that x is an upper bound for A. But $x < c$ which contradicts the fact that c is the *least* upper bound for A.

Hence the only possibility is $f(c) = 0$. □

Remark 5.26 (i) We have used exactly the same steps in this proof as we used earlier in the course to prove that there exists a number "$\sqrt{2}$" in $\mathbb{R}$ (Theorem 2.19). In fact, if we work through the proof of the IVT with the function $f(x) = x^2 - 2$, we get exactly the proof of Theorem 2.19.

(ii) We used the Completeness Axiom to prove the IVT and in fact this is necessary: the IVT is not true if we replace $\mathbb{R}$ by $\mathbb{Q}$ in the domain of f. For example $f(x) = x^2 - 2$ has no value $c \in \mathbb{Q}$ such that $f(c) = 0$.

(iii) The IVT is of course also true for functions that have $f(a) \geq 0$ and $f(b) \leq 0$ (just apply Theorem 5.23 to the function $-f$).

Example 5.27

Let $f(x) = x^5 + x + 1$. There is no formula for the solutions to a general equation of degree 5. There are formulas for the solutions of polynomial equations of degrees 2, 3 and 4, but is was proved by Abel and Galois in the early 19^{th} century that there cannot be a formula in terms of square roots, cube roots, etc. for a general polynomial equation of degree 5 and above (the Level 7 module *Further Topics in Algebra* concludes with a proof of this fact).

But we can use the IVT to prove that $f(x)$ has at least one real root! We have

- $f(10) = 10^5 + 10 + 1 > 0$,
- $f(-10) = (-10)^5 - 10 + 1 < 0$.

Since f is continuous on $[-10, 10]$ (being a polynomial), we can deduce that $\exists c \in [-10, 10]$ such that $f(c) = 0$.

We can generalise this example and prove that every polynomial of odd degree has a real root.

Corollary 5.28

Suppose $p(x)$ is the polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$$

where n is odd and $a_j \in \mathbb{R}$ for all j. Then $p(x) = 0$ has at least one solution $x \in \mathbb{R}$.

Proof. Let $M = \max\{1, |a_0| + |a_1| + \ldots + |a_{n-1}|\}$. Then we have

$$\begin{aligned}
p(M) &= M^n + a_{n-1}M^{n-1} + \ldots + a_1M + a_0 \\
&\geq M^n - |a_{n-1}|M^{n-1} - \ldots - |a_1|M - |a_0| \\
&\geq M^n - (|a_{n-1}| + \ldots + |a_0|)M^{n-1} \\
&\geq M^{n-1}(M - (|a_{n-1}| + \ldots + |a_0|)) \\
&\geq 0.
\end{aligned}$$

Here, we used $M \geq 1$ for the third line and $M \geq |a_0| + |a_1| + \ldots + |a_{n-1}|$ for the last line. Similarly, we obtain $p(-M) \leq 0$.

Now, p is continuous on $\mathbb{R}$ since p is a polynomial. Hence by the IVT, $\exists c \in [-M, M]$ with $p(c) = 0$. This proves the corollary. $\square$

Remark 5.29
A polynomial of even degree may not have a real root. For example $p(x) = x^2 + 1$ has no real roots. However, *every* polynomial has a complex root! This result is called the Fundamental Theorem of Algebra.

Another application of the IVT is the following.

Corollary 5.30 (A fixed point theorem)
Let $f : [0,1] \to [0,1]$ be a continuous function. Then f has a *fixed point*, i.e. there exists $c \in [0,1]$ such that $f(c) = c$.

Proof. Let g be the function defined by $g(x) = f(x) - x$. Then g is continuous since it is a sum of continuous functions. Moreover

- $g(0) = f(0) - 0 \geq 0$ (since $f(0) \geq 0$),
- $g(1) = f(1) - 1 \leq 0$ (since $f(1) \leq 1$).

Hence, by the IVT $\exists c \in [0,1]$ such that $g(c) = 0$. But this means $f(c) - c = 0$, i.e. $f(c) = c$. □

Remark 5.31
(i) We cannot replace $[0,1]$ by the open interval $(0,1)$ in Corollary 5.30. There exist continuous maps $(0,1) \to (0,1)$ which do *not* have a fixed point (see Problem Sheet 6).

(ii) Corollary 5.30 can be generalized to every dimension n, where it says: let D^n denote the unit disc in $\mathbb{R}^n$, i.e.

$$D^n := \{(x_1, x_2, \ldots, x_n) \ : \ x_1^2 + x_2^2 + \ldots + x_n^2 \leq 1\} \subseteq \mathbb{R}^n.$$

Then every continuous map $f : D^n \to D^n$ has at least one fixed point. This is called the Brouwer Fixed Point Theorem (proved by Brouwer about 100 years ago). The proof uses Algebraic Topology.

