Queen Mary University of London

Main Examination period 2022 – May/June – Semester B

MTH6112: Actuarial Financial Engineering

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

IFoA exemptions. For actuarial students, this module counts towards IFoA actuarial exemptions. To be eligible for IFoA exemption, **your must submit your exam within the first 3 hours of the assessment period**.

Examiners: C. Sutton, L. Fang

Question 1 [18 marks]. Sam is considering an investment in the shares of Derby Industries Inc but does not know much about the company. In a personal finance blog Sam reads that the author believes the current Derby Industries share price of \$23 per share accurately reflects the company's balance sheet and cashflow. Therefore Sam uses a stochastic process, X_t to model the share price where t is measured in days. The model used is

$$X_t = 23 + W_t$$

where W_t is a standard Wiener Process.

(a) Find the value of $\mathbb{E}[X_t]$ for $t \geq 0$. [**2**]

Solution:

$$\mathbb{E}[X_t] = \mathbb{E}[23 + W_t] = 23 + \mathbb{E}[W_t] = 23 + 0 = 23 \ as \ W_t \sim N(0, 1)$$

(b) Determine whether X_t is a martingale with respect to its natural filtration. [**4**]

Solution:
(a) X_t is a martingale with respect to the filtration F_t if $\mathbb{E}[X_t|F_s] = X_s$ for all $t > s$,
here $\mathbb{E}[X_t|F_s] = \mathbb{E}[23 + W_t|F_s] = 23 + \mathbb{E}[W_t|F_s] = 23 + \mathbb{E}[W_s + (W_t - W_s)|F_s] = 23 + W_s = X_s \ as \ (W_t - W_s) \sim N(0, t - s)$,
hence X_t is a martingale.

(c) What is the probability that the share price will be between \$22 and \$24 after 5 days under this model? [**4**]

Solution:

$$\begin{aligned}
&Pr(22 \leq X_5 \leq 24) = Pr(-1 \leq X_5 – X_0 \leq 1) \ and \\
&X_5 – X_0 \sim N(0, 5) \ therefore \\
&Pr(-1 \leq X_5 – X_0 \leq 1) = \Phi(1/\sqrt{5}) + \Phi(-1/\sqrt{5}) \\
&= 0.673 – (1 – 0.673) \ from \ standard \ normal \ statistical \ tables \\
&= 0.346
\end{aligned}$$

(d) What are the weaknesses of this model when evaluating the risk of investing in Derby Industries shares? [**8**]

Solution:
The weaknesses of this model are

- no drift parameter
- therefore no long term expected positive return from the share
- volatility parameter = 1
- therefore all volatility normally distributed

- ... and proportional to time
- this does not capture well the risk of investment in the shares
- studies have shown log normal better than normal for equity risk
- potential over-reliance on one data point for X_0

Parts a, b, c similar to exercise sheet, Part d unseen.
IFoA CM2 syllabus areas 4.4.1, 4.4.2, 4.4.3.

Question 2 [22 marks]. Consider the following stochastic differential equation

$$dX_t = Y_t dt + zW_t$$

where Y_t is a stochastic process, z is a constant and W_t represents standard Brownian Motion.

(a) Write down Ito's lemma for $f(X_t, t)$ where f is a suitable function. [**3**]

Solution:
By Ito's lemma

$$df(X_t, t) = [\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}A_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}z^2]dt + \frac{\partial f}{\partial x}zdW_t$$

(b) Determine $df(X_t, t)$ where $f(X_t, t) = e^{2tX_t}$, simplifying your answer where possible. [**3**]

Solution:
If $f(X_t, t) = e^{2tX_t}$,

$$\frac{\partial f}{\partial x} = 2te^{2tX_t},$$
$$\frac{\partial^2 f}{\partial x^2} = 4t^2 e^{2tX_t}$$
$$\frac{\partial f}{\partial t} = 2X_t e^{2tX_t}$$

inserting these in the formula for Ito's lemma in (a) we get

$$\begin{aligned} df(X_t, t) &= [2X_t e^{2tX_t} + 2te^{2tX_t}A_t + \frac{1}{2}4t^2 e^{2tX_t}z^2]dt + 2te^{2tX_t}zdW_t \\ &= 2e^{2tX_t}[(X_t + tA_t + t^2z^2)dt + tzdW_t] \end{aligned}$$

(c) If Y_t is replaced with a constant y, what function $g(X_t, t)$ is required such that the application of Ito's lemma leads to $g(X_t, t)$ representing Geometric Brownian Motion? [**1**]

Solution:
$g(X_t, t) = e^{X_t}$ leads to Geometric Brownian Motion

(d) State the probability distribution that X_t follows under the function in (c) above. **[3]**

Solution:
then X_t has a log normal distribution such that
$logX_t \sim N(logX_0 + (y - \frac{1}{2}z^2)t, z^2t)$.

