Queen Mary
University of London

Main Examination period 2023 – May/June – Semester B

MTH6112: Actuarial Financial Engineering

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

For actuarial students only: This module also counts towards IFoA exemptions. For your submission to be eligible, **you must submit within the first 3 hours.**

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

Examiners: L. Fang, F. Parsa

Question 1 [30 marks].

Let W_t be a standard Brownian Motion.

(a) The simplest version of the Ornstein-Uhlenbeck process X_t is defined by

$$X_t = \mathrm{e}^{-t} W_t, \quad \text{for some constant } \theta > 0.$$

(i) Does this process have independent increments? [**3**]

(ii) Is X_t a Brownian Motion? [**3**]

(iii) What is the distribution of the increment $X_t - X_s$ for $t > s$? [**3**]

(iv) Compute $\mu_m = \mathbb{E}[(X_t)^m]$ for all integer $m > 0$. [**3**]

(v) Compute $\mathrm{Cov}(X_t, X_s)$. [**3**]

Solution

(i) Increments $X_{t_{i+1}} - X_{t_i}$ can be expressed in terms of the Brownian motion as follows:

$$\begin{aligned} X_{t_{i+1}} - X_{t_i} &= \mathrm{e}^{-t_{i+1}} W_{t_{i+1}} - \mathrm{e}^{-t_i} W_{t_i} \\ &= \mathrm{e}^{-t_{i+1}} (W_{t_{i+1}} - W_{t_i}) + (\mathrm{e}^{-t_{i+1}} - \mathrm{e}^{-t_i}) W_{t_i}. \end{aligned}$$

It is clear that the first term in this expression is independent from all previous history of the Brownian motion (see properties of a Brownian motion). However, the second one is not. To formally prove that the increments are not independent, let us take three different times $t < s < r$ and calculate the covariance $\mathrm{Cov}\left[X_r - X_s, X_s - X_t\right]$.

$$\begin{aligned} \mathrm{Cov}\left[X_r - X_s, X_s - X_t\right] &= \mathrm{Cov}\left[X_r, X_s\right] + \mathrm{Cov}\left[X_s, X_t\right] \\ &\quad - \mathrm{Cov}\left[X_r, X_t\right] - \mathrm{Cov}\left[X_s, X_s\right] \\ &= \mathrm{e}^{-r-s} \mathrm{Cov}\left[W_r, W_s\right] + \mathrm{e}^{-t-s} \mathrm{Cov}\left[W_t, W_s\right] \\ &\quad - \mathrm{e}^{-t-r} \mathrm{Cov}\left[W_t, W_r\right] - \mathrm{e}^{-2s} \mathrm{Var}\left[W_s\right] \\ &= \mathrm{e}^{-s-r} s + \mathrm{e}^{-t-s} t - \mathrm{e}^{-t-r} t - \mathrm{e}^{-2s} s \\ &= \left(\mathrm{e}^{-r} - \mathrm{e}^{-s}\right)\left(s\mathrm{e}^{-s} - t\mathrm{e}^{-t}\right) > 0. \end{aligned}$$

(ii) It follows from the above that X_t is not a Brownian Motion.

(iii) An increment $X_t - X_s$ can be written as a sum of two independent Gaussian random variables. Indeed

$$X_t - X_s = \mathrm{e}^{-t}\left[W_t - W_s\right] + \left(\mathrm{e}^{-t} - \mathrm{e}^{-s}\right)\left[W_s - W_0\right].$$

Thus, $X_t - X_s$ is a Gaussian random variable with mean zero and variance

$$\mathrm{Var}\left[X_t - X_s\right] = \mathrm{e}^{-2t}(t-s) + \left(\mathrm{e}^{-t} - \mathrm{e}^{-s}\right)^2 s = te^{-2t} + se^{-2s} - 2se^{-t-s}.$$

(iv) For the m-th moment we use the formula derived in lectures:

$$\mu_m = \mathbb{E}\left[\mathrm{e}^{-mt}W_t^m\right] = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ \mathrm{e}^{-2pt}\frac{(2p)!}{2^p p!}t^p, & \text{if } m = 2p \text{ (is even).} \end{cases}$$

(v) Finally, for the covariance (which in fact has been computed above) one has:

$$\mathrm{Cov}\left[X_t, X_s\right] = \mathrm{e}^{-t-s}\mathrm{Cov}\left[W_t, W_s\right] = \mathrm{e}^{-t-s}min(t,s).$$

(b) Consider a Brownian Motion $B_t = \mu t + \sigma W_t$, where W_t is the standard Wiener Process and μ, and σ are the parameters of the Brownian Motion. We also define the related Geometric Brownian S_t by $S_t = \mathrm{e}^{B_t}$.
Are the following processes martingale or not, with respect to the natural filtration, i.e. the one associated with W_t?

