Partial Differential Equations (MTH6151)

Course notes

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Chapter 1

Introduction to partial differential equations

1.1 Basic concepts

What is a **partial differential equation (pde)**? It is an equation for a function $U = U(x_1, \dots, x_n)$ of $n \geq 2$ variables involving partial derivatives of U. If the equation depends on only one variable one speaks of an **ordinary differential equation**. Partial differential equations are key to describing the fundamental interactions of Nature and in the modelling of a wide range of systems (economics, finance, population dynamics, ecology, ...).

Question. What do we want to know of a pde?

Besides finding a solution, we want to know how many solutions are there. We also want to know under which circumstances one can expect a solution to exist. Also, one would like to understand the properties of the solutions and also, if necessary know how to compute solutions numerically.

1.1.1 Some terminology

Definition 1.1.1. A **solution** to a pde is a function $U = U(x_1, \dots, x_n)$ which satisfies the equation in some region $\Omega \subset \mathbb{R}^n$.

Note. In this course we will be mostly interested in the case $n = 2$ so that $(x_1, x_2) = (x, y)$ or $(x_1, y_1) = (x, t)$ —the latter choice used in problems involving time.

Definition 1.1.2. The **order** of a pde is the highest derivative which appears in the equations.

Example 1.1.3. The most general second order pde in two independent variables (x, y) is given for $U = U(x, y)$ by

$$F\left(x, y, U(x,y), \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial^2 U}{\partial x^2}, \frac{\partial^2 U}{\partial x \partial y}, \frac{\partial^2 U}{\partial y^2}\right) = 0, \tag{1.1}$$

with F some smooth (i.e. nice) function of the arguments.

Notation. In this course we will systematically use the shorthand notation

$$U_x \equiv \frac{\partial U}{\partial x}, \qquad U_y \equiv \frac{\partial U}{\partial y}, \qquad U_{xx} \equiv \frac{\partial^2 U}{\partial x^2}, \qquad U_{xy} \equiv \frac{\partial^2 U}{\partial x \partial y}, \qquad \ldots$$

In terms of the above notation, equation (1.1) can be rewritten as

$$F(x, y, U, U_x, U_y, U_{xx}, U_{xy}, U_{yy}) = 0.$$

Concrete examples of pde's to be considered in this course are

$$U_x \pm U_t = 0 \qquad \text{(advection equation in } 1+1 \text{ dimensions)}, \tag{1.2a}$$
$$U_{tt} - U_{xx} = 0 \qquad \text{(wave equation in } 1+1 \text{ dimensions)}, \tag{1.2b}$$
$$U_{xx} + U_{yy} = 0 \qquad \text{(Laplace equation in } 2 \text{ dimensions)}, \tag{1.2c}$$
$$U_t - U_{xx} = 0 \qquad \text{(heat equation in } 1+1 \text{ dimensions)}. \tag{1.2d}$$

The advection equation (1.2a) is first order, while equations (1.2b)-(1.2d) are second order.

Note. In this course we will only consider equations of first and second order.

Operators

Given a pde, one often uses the language of **operators** to write it in the form

$$\mathcal{L}U = 0.$$

Note. Just as a function $f : x \to y$ sends a point x to a point y, an operator $L : U \to V$ sends a function to a function. Thus, one can think of an operator as a **function of functions**.

Example 1.1.4. The Laplace equation

$$U_{xx} + U_{yy} = 0$$

can be written as

$$\mathcal{L}U = 0 \qquad \text{with} \qquad \mathcal{L} \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Example 1.1.5. The operator associated to the equation

$$U_x + yU_y = 0,$$

is given by

$$\mathcal{L} \equiv \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Linear pde's

Definition 1.1.6. An operator is **linear** if

(i) $\mathcal{L}(U+V) = \mathcal{L}U + \mathcal{L}V$,

(ii) $\mathcal{L}(\alpha U) = \alpha \mathcal{L}U$

for any functions U, V and constant α.

A partial differential equation $\mathcal{L}U = 0$ is called **linear** whenever $\mathcal{L}$ is linear. Alternatively, a pde is linear if it is linear in U, U_x, U_y, $U_{xx}, \dots$. If the equation is not linear, we say it is **non-linear**.

Example 1.1.7. Equations (1.2a), (1.2b), (1.2c) and (1.2d) are linear.

Example 1.1.8. The equation

$$U_{tt} - U_{xx} + U^2 = 0$$

is non-linear.

Example 1.1.9. The equation

$$U_{tt} - U_{xx} = \sin^2 U$$

is non-linear.

Note. In this course we will focus on linear equations.

Homogeneous and inhomogeneous equations

A concept which will be important in our discussion is the following:

Definition 1.1.10. Given a pde operator $\mathcal{L}$, an equation of the form

$$\mathcal{L}U = 0$$

is said to be **homogeneous**. An equation of the form

$$\mathcal{L}U = f,$$

with $f \neq 0$ a function is called **inhomogeneous**.