To get full marks for this part students need to specify the full distribution parameters not just state log normal and express in terms of y, z not μ and σ.

(e) If $y = 0.052, z = 0.149$ and $X_0 = 150$, find a 95% confidence interval for X_{15}. **[5]**

Solution:
Now $logX_t \sim N(logX_0 + (y - \frac{1}{2}z^2)t, z^2t)$,
so $logX_{15} \sim N(log150 + 15(0.052 - \frac{1}{2} \times 0.149^2), 0.149^2 \times 15)$
$logX_{15} \sim N(5.624128, 0.333015)$
and a 95% confidence interval for $logX_{15}$ is
$5.624128 \pm 1.96(0.149)\sqrt{15}$
$= (4.493062, 6.755194)$
and a 95% confidence interval for X_{15} is
$(e^{4.493062}, e^{6.755194}) = (89.39, 858.51)$

(f) If X_t is to be used to model the value of an equity portfolio, how realistic is it to use a constant y rather than a stochastic process Y_t? **[7]**

Solution:
constant y rather than stochastic Y_t

- helps by allowing log normal assumptions using GBM
- however does not allow for number of real life properties of drift
- different risk premia observed at different times
- risk premia varying alongside default risk
- wealth effects
- ompensation for varying volatility related risks over time

Parts a, b, e similar to seminar, Part c, d from lectures, f unseen and higher order skills. IFoA CM2 syllabus areas 4.4.1, 4.4.4, 4.4.5.

Question 3 [15 marks]. A European call option, with value c_t at time t, is written on a non-dividend paying stock, with price S_t at time t. The call option matures at time T and the strike price is K. The continuously compounded risk-free rate is r.

A portfolio contains one call option and $Ke^{-(T-t)r}$ cash.

(a) Prove that, at time T, the value of the portfolio will always be greater than or equal to the value of the share, S_T. **[3]**

Solution:
At time T, the call either expires worthless or is exercised.
If the call is not exercised, then this is because $K > S_T$. Then the value of the portfolio is $K > S_T$.
If the call is exercised, then the portfolio value is $S_T - K + K = S_T$.
In either case, the portfolio value is greater than or equal to the share.

(b) State the upper and lower bound for the value of the call option, c_t. **[4]**

Solution:
Upper bound: $c_t \leq S_t$.
Lower bound:
The portfolio is greater than or equal to the value of the share at time T.
By the principle of no arbitrage, it must also be the case that at time t, the portfolio value is greater than or equal to the value of the share:
$c_t + Ke^{-(T-t)r} \leq S_t$.

The prices of a stock follow a geometric Brownian motion with parameters $\mu = 0.3$ and $\sigma = 0.2$. Presently, the stock's price is £50. Consider a call option having nine months until its expiration time and having a strike price of £45.

(c) What is the probability that the call option will be exercised? **[4]**

Solution:
Let $S(t)$ denote the price of the Bancroft Stock at time t, where t is measured in years. We are told that $S(t)$ is geometric Brownian motion with drift parameter $\mu = 0.3$, volatility parameter $\sigma = 0.2$ and starting parameter $S = 50$, that is

$$S(t) = S\exp(\mu t + \sigma W(t)),$$

where $W(t)$ denotes the Wiener process. Let $K = 45$ denote the strike price and let $t = 3/4$ denote the expiration time of the call option.
The option will be exercised if $S(t) > K$. Thus, the desired probability is

$$\mathbb{P}(S(t) > K) = \mathbb{P}(S\exp(\mu t + \sigma W(t)) > K) = \mathbb{P}\left(\frac{W(t)}{\sqrt{t}} > \frac{\log\frac{K}{S} - \mu t}{\sigma\sqrt{t}}\right)$$
$$= \mathbb{P}\left(\frac{W(t)}{\sqrt{t}} > -0.91\right) = 1 - \Phi(-0.91) = 1 - 0.1814 = 0.8186.$$

Here we have used the fact that

$$\frac{W(t)}{\sqrt{t}} \sim \mathrm{N}(0,1).$$

Thus the probability that the call option will be exercised is 0.8186.