(i) $Z_t = 3W_t$; **[5]**

(ii) $Z_t = W_t^2 - 2t$; **[5]**

(iii) $Z_t = \mathrm{e}^{-\mu t - \frac{\sigma^2 t}{2}}S_t$. **[5]**

Solution For the process X_t to be a martingale with respect to filtration $\mathcal{F}_s$ it is sufficient to satisfy

$$\mathbb{E}\left[|X_t|\right] < \infty, \forall t$$
$$\mathbb{E}\left[X_t|\mathcal{F}_s\right] = X_s, \forall t > s.$$

(i) Yes. Using the properties of conditional expectation discussed in the lecture we have

$$\begin{aligned} \mathbb{E}\left[3W_t|\mathcal{F}_s\right] &= \mathbb{E}\left[3(W_t - W_s) + 3W_s|\mathcal{F}_s\right] \\ &\overset{linearity}{=} 3\mathbb{E}\left[W_t - W_s|\mathcal{F}_s\right] + 3\mathbb{E}\left[W_s|\mathcal{F}_s\right] \\ &\overset{independence}{=} 3\mathbb{E}\left[W_t - W_s\right] + 3\mathbb{E}\left[W_s|\mathcal{F}_s\right] \\ &= 3\mathbb{E}\left[W_s|\mathcal{F}_s\right] \overset{measurability}{=} 3W_s. \end{aligned}$$

To estimate the expectation we use Schwartz inequality

$$\mathbb{E}\left[|W_t|\right] \le \sqrt{\mathrm{Var}\left[W_t\right]} = \sqrt{t} < \infty.$$

Thus W_t is a martingale with respect to the natural filtration $\{\mathcal{F}_t\}_{t\ge 0}$.

(ii) No. Using the properties of conditional expectation discussed in the lecture we have

$$\begin{aligned}
\mathbb{E}\left[W_t^2 - 2t|\mathcal{F}_s\right] &= \mathbb{E}\left[(W_t - W_s + W_s)^2 - 2t|\mathcal{F}_s\right] \\
&\overset{linearity}{=} \mathbb{E}\left[(W_t - W_s)^2|\mathcal{F}_s\right] + \mathbb{E}\left[W_s^2 - 2t|\mathcal{F}_s\right] \\
&\overset{measurability}{=} \mathbb{E}\left[(W_t - W_s)^2|\mathcal{F}_s\right] + W_s^2 - 2t \\
&\overset{independence}{=} \mathbb{E}\left[(W_t - W_s)^2\right] + W_s^2 - 2t \\
&= t - s + W_s^2 - 2t = W_s^2 - s - t \ne W_s^2 - 2s.
\end{aligned}$$

Thus $W_t^2 - t$ is not a martingale with respect to a natural filtration $\{\mathcal{F}_t\}_{t\ge 0}$.

(iii) Yes. Using the properties of conditional expectation discussed in the lecture we have

$$\begin{aligned}
\mathbb{E}\left[e^{-\mu t - \frac{\sigma^2 t}{2}} e^{\mu t + \sigma W_t}|\mathcal{F}_s\right] &= \mathbb{E}\left[e^{-\frac{\sigma^2 t}{2} + \sigma W_t}|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[e^{\sigma(W_t - W_s) - \frac{\sigma^2 t}{2} + \sigma W_s}|\mathcal{F}_s\right] \\
&\overset{measurability}{=} \mathbb{E}\left[e^{\sigma(W_t - W_s)}|\mathcal{F}_s\right] e^{\sigma W_s - \frac{\sigma^2 t}{2}} \\
&\overset{independence}{=} \mathbb{E}\left[e^{\sigma(W_t - W_s)}\right] e^{\sigma W_s - \frac{\sigma^2 t}{2}} \\
&= \mathbb{E}\left[e^{\mathcal{N}\left(0, \sigma^2(t-s)\right)}\right] e^{\sigma W_s - \frac{\sigma^2 t}{2}} \\
&= e^{\frac{\sigma^2(t-s)}{2} + \sigma W_s - \frac{\sigma^2 t}{2}} = e^{-\frac{\sigma^2 s}{2} + \sigma W_s} \\
&= e^{-\mu s - \frac{\sigma^2 s}{2}} S_s.
\end{aligned}$$