The principle of superposition

Some important observations which will be used repeatedly are the following:

- If U_1, U_2, $\dots$, U_N are solutions to $\mathcal{L}U = 0$, a linear pde, then

  $$U_1 + \cdots + U_N$$

  is also a solution. This observation is called the **principle of superposition** and is a key property of linear pde's. More about this later!

- If U solves the homogeneous linear equation $\mathcal{L}U = 0$ and V solves the inhomogeneous linear equation $\mathcal{L}V = g$ then $U + V$ solves the inhomogeneous equation. This can be seen from
$$\mathcal{L}(U + V) = \mathcal{L}U + \mathcal{L}V = 0 + g = g.$$

Question. Consider the pde
$$(\cos x^3)U_x + (xy)U_y = \tan(3x + 5y^3).$$
Is this equation linear or non-linear?

1.2 Solving some basic pde's

Start by looking at a very basic example, an ordinary differential equation (ode).

Example 1.2.1. Consider the ordinary differential equation for the function $U = U(t)$
$$\frac{dU}{dt} = 0.$$
The solution is given by
$$U(t) = c$$
with c a constant.

Consider now a function $U = U(x, y)$ of two variables.

Example 1.2.2. The solution of the pde
$$U_x = \frac{\partial U}{\partial x} = 0$$
is given (by integrating with respect to x) by
$$U(x, y) = f(y)$$
where f is a function of y only.

> **Note.** Whereas ode's have general solutions involving arbitrary constants, pde's have general solutions involving arbitrary functions of some of the coordinates.

Consider now an extension of the previous example:

Example 1.2.3. Let
$$U_{xx} = \frac{\partial^2 U}{\partial U^2} = 0.$$
Integrating once with respect to x one finds that
$$U_x = f(y)$$
as in the previous example. Integrating once more one finds
$$U(x, y) = xf(y) + g(y)$$
with f, g arbitrary functions of y.

A more sophisticated example is:

Example 1.2.4. Consider the equation

$$U_{xy} = 0.$$

Integrating once with respect to x one finds that

$$U_y = f(y).$$

Now, integrating with respect to y one has

$$U(x,y) = g(x) + \int f(y)dy.$$

where g is a function of x only. But $\int f(y)dy$ is, in fact, a function of y so we can actually write

$$U(x,y) = g(x) + F(y)$$

with $F(x) \equiv \int f(y)dy$. We can readily check that the above is, indeed, a solution by direct differentiation.

> **Note.** Recall that if a function $U(x,y)$ can be differentiated twice, then
>
> $$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x},$$
>
> or in terms of our new notation
>
> $$U_{xy} = U_{yx}.$$

Example 1.2.5. Consider the equation

$$U_{xx} + U = 0.$$

It can be checked that the solution is given by

$$U(x,y) = f(y)\cos x + g(y)\sin x.$$

The above equation should be compared with the ode

$$z'' + z = 0.$$

> **Note.** The above example shows that often it is useful to pretend that $U(x,y) = U(x)$ and then see what ode arises.

A similar example to the previous one is:

Example 1.2.6. Let

$$U_x = 2x \sin y + e^{xy}.$$

Direct integration gives

$$U(x, y) = x^2 \sin y + \frac{e^{xy}}{y} + f(y).$$

And finally two more examples which will be further elaborated during the course:

Example 1.2.7. One can readily verify by direct computation that

$$U(x, y) = \sin(nx) \sinh(ny)$$

solves

$$U_{xx} + U_{yy} = 0.$$

Example 1.2.8. If f is a differentiable function of one variable and $c \neq 0$ is a constant, then

$$U(x, t) = f(x - ct)$$

satisfies the advection equation

$$U_t + cU_x = 0.$$

For example, if $f(z) = \sin z$ then

$$f(x - ct) = \sin(x - ct).$$

The assertion can be verified using the chain rule for ordinary derivatives.

Note. Recall that if $f = f(x)$ and $g = g(x)$ are two differentiable functions of x then the derivative of the composition $f \circ g$ is given by

$$\frac{df \circ g}{dx} = \frac{d}{dx} f(g(x)) = \frac{df(g(x))}{dg} \frac{dg}{dx}.$$

Progress Check

1. What is a partial differential equation (pde)?
2. What is the difference between a pde and an ordinary differential equation (ode)?
3. What is the order of a pde?
4. What is an operator? What is a linear operator?
5. When do we say that a pde is linear?
6. When do we say that an equation is homogeneous/inhomogeneous?
7. What is the principle of superposition for linear pde's?
8. How do you verify that a given function is a solution to a given pde?
9. What does the chain rule for functions of one variable say?