(d) If the interest rate is 3%, find the price of the call option using the Black-Scholes formula. [**4**]

Solution:

Let $r = 0.03$ denote the nominal interest rate. The Black-Scholes price C of the call is given by the Black-Scholes Formula

$$C = S\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t}),$$

where

$$\omega = \frac{rt + \frac{\sigma^2 t}{2} - \log\frac{K}{S}}{\sigma\sqrt{t}}.$$

Now

$$\omega = 0.824806 \quad \text{and} \quad \omega - \sigma\sqrt{t} = 0.651600,$$

so

$$\begin{aligned} C = 50\Phi(0.82) - 45e^{-0.03\times 3/4}\Phi(0.65) &= 50\Phi(0.82) - 43.998806\Phi(0.65) \\ &= 50 \times 0.7939 - 43.998806 \times 0.7422 = 7.0391. \end{aligned}$$

Thus, the Black-Scholes price of the call option is 7.0391.

Part a from lectures, Part c, d similar to seminar, Part b unseen and higher order skills. IFoA CM2 syllabus areas 6.1.8, 6.1.9.

Question 4 [25 marks]. A short rate of interest is governed by the Vasicek model, i.e.

$$dr_t = -a(r_t - \mu)dt + \sigma dB_t$$

where B_t is a standard Brownian motion and $a, \mu > 0$ are constants.

A stochastic process $\{X_t : t \geq 0\}$ is defined by $X_t = e^{at+b} \cdot r_t$, where b is a constant.

(a) Derive an equation for dX_t. [**5**]

Solution:

$$dX_t = \frac{\partial X_t}{\partial t}dt + \frac{\partial X_t}{\partial r_t}dr_t + \frac{\partial^2 X_t}{\partial {r_t}^2}(dr_t)^2$$

$$\frac{\partial X_t}{\partial t} = ae^{at+b}r_t, \ \frac{\partial X_t}{\partial r_t} = e^{at+b}, \ \frac{\partial^2 X_t}{\partial {r_t}^2} = 0$$

Therefore,

$$\begin{aligned} dX_t &= ae^{at+b}r_t dt + e^{at+b}(-a(r_t - \mu)dt + \sigma dB_t) \\ &= a\mu e^{at+b}dt + \sigma e^{at+b}dB_t \end{aligned}$$

(b) Solve the equation to find X_t. [**5**]

Solution:
Integrate both sides from 0 to t:

$$\begin{aligned} X_t - X_0 &= \int_0^t a\mu e^{as+b} ds \ + \int_0^t \sigma e^{as+b} dB_s \\ &= \mu e^b(e^{at} - 1) + \sigma e^b \int_0^t e^{as} dB_s \end{aligned}$$

$\Rightarrow \mathrm{X}_t = X_0 + \mu e^b(e^{at} - 1) + \sigma e^b \int_0^t e^{as} dB_s$

(c) Prove that:

$$r_t = \mu + (r_0 - \mu)e^{-at} + \sigma \int_0^t e^{a(s-t)} dB_s.$$

[**5**]

Solution:
We shall be looking for a function $u(t)$ such that

$$r_t - \mu = u(t)e^{-at} \tag{1}$$

Then $u(t) = e^{at}(r_t - \mu) = f(t, r)$. $\Rightarrow u(0) = r_0 - \mu$.
Use the chain rule,

$$\begin{aligned} du &= f_t^{'} dt + f_r^{'} dr + \frac{1}{2} f_{rr}^{''} (dr)^2 \\ f_t^{'} &= \frac{\partial}{\partial t}(e^{at}(r - \mu)) = a e^{at}(r - \mu) \\ f_r^{'} &= \frac{\partial}{\partial r}(e^{at}(r - \mu)) = e^{at} \\ f_{rr}^{''} &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} du(t) &= a e^{at}(r - \mu)dt + e^{at} dr \\ &= a e^{at}(r - \mu)dt + e^{at}(-a(r - \mu)dt + \sigma dB_t) \\ &= \sigma e^{at} dB_t \end{aligned}$$

Hence $\int_0^t du(s) = \sigma \int_0^t e^{as} dB_s$, or equivalently,

$$u(t) - u(0) = \sigma \int_0^t e^{as} dB_s \tag{2}$$

It follows from (1) that $r_0 - \mu = u(0)$. So (2) can be rewritten as

$$u(t) = u(0) + \sigma \int_0^t e^{as} dB_s = r_0 - \mu + \sigma \int_0^t e^{as} dB_s$$

and we obtain (again due to (1)) that

$$\begin{aligned}
r_t &= \mu + e^{-at}\left(r_0 - \mu + \sigma\int_0^t e^{as}dB_s\right)\\
&= r_0e^{-at} + \mu(1-e^{-at}) + \sigma e^{-at}\int_0^t e^{as}dB_s\\
&= (r_0-\mu)e^{-at} + \mu + \sigma e^{-at}\int_0^t e^{as}dB_s \ \square
\end{aligned}$$

(d) Determine the probability distribution of r_t and the limiting distribution for large t. **[5]**