The "discounted" Geometric Brownian Motion is a positive valued process and thus

$$\mathbb{E}\left[\left|e^{-\mu t - \frac{\sigma^2 t}{2}} S_t\right|\right] = e^{-\mu t - \frac{\sigma^2 t}{2}} \mathbb{E}\left[S_t\right] = e^{-\mu t - \frac{\sigma^2 t}{2}} e^{\mu t + \frac{\sigma^2 t}{2}} = 1 < \infty.$$

Part a, b similar to lecture and seminar.
IFoA CM2 syllabus areas 4.4.1, 4.4.2, 4.4.3, 4.5.7.

Question 2 [20 marks]. A short rate of interest is governed by the Vasicek model, i.e.

$$dr_t = -a(r_t - \mu)dt + \sigma dB_t$$

where B_t is a standard Brownian motion and $a, \mu > 0$ are constants.

You are given that r_t has the following explicit expression:

$$r_t = \mu + (r_0 - \mu)e^{-at} + \sigma \int_0^t e^{a(s-t)} dB_s.$$

(a) Find the probability $\mathbb{P}[r_t < 0]$ when $t \to \infty$. Please show the detailed calculation , rather than use the result on relevant slides directly. [**10**]

Solution:

$$\begin{aligned} &dB_s \sim N(0, ds) \\ \Rightarrow &\sigma e^{a(s-t)} dB_s \sim N(0, \sigma^2 e^{2a(s-t)} ds) \\ \Rightarrow &\int_0^t \sigma e^{a(s-t)} dB_s \sim N(0, \int_0^t \sigma^2 e^{2a(s-t)} ds) \end{aligned}$$

The distribution of r_t is given by:

$$r_t \sim N(\mu + e^{-at}(r_0 - b), \int_0^t \sigma^2 e^{2a(s-t)} ds) = N(\mu + e^{-at}(r_0 - \mu), \frac{\sigma^2}{2a}(1 - e^{-2at}))$$

As $t \to \infty$, $e^{-2at} \to 0$, so we get:

$$r_t \sim N(\mu, \frac{\sigma^2}{2a})$$

Therefore,

$$\lim_{t\to\infty} \mathbb{P}(r_t < 0) = \Phi\left(\frac{-\mu\sqrt{2a}}{|\sigma|}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-\mu\sqrt{2a}}{|\sigma|}} e^{-\frac{x^2}{2}} dx.$$

(b) State what happens to $\mathbb{P}[r_t < 0]$ as $|\sigma| \to 0$. [**5**]

Solution:
When $|\sigma| \to 0$,

$$\lim_{t\to\infty} \mathbb{P}(r_t < 0) = \Phi(-\infty) = 0,$$

i.e. this probability decreases to 0 as $|\sigma| \to 0$

(c) Critically evaluate the Vasicek model. [**5**]

Solution:
According to the result of (b), the unfortunate property of this model is that r_t can be negative.
This is unrealistic because interest rate rarely become negative.
However, the probability of such an event is small when σ is small.
The most important good feature of this model is the "mean reversion" property of r_t: the value r_t will eventually return to its long-term mean μ.
Vasicek model is a one-factor model, and there are some short-comings:

- Single factor short-rate models mean that all maturities behave in the same way - there is no independence.
- There is little consistency in valuation between the models.
- They are difficult to calibrate.

Part a, b from lectures, c similar to seminar.
IFoA CM2 syllabus areas 4.5.2, 4.5.6, 4.5.7.

Question 3 [20 marks]. The company F. Bancroft & Sons issued zero-coupon bonds with expiration time of 5 years today, and the total nominal value of £1 million. The total value of the company now stands at £1.2 million. A continuously compounded interest rate is 3% per annum. The total value of the company follows the Geometric Brownian motion with parameters $\mu = 0.3$ and $\sigma = 0.1$.