Chapter 2

First order pde's

In this part of the course we study first order partial differential equations. We mainly focus on linear equations, but the main method of study is, in principle, also applicable to non-linear equations.

2.1 Refresher

In the following we will need some results from Calculus.

2.1.1 The chain rule for partial derivatives

An important tool in the analysis of pde's is the chain rule for partial derivatives. Given the usual coordinates (x, y) on $\mathbb{R}^2$ consider new coordinates $(\tilde{x}, \tilde{y})$ given by an expression of the form

$$\tilde{x} = \tilde{x}(x, y), \qquad \tilde{y} = \tilde{y}(x, y).$$

That is, we assume that $(\tilde{x}, \tilde{y})$ can be written as functions of the old coordinates (x, y). One is then interested in the relation between the partial derivatives $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial \tilde{x}$, $\partial/\partial \tilde{y}$. This is given by the **chain rule** for partial derivatives which, in the language of operators takes the form:

$$\frac{\partial}{\partial x} = \frac{\partial \tilde{x}}{\partial x}\frac{\partial}{\partial \tilde{x}} + \frac{\partial \tilde{y}}{\partial x}\frac{\partial}{\partial \tilde{y}},$$
$$\frac{\partial}{\partial y} = \frac{\partial \tilde{x}}{\partial y}\frac{\partial}{\partial \tilde{x}} + \frac{\partial \tilde{y}}{\partial y}\frac{\partial}{\partial \tilde{y}}.$$

Note. Observe the pattern in the above seemingly complicated equations which helps to remember the formulae.

2.1.2 Gradient and directional derivatives

Given a function $f = f(x, y)$ the **gradient** ∇f is the vector defined by

$$\nabla f \equiv \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (f_x, f_y).$$

Geometrically, $f = f(x, y)$ can be thought of as a surface in $\mathbb{R}^3$ where the z coordinate is given by the function f. At a given point (x, y), the gradient gives the direction of maximum growth (**steepest slope**) of f.

Now, given a vector $\vec{v} = (v_1, v_2)$ on $\mathbb{R}^2$, the **directional derivative** $\nabla_{\vec{v}} f$ of the function $f = f(x, y)$ in the direction of f is defined by

$$\nabla_{\vec{v}} f \equiv \vec{v} \cdot \nabla f = v_1 f_x + v_2 f_y,$$

where $\cdot$ denotes the **inner product (dot product)**. This derivative gives the change of f in the direction of $\vec{v}$.

2.1.3 Fundamental theorem of Calculus

In a nutshell the **Fundamental theorem of Calculus** says that integration and differentiation are inverse operations. In a more precise manner, one has that

$$f(x) - f(x_\star) = \int_{x_\star}^{x} f'(s) ds,$$

where $f = f(x)$ is a function with derivative f'.

2.2 First order pde's with constant coefficients

In this section we discuss how to obtain the solutions of the partial differential equation

$$aU_x + bU_y = 0, \tag{2.1}$$

with a, $b \neq 0$ some constants. This equation is a first order homogeneous equation. We will analyse two methods to obtain the solution to this equation.

2.2.1 Solution by change of coordinates (analytic approach)

A general observation which is often very useful is that a change of variables can turn a seemingly hard problem into an easy one. We try this approach here.

In what follows we consider the change of variables given by

$$\begin{aligned} \tilde{x}(x, y) &= ax + by, \\ \tilde{y}(x, y) &= bx - ay. \end{aligned}$$

We now express equation (2.1) in terms of the coordinates $(\tilde{x}, \tilde{y})$. For this, we make use of the chain rule. One has that

$$\begin{aligned} U_x &= \frac{\partial U}{\partial x} = \frac{\partial \tilde{x}}{\partial x}\frac{\partial U}{\partial \tilde{x}} + \frac{\partial \tilde{y}}{\partial x}\frac{\partial U}{\partial \tilde{y}} = aU_{\tilde{x}} + bU_{\tilde{y}}, \\ U_y &= \frac{\partial U}{\partial y} = \frac{\partial \tilde{x}}{\partial y}\frac{\partial U}{\partial \tilde{x}} + \frac{\partial \tilde{y}}{\partial y}\frac{\partial U}{\partial \tilde{y}} = bU_{\tilde{x}} - aU_{\tilde{y}}. \end{aligned}$$

Substituting these expressions into the left hand side of equation (2.1) one has that

$$\begin{aligned} aU_x + bU_y &= a(aU_{\tilde{x}} + bU_{\tilde{y}}) + b(bU_{\tilde{x}} - aU_{\tilde{y}}) \\ &= (a^2 + b^2)U_{\tilde{x}}. \end{aligned}$$

Thus, one concludes that in terms of the coordinates $(\tilde{x}, \tilde{y})$, equation (2.1) takes the form

$$U_{\tilde{x}} = 0.$$

We already know how to solve this equation. Namely one has that

$$U(\tilde{x}, \tilde{y}) = f(\tilde{y}),$$

where f is a function only of the coordinate $\tilde{y}$. We can rewrite this expression in terms of the coordinates (x, y) as

$$U(x, y) = f(bx - ay). \tag{2.2}$$

That is, $U(x, y)$ depends only on the combination $bx - ay$. The formula (2.2) is the **general solution** of equation (2.1). Observe that it involves an arbitrary function.