We end this section with a more general version of the Intermediate Value Theorem.

Corollary 5.32 (More general form of IVT)
Suppose $f : [a,b] \to \mathbb{R}$ is continuous and y is a real number between $f(a)$ and $f(b)$. Then $\exists c \in [a,b]$ such that $f(c) = y$.

Proof. Suppose without loss of generality that $f(a) < f(b)$. We are given a value y with $f(a) < y < f(b)$. Define g by $g(x) = f(x) - y$.

- g is continuous on $[a, b]$ as the sum of two continuous functions,
- $g(a) = f(a) - y < 0$ (since $f(a) < y$),
- $g(b) = f(b) - y > 0$ (since $y < f(b)$).

So by the IVT, $\exists c \in [a, b]$ with $g(c) = 0$, i.e. $f(c) = y$. □

5.E Continuous functions on closed intervals and the "boundedness principle"

Our aim in this section is to prove the following results for a continuous function f on a closed interval $[a, b] \subseteq \mathbb{R}$.

(i) f is bounded on $[a, b]$, so by the Completeness Axiom the set $f([a, b]) = \{f(x) : x \in [a, b]\}$ has a least upper bound M and a greatest lower bound m.

(ii) M and m are attained by f on $[a, b]$, i.e. $\exists c, d \in [a, b]$ with $f(c) = M$ and $f(d) = m$ (that is: M and m are maximum and minimum of $f([a, b])$).

(iii) $f([a, b])$ is the closed interval $[m, M]$. (This is the easy step: it follows directly from the general IVT, Corollary 5.32.)

To prove (i), we first prove the following consequence of the Bolzano–Weierstrass theorem (Theorem 3.42), which stated that every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Corollary 5.33 (Bolzano–Weierstrass for a closed interval $[a, b] \subseteq \mathbb{R}$)
If $(x_n)_{n=1}^{\infty}$ is a sequence in $[a, b]$ then $(x_n)_{n=1}^{\infty}$ has a subsequence $(x_{r_j})_{j=1}^{\infty}$ which converges to a point in $[a, b]$.

Proof. A sequence in $[a, b]$ is bounded (it is bounded above by b and bounded below by a). So by Theorem 3.42 the sequence $(x_n)_{n=1}^{\infty}$ has a subsequence $(x_{r_j})_{j=1}^{\infty}$ which converges to a point $x \in \mathbb{R}$.

But for all j we know that $x_{r_j} \geq a$, so comparing $(x_{r_j})_{j=1}^{\infty}$ with the constant sequence $(a)_{j=1}^{\infty}$, we see that $\lim_{j\to\infty} x_{r_j} \geq a$ by Lemma 3.27. Similarly, $x_{r_j} \leq b$ for all j and hence $\lim_{j\to\infty} x_{r_j} \leq b$. Hence, $x = \lim_{j\to\infty} x_{r_j} \in [a, b]$. □

Remark 5.34
We do need $[a, b]$ to be closed. For example the sequence $x_n = \frac{1}{n}$ has all its elements in $(0, 1)$ but its limit is $x = 0 \notin (0, 1)$.

Examples 5.35 (i) $x_n = (-1)^n(1 - \frac{1}{n})$. This is the sequence $0, \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}, \ldots$. Thus $x_n \in [-1, +1]$ for all n and hence it must have a subsequence which converges to a limit in $[-1, 1]$.

Indeed, $x_1, x_3, x_5, x_7, \ldots$ is a subsequence which converges to -1. Moreover, $x_2, x_4, x_6, x_8, \ldots$ is a subsequence which converges to $+1$.

(ii) Write down all the rational numbers in $[0, 1] \in \mathbb{R}$ in order of increasing denominator:

$$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \ldots$$

This sequence must have a subsequence which converges with limit in $[0, 1]$. Indeed, the subsequence given by the elements $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ converges to zero while the subsequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ converges to 1. One can show that in fact, for *every* $x \in [0, 1]$ there exists a subsequence of the above sequence which converges to x.

Definition 5.36
Suppose $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ is continuous. We say that

- f is *bounded above* on D if $\{f(x) \ : \ x \in D\}$ is bounded above,
- f is *bounded below* on D if $\{f(x) \ : \ x \in D\}$ is bounded below,
- f is *bounded* on D if $\{f(x) \ : \ x \in D\}$ is bounded.

Examples 5.37 (i) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin(x)$ is bounded above by $+1$ and below by -1.

(ii) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is bounded below by 0 but not bounded above.