Solution:

$$\begin{aligned}
&dB_s \sim N(0, ds)\\
\Rightarrow &\sigma e^{a(s-t)}dB_s \sim N(0, \sigma^2e^{2a(s-t)}ds)\\
\Rightarrow &\int_0^t \sigma e^{a(s-t)}dB_s \sim N(0, \int_0^t \sigma^2 e^{2a(s-t)}ds)
\end{aligned}$$

The distribution of r_t is given by:

$$r_t \sim N(\mu + e^{-at}(r_0 - b), \int_0^t \sigma^2 e^{2a(s-t)}ds) = N(\mu + e^{-at}(r_0-\mu), \frac{\sigma^2}{2a}(1-e^{-2at}))$$

As $t\to\infty$, $e^{-2at}\to 0$, so we get:

$$r_t \sim N(\mu, \frac{\sigma^2}{2a})$$

(e) Derive, in the case where $s < t$, the conditional expectation $E[r_t|F_s]$, where $\{F_s : s \geq 0\}$ is the filtration generated by the Brownian motion B_s. **[5]**

Solution:

$$\begin{aligned}
E[r_t|F_s] &= E[\mu + (r_0-\mu)e^{-at} + \sigma\int_0^t e^{a(u-t)}dB_u|F_s]\\
&= \mu + (r_0-\mu)e^{-at} + E[\sigma\int_0^t e^{a(u-t)}dB_u|F_s]\\
&= \mu + (r_0-\mu)e^{-at} + E[\sigma\int_0^s e^{a(u-t)}dB_u|F_s] + E[\sigma\int_s^t e^{a(u-t)}dB_u|F_s]\\
&= \mu + (r_0-\mu)e^{-at} + E[\sigma\int_0^s e^{a(u-t)}dB_u|F_s]\\
&= \mu + (r_0-\mu)e^{-at} + e^{-a(t-s)}E[\sigma\int_0^s e^{a(u-s)}dB_u|F_s]\\
&= \mu + (r_0-\mu)e^{-at} + e^{a(s-t)}[r_s - \mu(r_0-b)e^{-as}]\\
&= \mu(1-e^{a(s-t)}) + e^{a(s-t)}r_s
\end{aligned}$$

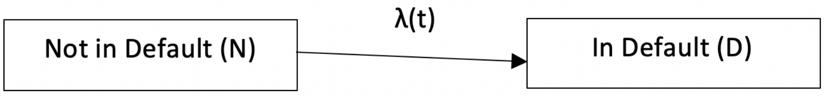
The last step is from (c) with s replaced by u, and t replaced by s.

Part c, d from lectures, Part a, b similar to seminar, Part e application of lecture material.
IFoA CM2 syllabus areas 4.5.3, 4.5.4, 4.5.6, 4.4.2, 4.4.6.

Question 5 [20 marks]. A credit analyst wishes to model the probability that a bond issued by a shipping company defaults, using a time varying default transition intensity $\lambda(t)$.

(a) Draw diagram representing a two-state model that could be used in this scenario. [**2**]

Solution:
Two-state model:



The credit analyst uses a quadratic form for the default transition intensity

$$\lambda(t) = \frac{2 + 10t - t^2}{300}$$

where $1 \leq t \leq 10$.

(b) Calculate the probability that the bond does not default between times 2 and 6. [**6**]

Solution:
Probability that bond does not default between t=2 and t=6

$$\begin{aligned} &= exp[-\int_2^6 \lambda(t)dt] \\ &= exp[-\frac{1}{300}\int_2^6 (2 + 10t - t^2)dt] \\ &= exp[-\frac{1}{300}(2t + 5t^2 - \frac{1}{3}t^3)|_2^6] \\ &= exp(-0.8622) = 0.422 \end{aligned}$$

(c) Explain how the model could be modified to allow for the default transition intensity to depend on economic growth as well as time. [**6**]

Solution:
For the default transition intensity to depend on economic growth as well as time we would need a new function λ in the form $\lambda(X_t, t)$ where X_t is a stochastic variable used to model economic growth over time. It is likely that we would want X_t to have properties of or similar to geometric Brownian motion with a drift parameter linked to long term economic growth.

A credit rating agency has given the bond a B rating.

(d) Explain how the two-state model could be extended to use information from the credit rating agency including how default transition intensities are estimated. [**6**]

Solution:

A multi-state model would have each rating given by the agency as a state along with the in-default state. A transition probability matrix (usually one year probabilities) could then be developed from past experience data of the ratings agency and these probabilities in a Markov jump process used to calculated default probabilities given a starting credit rating.

Parts a from lecture, Part b similar to seminar, Part c unseen and higher order skills, Part d application of lecture material.
IFoA CM2 syllabus areas 4.6.2, 4.6.4, 4.6.6.

End of Paper.