Question. Where does the change of variables we have used to solve the pde comes from? Could we have guessed it?

2.2.2 Geometric approach

By taking a **geometric approach**, one can understand where do the change of variables we used comes from.

The basic observation is the following:

$$\begin{aligned} aU_x + bU_y &= (a, b) \cdot (U_x, U_y) \\ &= (a, b) \cdot \nabla U \\ &= \nabla_{\vec{v}} U, \end{aligned}$$

where $\vec{v} \equiv (a, b)$. Thus, equation (2.1) means geometrically that the function U is constant in the direction of $\vec{v}$.

Question. What curves have tangent given by the **constant** vector $\vec{v} = (a, b)$?

The curves necessarily have to be lines! The lines have slope $dy/dx = b/a$ so that their equation is of the form

$$y = \frac{b}{a} x + c, \qquad c \quad \text{a constant.}$$

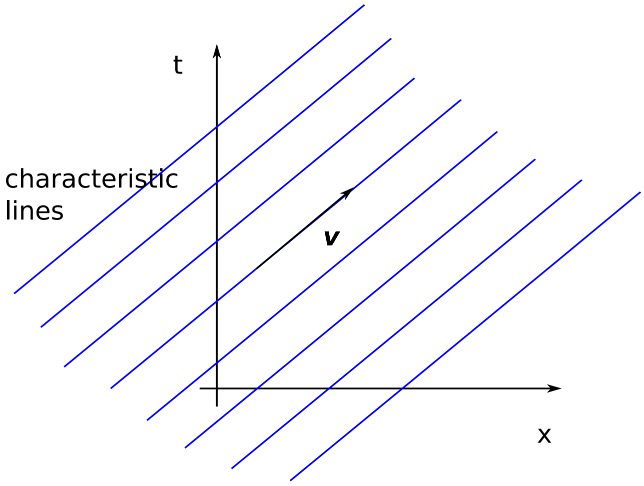
The last expression can be rewritten as

$$bx - ay = c. \tag{2.3}$$

From the previous discussion it follows that the solution is constant along these lines —we call these lines **characteristic lines**. Thus, the function $U(x, y)$ depends on the value of c only and one can write

$$U(x, y) = f(c) = f(bx - ay)$$

where in the last equality we have used (2.3) and f is, again, an arbitrary function of its argument. Observe that the result we have obtained coincides with what we had in (2.2).



Question. How do we specify the function f?

For this, one needs to impose **initial** and/or **boundary conditions**. What are these?

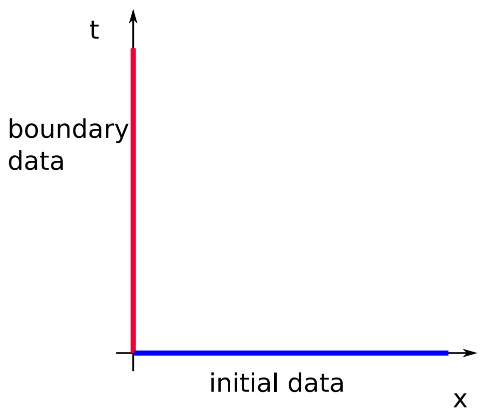
Notation. In what follows it will be conceptually convenient to use coordinates (x, t) rather (x, y) and think of t as a time —that is, the equations we will analyse describe some process of evolution in time. Conventionally, the time coordinate is assigned to the y-axis.

2.2.3 Initial and boundary conditions

Definition 2.2.1.

i. A prescription of the value of the solution to a pde at $t = 0$ (i.e. along the x-axis) will be called an **initial condition.**

ii. A prescription of the value of the solution to a pde at $x = 0$ (i.e. along the t-axis) will be called a **boundary condition**.



Note. More generally, boundary conditions can be prescribed on any line parallel to the t-axis —i.e. lines of the form $x = x_\bullet$ with $x_\bullet$ a constant. More generally, one can have combinations of boundary and initial data. Initial and boundary data arise from physical, geometric and/or commonsensical considerations.