(iii) $f : (0, 1) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is bounded below by 1 but not bounded above.

Theorem 5.38 (Boundedness Principle I)
If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded on $[a, b]$.

Remark 5.39
As Example (iii) above shows, we need $[a, b]$ rather than (a, b).

Proof. We assume for a contradiction that $f : [a, b] \to \mathbb{R}$ is continuous but *not* bounded above. As f is not bounded above, we can choose $x_1 \in [a, b]$ with $f(x_1) \geq 1$ and $x_2 \in [a, b]$ with $f(x_2) \geq 2$, etc. That is, we have a sequence $(x_n)_{n=1}^{\infty}$ in $[a, b]$ with $f(x_n) \geq n$.

By Corollary 5.33, this sequence $(x_n)_{n=1}^{\infty}$ has a subsequence $(x_{r_j})_{j=1}^{\infty}$ which converges to some $x \in [a, b]$. By Theorem 5.19, the sequence $(f(x_{r_j}))_{j=1}^{\infty}$ converges to $f(x)$ since f is continuous at x. But on the other hand, by construction, $(f(x_{r_j}))_{j=1}^{\infty}$ tends to infinity. Contradiction.

This shows that f is bounded above. The fact that f is bounded below follows analogously. □

Theorem 5.40 (Boundedness Principle II)
If $f : [a, b] \to \mathbb{R}$ is continuous, then f attains a maximum and a minimum on $[a, b]$, i.e. $\exists c \in [a, b]$ s.t. $f(x) \leq f(c)$ for all $x \in [a, b]$ and $\exists d \in [a, b]$ s.t. $f(x) \geq f(d)$ for all $x \in [a, b]$.

Remark 5.41
This is only true in general for a *closed* interval $[a, b]$. For example the function $f(x) = \frac{1}{x}$ on the open interval $(0, 1)$ has a greatest lower bound which is 1, but the value 1 is not attained inside the open interval $(0, 1)$, so the function has no minimum on this open interval.

Proof. Let $M = \sup\{f(x) : x \in [a, b]\}$. (This exists according to Theorem 5.38 and the Completeness Axiom.) Suppose towards a contradiction that M is not attained by f at any $x \in [a, b]$.

Now define $g : [a, b] \to \mathbb{R}$ by $g(x) = \frac{1}{M - f(x)}$. This is well-defined since we assume $f(x) \neq M$ for any $x \in [a, b]$. Since f is continuous, so is g (since it is the quotient of continuous functions and the denominator is not zero). Hence by Theorem 5.38 also $g : [a, b] \to \mathbb{R}$ is bounded. In particular, there is some $M' > 0$ such that $g(x) \leq M'$ for all $x \in [a, b]$.

But this means that

$$\begin{aligned} \frac{1}{M - f(x)} \leq M' &\qquad \forall x \in [a, b] \\ \Rightarrow \frac{1}{M'} \leq M - f(x) &\qquad \forall x \in [a, b] \\ \Rightarrow f(x) \leq M - \frac{1}{M'} &\qquad \forall x \in [a, b]. \end{aligned}$$

In particular, $M - \frac{1}{M'} < M$ is an upper bound for f, contradicting the fact that M was chosen to be the least upper bound.

Hence M must be attained by f for some $c \in [a, b]$ (i.e. $f(c) = M$). Applying exactly the same reasoning to the function $-f$, we see that $-f$ attains a maximum on $[a, b]$ and hence f attains a minimum at some point $d \in [a, b]$. □

Theorem 5.42 (The Interval Theorem)
If $f : [a, b] \to \mathbb{R}$ is continuous, then $\{f(x) : x \in [a, b]\}$ is a closed interval.

Proof. Let $m = \inf\{f(x) : x \in [a, b]\}$ and $M = \sup\{f(x) : x \in [a, b]\}$. By the Boundedness Principle I (Theorem 5.38), we know that M and m exist and by the Boundedness Principle II (Theorem 5.40) we know that they are attained, i.e. $\exists c, d \in [a, b]$ with $f(c) = m$ and $f(d) = M$.

Without loss of generality, assume $c < d$. f is continuous on $[c, d]$ (as $[c, d] \subseteq [a, b]$). So, given any $y \in [m, M] = [f(c), f(d)]$, by the Intermediate Value Theorem (in the general form of Corollary 5.32), there exists some $x \in [a, b]$ with $f(x) = y$. Hence $\{f(x) : x \in [a, b]\} = [m, M]$. □

Where to from here?

The Semester B module MTH5105: Differential and Integral Analysis takes these ideas further, developing the modern edifice of calculus on rigorous foundations.