(a) Give three examples of credit risk models. Which of them are structural model(s)? [**4**]

(b) Under the **Merton model**, find the current value of the shareholders' equity. [**8**]

(c) In 2 years time, the company's value drops by 10%. What is the probability of F. Bancroft & Sons's default on its obligation to bondholders? [**8**]

Solution:

(a) Three examples: The Merton model, two-state model for credit ratings, and The JLT model. The Merton model is the simplest example of a structural model.

(b) According to the Merton model, the shareholders can be treated as having a European Call option on the assets of the company with strike price $L_0 = £1$ million and maturity $T = 5$ years. Thus the value of shareholders' equity is equal to

$$E_0 = F_0 \Phi(\omega) - L_0 e^{-rT} \Phi\left(\omega - \sigma\sqrt{T}\right).$$

First we calculate ω by using the formula

$$\begin{aligned} \omega &= \frac{\log \frac{F_0}{L_0} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} = 1.5980, \\ \omega - \sigma\sqrt{T} &= \frac{\log \frac{F_0}{L_0} + rT}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T} = 1.3744. \end{aligned}$$

Plugging the numbers into the above formula gives

$$E_0 = 1.2 \times 0.944979 - 0.8481 \times 0.915341 = £\, 0.3577 \,\text{millions}.$$

(c) In two years the value of the company drops to $F_2 = F_0 \times 0.9 = £\, 1.08$ millions. The company's value, under the assumptions of the Black-Scholes theory, follows the Geometric Brownian Motion.

$$F_{2+t} = F_2 e^{\mu t + \sigma W_t}.$$

The company would default if the value of the company drops below the repayment value L_0.

$$\begin{aligned} \mathbb{P}(\text{default}) &= \mathbb{P}(F_5 < L_0) = \mathbb{P}\left(F_2 e^{3\mu + \sigma W_3} < L_0\right) = \mathbb{P}\left(W_3 < \frac{\ln \frac{L_0}{F_2} - 3\mu}{\sigma}\right) \\ &= \Phi\left(\frac{\ln \frac{L_0}{F_2} - 3\mu}{\sigma\sqrt{T}}\right) = \Phi(-4.5879) = 2.2391 \times 10^{-6}. \end{aligned}$$

Part a from lectures, b, c similar to seminar.
IFoA CM2 syllabus areas 4.6.2, 4.6.3.

Question 4 [30 marks].
The price of a share $S(t)$ evolves according to a Geometric Brownian Motion with parameters $S,\ \mu,\ \sigma$, i.e. $S(t) = Se^{\mu t+\sigma W(t)}$. The continuously compounded interest rate is r.
An exotic derivative on this share has the payoff function

$$R(T) = \frac{1}{T}\int_0^T S(t)(S(T)-c)dt.$$

Where c is a constant. The payoff time is T.

(a) Show that $\mathbb{E}\left(e^{aW(t)+bW(t+s)}\right) = e^{\frac{(a+b)^2}{2}t+\frac{b^2}{2}s}$, where $t > 0$ and $s > 0$. **[10]**

Solution:
Denote $Y = aW(t) + bW(t+s)$. Notice that

$$Y = (a+b)W(t) + b(W(t+s) - W(t)).$$

So, Y is a sum of 2 independent random variables and hence $e^Y = e^{(a+b)W(t)} \times e^{b(W(t+s)-W(t))}$ is a product of two independent random variables. It follows that

$$\mathbb{E}\left(e^{aW(t)+bW(t+s)}\right) = \mathbb{E}\left[e^{(a+b)W(t)} \times e^{b(W(t+s)-W(t))}\right] = \mathbb{E}\left[e^{(a+b)W(t)}\right]\times\mathbb{E}\left[e^{b(W(t+s)-W(t))}\right].$$

We know that $\mathbb{E}\left(e^{\sigma W(t)}\right) = e^{\frac{\sigma^2}{2}t}$. Since $W(t+s) - W(t) \sim \mathcal{N}(0, s)$ and therefore

$$\mathbb{E}\left[e^{b(W(t+s)-W(t))}\right] = \mathbb{E}\left[e^{bW(s)}\right] = e^{\frac{b^2}{2}s}.$$