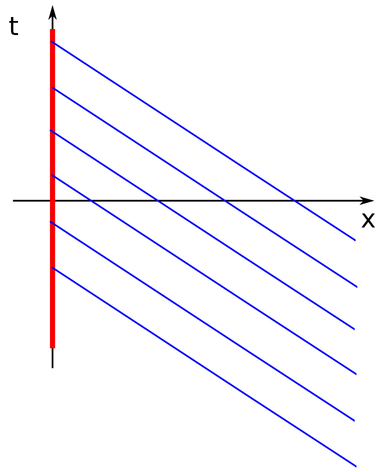
We exemplify these concepts with a couple of examples.

Example 2.2.2. Solve

$$
\begin{aligned}
&4U_x - 3U_t = 0, \\
&U(0,t) = t^3 \qquad \text{(boundary conditions)}.
\end{aligned}
$$

From the previous discussion one has that $a = 4$, $b = -3$ and the solution to the equation is constant along the lines

$$-3x - 4t = c.$$



Thus, the solution is of the form

$$U(x,t) = f(c) = f(-3x - 4t).$$

We now make use of the boundary condition to determine the function f. On the t-axis one has that $x = 0$ so that $c = -4t$. Thus, one can write $t = -c/4$. Using the latter one can write, on the one hand, that

$$U(0,t) = t^3 = -\frac{1}{64}c^3.$$

On the other hand, from the general solution one has that

$$U(0,t) = f(c).$$

Hence, one concludes that

$$f(c) = -\frac{1}{64}c^3.$$

Thus, the solution determined by the prescribed boundary data is given by

$$U(x,t) = \frac{1}{64}(3x + 4t)^3.$$

One can verify that the above expression is indeed a solution to the original problem by direct evaluation.

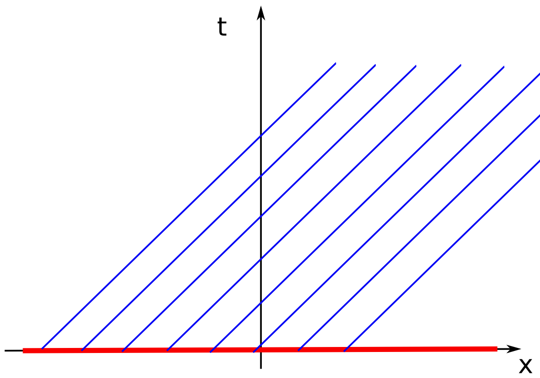
Note. Observe that after prescribing boundary data one has obtained a **unique** solution.

Example 2.2.3. Solve

$$\begin{aligned} &3U_x + 2U_t = 0, \\ &U(x,0) = \sin x \qquad \text{(initial condition)}. \end{aligned}$$

In this case, following the general discussion gives $a = 3$, $b = 2$ so that the solution is constant along the line

$$2x - 3t = c.$$



The general solution is then given by

$$U(x,t) = f(c) = f(2x - 3t).$$

The characteristic lines intersect the x-axis at $x = c/2$ (i.e. $c = 2x$). Now,

$$U(x,0) = \sin x = \sin \frac{c}{2}.$$

However, one also has that

$$U(x,0) = f(c).$$

Accordingly, one concludes that

$$f(c) = \sin \frac{c}{2},$$

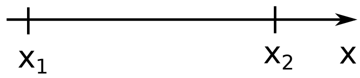
and the solution for the given initial data is given by

$$U(x,t) = \sin \frac{1}{2}(2x - 3t).$$

2.2.4 An application: traffic models

Equations like (2.1) arise in many models. In this section we consider a traffic model. This theory was invented in Manchester by Sir J. Lighthill and G. B. Whitham in 1955. The ideas of this model are also applicable in the discussion of glacier flows and sedimentation in river deltas.

In what follows we are interested in describing the traffic along a one-directional road. We assume the road to be straight —although this is not key for the discussion. A position along the road is described by the coordinate x. In this model the **traffic density** $\rho(x,t)$ is defined as the number of cars (or other vehicles) per unit distance at time t and position x. The traffic density is a type of average.



The problem one is interested in solving is to find $\rho(x,t)$ assuming that the initial density $\rho(x,0)$ is known —that is, we know the initial distribution of cars along the road.

We construct an equation for the model by the following considerations: the number of cars between two (arbitrary) fixed points x_1 and another x_2 at time t is given by

$$\int_{x_1}^{x_2} \rho(x,t)dx.$$

The rate of change of the number of cars between x_1 and x_2 with respect to time is given by

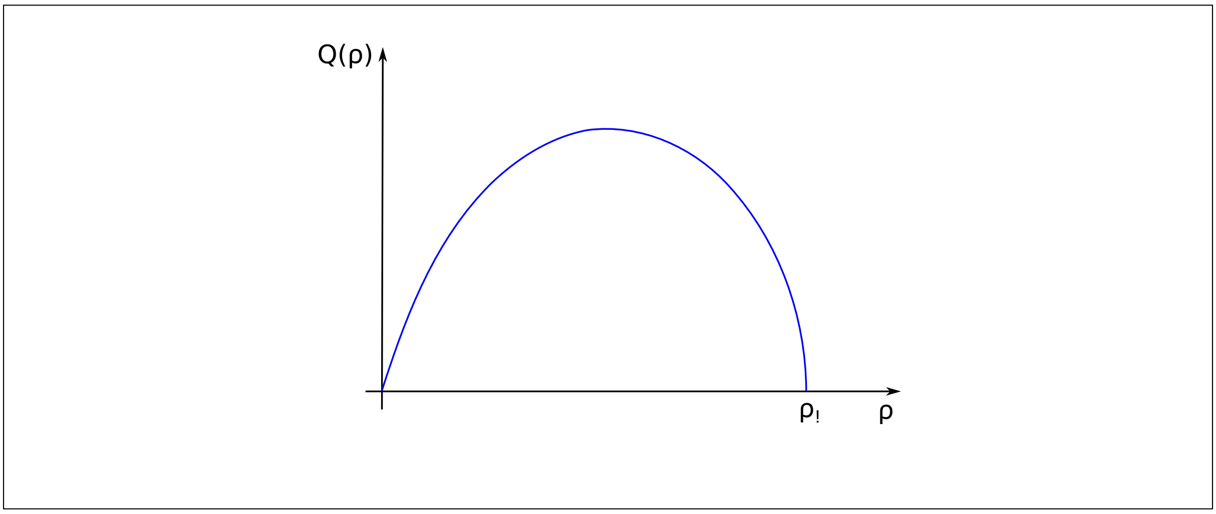
$$\begin{aligned}
\frac{\partial}{\partial t}\int_{x_1}^{x_2} \rho(x,t)dx = \int_{x_1}^{x_2} \frac{\partial}{\partial t}\rho(x,t)dx \\
&\approx (\text{Number of cars through } x_1 \text{ between } t \text{ and } \Delta t) \\
&\qquad -(\text{Number of cars through } x_2 \text{ between } t \text{ and } \Delta t) \\
&= Q(x_1,t) - Q(x_2,t).
\end{aligned}$$

We call $Q(x,t)$ the **flow** at the position x and time t. The above expression says in plain words that the change in the number of cars between x_1 and x_2 is due to cars entering and leaving the section of the road under consideration —we can call this the **law of conservation of cars**. To complete the model one needs to say something about the flow $Q(x,t)$. A reasonable assumption is the following:

Assumption. The flow of cars depends on the density of cars —that is, one has

$$Q(x,t) = Q(\rho(x,t)) = Q(\rho).$$

The precise form of $Q(x,t)$ is determined by observations. A typical form for the flow is given in the figure below. Observe that if there are no cars then the flow is zero. As the number (i.e. density) of cars starts increasing so does the flow. It reaches a maximum and then starts decreasing (as the road starts getting clogged) and is again zero if there are too many cars in the road.



Using the Fundamental Theorem of Calculus (keeping t fixed) one has that

$$Q(x_2,t) - Q(x_1,t) = \int_{x_1}^{x_2} \frac{\partial}{\partial x} Q(x,t)dx.$$

Thus, one has that

$$\int_{x_1}^{x_2} \frac{\partial}{\partial t}\rho(x,t)dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x} Q(x,t)dx.$$

However, the points x_1 and x_2 defining the section of the road under consideration are arbitrary so

$$\frac{\partial}{\partial t}\rho(x,t) = -\frac{\partial}{\partial x} Q(x,t).$$

Now, to conclude, we need to compute $\partial Q(x,t)/\partial x$ given that $Q(x,t) = Q(\rho)$. For this we use the chain rule:

$$\frac{\partial Q}{\partial x}(\rho) = \frac{d}{d\rho} Q \frac{d\rho}{dx} = Q'\rho_x.$$

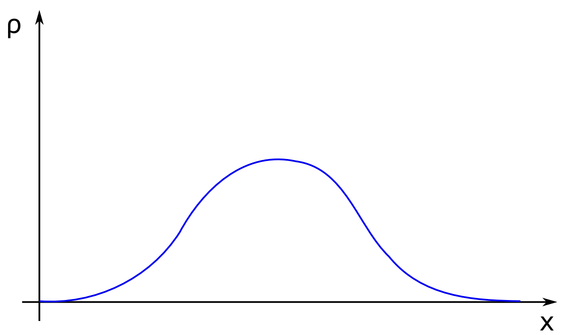
Hence, one obtains the equation

$$\rho_t + Q'(\rho)\rho_x = 0.$$

If the density of cars is very small (i.e. $\rho << 1$) then $Q(\rho) \approx k\rho$ with $k > 0$ a constant —cf. the plot of $Q(\rho)$. Thus, $Q' = k$ and the one obtains