Hence

$$\mathbb{E}\left(e^{aW(t)+bW(t+s)}\right) = e^{\frac{(a+b)^2}{2}t} \times e^{\frac{b^2}{2}s} = e^{\frac{(a+b)^2}{2}t+\frac{b^2}{2}s}. \quad (1)$$

(b) Use the result obtained in (a), calculate the no-arbitrage price of this exotic derivative. **[10]**

Solution:
By Theorem 5.2,

$$C = \mathrm{e}^{-rT}\tilde{\mathbb{E}}\left(\frac{1}{T}\int_0^T S(t)(S(T)-c)\mathrm{d}t\right) = \frac{\mathrm{e}^{-rT}}{T}\tilde{\mathbb{E}}\left(\int_0^T S(t)S(T)\mathrm{d}t - c\int_0^T S(t)\mathrm{d}t\right). \quad (2)$$

To compute this expectation over the risk-neutral probability, we have to turn $\tilde{\mathbb{E}}$ into $\mathbb{E}$ by replacing $S(t)$ and $S(T)$ by $\tilde{S}(t)$ and $\tilde{S}(T)$. Thus, by Theorem 5.3,

$$\tilde{\mathbb{E}}\left(\int_0^T S(t)S(T)\mathrm{d}t\right) = \mathbb{E}\left(\int_0^T \tilde{S}(t)\tilde{S}(T)\mathrm{d}t\right).$$

It is possible to change the order of the two operations (Slide 37, Week 3-4):

$$\mathbb{E}\left(\int_0^T \tilde{S}(t)\tilde{S}(T)\mathrm{d}t\right) = \int_0^T \mathbb{E}\left(\tilde{S}(t)\tilde{S}(T)\right)\mathrm{d}t.$$

In words, rather than first computing the integral and then the expectation, we can first compute the expectation and after that compute the integral.

We have that

$$\tilde{S}(t)\tilde{S}(T) = S\mathrm{e}^{\tilde{\mu}t+\sigma W(t)} \times S\mathrm{e}^{\tilde{\mu}T+\sigma W(T)} = S^2\mathrm{e}^{\tilde{\mu}(t+T)+\sigma W(t)+\sigma W(T)}$$

and hence

$$\mathbb{E}\left(\tilde{S}(t)\tilde{S}(T)\right) = S^2\mathrm{e}^{\tilde{\mu}(t+T)}\mathbb{E}\left(\mathrm{e}^{\sigma(W(t)+W(T))}\right).$$

Using the result stated in (a) with $a = b = \sigma$ and $s = T - t$ we obtain

$$\mathbb{E}\left(\tilde{S}(t)\tilde{S}(T)\right) = S^2\mathrm{e}^{\tilde{\mu}(t+T)+2\sigma^2 t+\frac{\sigma^2}{2}(T-t)} = S^2\mathrm{e}^{\tilde{\mu}(t+T)+1.5\sigma^2 t+\frac{\sigma^2}{2}T}.$$

Since $\tilde{\mu} = r - \frac{\sigma^2}{2}$, we have

$$\mathbb{E}\left(\tilde{S}(t)\tilde{S}(T)\right) = S^2\mathrm{e}^{rT+(r+\sigma^2)t}.$$

Integrating the last expression, we obtain

$$\mathbb{E}\left(\int_0^T \tilde{S}(t)\tilde{S}(T)\mathrm{d}t\right) = S^2\mathrm{e}^{rT}\int_0^T \mathrm{e}^{(r+\sigma^2)t}\mathrm{d}t = S^2\mathrm{e}^{rT}\frac{1}{r+\sigma^2}(\mathrm{e}^{(r+\sigma^2)T} - 1).$$

Similarly, for the second half of (2):

$$\tilde{\mathbb{E}}\left(c\int_0^T S(t)\mathrm{d}t\right) = c\mathbb{E}\left(\int_0^T \tilde{S}(t)\mathrm{d}t\right).$$

Since

$$\tilde{S}(t) = S\mathrm{e}^{\tilde{\mu}t+\sigma W(t)},$$

we have

$$\mathbb{E}\left(\tilde{S}(t)\right) = S\mathrm{e}^{(\tilde{\mu}+\frac{1}{2}\sigma^2)t},$$

so

$$\mathbb{E}\left(\int_0^T \tilde{S}(t)\mathrm{d}t\right) = \mathbb{E}\left(\int_0^T S\mathrm{e}^{(\tilde{\mu}+\frac{1}{2}\sigma^2)t}\mathrm{d}t\right) = \frac{S}{r}\left(e^{r-\frac{1}{2}\sigma^2+\frac{1}{2}\sigma^2 T} - 1\right).$$