$$\rho_t + k\rho_x = 0. \tag{2.4}$$

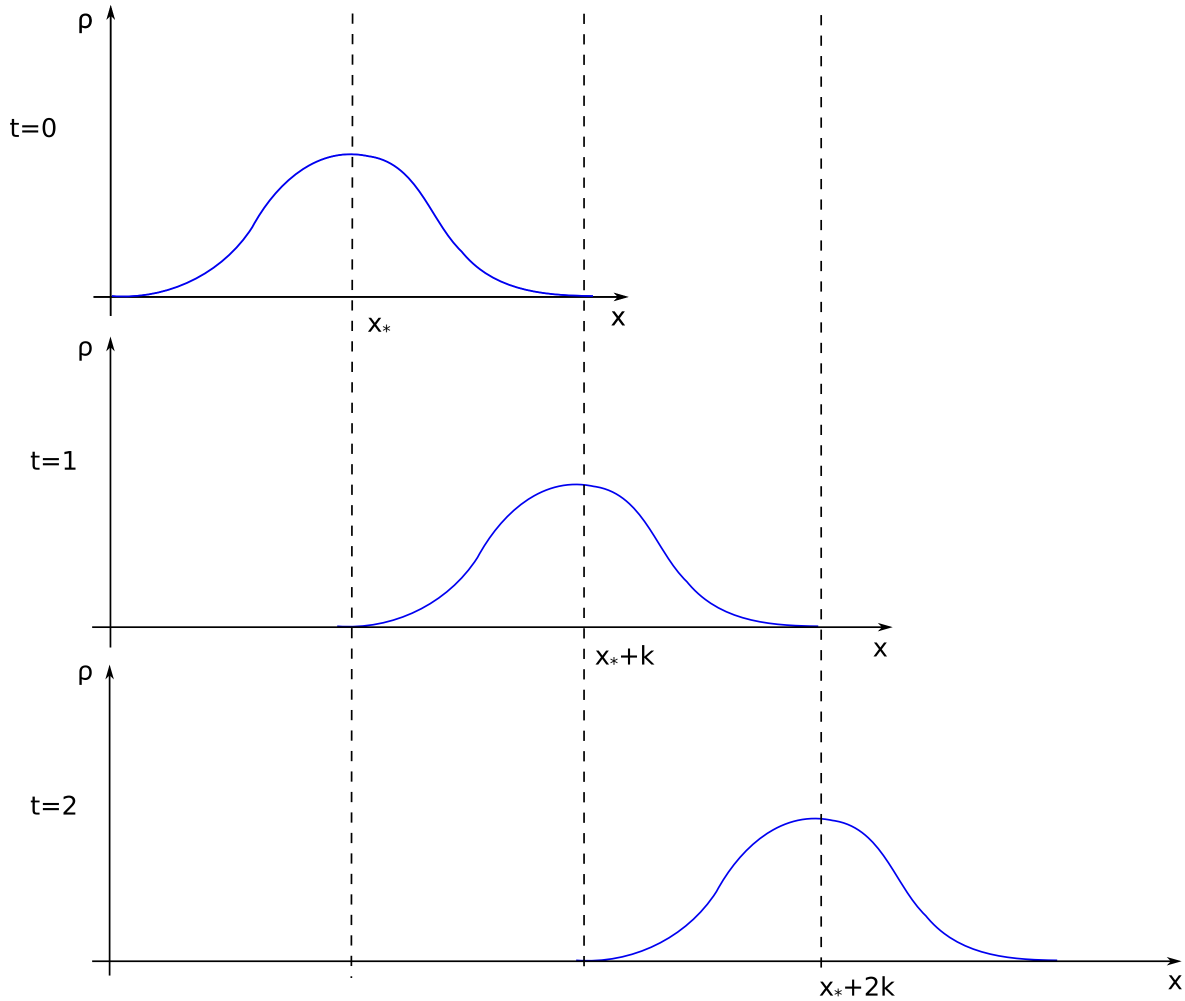
This is an equation of the same form as (2.1) —so, we know how to solve it! Now, assume that one is given an initial density of cars $\rho(x,0) = f(x)$ having the form of a bump as in the figure below:



Using the method discussed in the previous sections we find that the solution for the given initial data is given by

$$\rho(x,t) = f(x - kt).$$

The interpretation of this solution is as follows: as time increases, the initial bump moves to the right (keeping its shape) —see the figure below. In other words, if there is little traffic in the road, the cars move basically in formation.



2.3 The general linear first order pde with variable coefficients

In this section we will discuss how to solve the equation

$$a(x,y)U_x + b(x,y)U_y = c(x,y)U + d(x,y) \tag{2.5}$$

where a, b, c and d are functions of the coordinates (x,y). The **method of characteristics** used to analyse the equation with constant coefficients can be extended to consider this type of equation. The point of departure is the geometric perspective we followed in the previous section.