Finally we obtain from (2):

$$C = \frac{S^2}{(r+\sigma^2)T}(\mathrm{e}^{(r+\sigma^2)T} - 1) - \frac{cSe^{-rT}}{rT}\left(e^{r-\frac{1}{2}\sigma^2+\frac{1}{2}\sigma^2 T} - 1\right).$$

(c) Consider another exotic call option on the same share with expiration time t.
Its strike price K depends on $S(s)$ and $W(s)$, where $s < t$, i.e.

$$K = e^{-\sigma W(s)} \times (S(s))^2.$$

Denote by $\tilde{C}$ the no-arbitrage price of this option.
Denote by $C(S,T,K,\sigma,r)$ the Black-Scholes price of the standard European call option.

Using the properties of the Wiener process, write down the expression for the price $\tilde{C}$ in terms of the expectation of the risk-neutral process $\tilde{S}(t)$ and $\tilde{\mu}$, and show $\tilde{C} = C(S',T',K',\sigma',r')$. (Please write down the explicit expression of S',T',K',σ',r'). [**10**]

Solution:
In our case, by the definition of our call option, the expiration time $T = t$ and the payoff function G depends on 2 variables, $S(s)$ and $S(t)$, as follows:

$$G((S(s),S(t)) = (S(t)-K)^+ = (S(t) - e^{-\sigma W(s)} \times (S(s))^2)^+.$$

The price $\tilde{C}$ is

$$\begin{aligned}\tilde{C} &= \mathrm{e}^{-rt}\mathbb{E}(\tilde{S}(t) - e^{-\sigma W(s)} \times (\tilde{S}(s))^2)^+ = \mathrm{e}^{-rt}\mathbb{E}(Se^{\tilde{\mu}t+\sigma W(t)} - e^{-\sigma W(s)}S\mathrm{e}^{2\tilde{\mu}s+2\sigma W(s)})^+ \\ &= \mathrm{e}^{-rt}\mathbb{E}\left[\mathrm{e}^{\tilde{\mu}s+\sigma W(s)}(Se^{\tilde{\mu}(t-s)+\sigma(W(t)-W(s))} - Se^{\tilde{\mu}s})^+\right].\end{aligned}$$

Since the random variables $W(t)-W(s)$ and $W(s)$ are independent (as increments of the Wiener process), also the random variables $\mathrm{e}^{\tilde{\mu}s+\sigma W(s)}$ and $(Se^{\tilde{\mu}(t-s)+\sigma(W(t)-W(s))} - Se^{\tilde{\mu}s})^+$ are independent and therefore the expectation of their products splits into the product of their expectations:

$$\tilde{C} = \mathrm{e}^{-rt}\mathbb{E}\left(\mathrm{e}^{\tilde{\mu}s+\sigma W(s)}\right) \times \mathbb{E}(Se^{\tilde{\mu}(t-s)+\sigma(W(t)-W(s))} - Se^{\tilde{\mu}s})^+. \tag{3}$$

We know that $\mathbb{E}\left(\mathrm{e}^{\tilde{\mu}s+\sigma W(s)}\right) = \mathrm{e}^{rs}$. Also, by the definition of the Wiener process, $W(t)-W(s)$ has the same distribution as $W(t-s)$ and so

$$\mathbb{E}(Se^{\tilde{\mu}(t-s)+\sigma(W(t)-W(s))} - Se^{\tilde{\mu}s})^+ = \mathbb{E}(Se^{\tilde{\mu}(t-s)+\sigma W(t-s))} - Se^{\tilde{\mu}s})^+.$$

Plugging the last two relation into (3), we obtain

$$\tilde{C} = \mathrm{e}^{-r(t-s)}\mathbb{E}(Se^{\tilde{\mu}(t-s)+\sigma W(t-s)} - Se^{\tilde{\mu}s})^+ = C(S, t-s, Se^{\tilde{\mu}s}, \sigma, r).$$

Part a similar to seminar, Part b, c application of lecture material.
IFoA CM2 syllabus areas 4.4.2, 6.1.8, 6.1.9.

End of Paper.