2.3.1 Geometric approach

Equation (2.5) can be written as

$$\big(a(x,y), b(x,y)\big) \cdot \nabla U = c(x,y)U + d(x,y)$$

so that

$$\nabla_{\vec{v}} U = c(x,y)U + d(x,y), \qquad \text{with} \quad \vec{v} \equiv \big(a(x,y), b(x,y)\big).$$

In this case the vector $\vec{v}$ is no longer constant. This means that the characteristics are no longer lines but curves. If the characteristics are of the form $y = y(x)$ then they satisfy the ordinary differential equation

$$\frac{dy}{dx} = \frac{b(x,y(x))}{a(x,y(x))}. \tag{2.6}$$

Question. Can an ode always be solved?

There is a general result known as the **Picard-Lindelöf theorem** that basically says that an ode always has a solution for given initial conditions. Of course, this result does not say (at least directly) how to solve the equation.

In what follows we assume that we can solve the ode (2.6). This gives the solution in the form $y = y(x)$. Now, we compute the derivative of U along the characteristics —for this we use the chain rule as follows:

$$\begin{aligned}\frac{d}{dx}U(x,y(x)) &= \frac{dx}{dx}\frac{\partial U}{\partial x} + \frac{dy}{dx}\frac{\partial U}{\partial y} \\ &= U_x + \frac{b(x,y(x))}{a(x,y(x))}U_y.\end{aligned}$$

Finally, using equation (2.5) one obtains

$$\frac{d}{dx}U(x,y(x)) = \frac{c(x,y(x))}{a(x,y(x))}U + \frac{d(x,y(x))}{a(x,y(x))}.$$

This equation is, again, another ode. Its solutions yields the value of $U(x,y)$ along a given characteristic curve.

Note. If the characteristic curves cover the whole plane $\mathbb{R}^2$, then one obtains a solution to equation (2.5) on the whole of $\mathbb{R}^2$. On the other hand, if the characteristics do not exist somewhere, then the solution breaks down there —intuitively, one can say that the solution does not know where to go!

2.3.2 Examples

We now exemplify the general theory of the previous subsection with a number of examples.

Example 2.3.1. Find the general solution of

$$U_x + yU_y = 0.$$

The equation can be rewritten as

$$(1, y) \cdot \nabla U = 0,$$

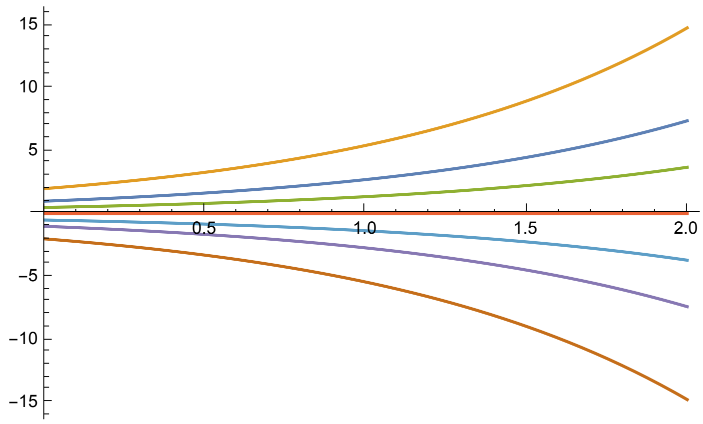
so that U is constant along the curves with tangent given by $\vec{v} = (1, y)$. The slope of the curves is $y/1$ and, hence, the ordinary differential equation to be solved is

$$\frac{dy}{dx} = y.$$

The solutions are given

$$y(x) = Ce^x, \qquad \text{with} \quad C \quad \text{a constant.}$$

These are the characteristics of the pde. A plot for various values of C is given below.



It is observed that, in fact, the whole planes can be covered by these curves by varying C —i.e.

$$\mathbb{R}^2 = \{(x, y) \,|\, y = Ce^x, \ C \in \mathbb{R}\}.$$

Now, observe that for $U(x, y(x)) = U(x, Ce^x)$ one has that

$$\begin{aligned} \frac{d}{dx}U(x, Ce^x) &= U_x + Ce^x U_y \\ &= U_x + yU_y = 0. \end{aligned}$$

Thus, along each characteristic curve the solution is a constant and the solution can only depend on C —that is,

$$U(x, y) = f(C).$$

However, as $y = Ce^x$, one has that $C = ye^{-x}$. Hence, one can write the general solution as

$$U(x, y) = f(ye^{-x}),$$

where f is an arbitrary function of a single variable.

Example 2.3.2. Find the solution to the boundary value problem

$$U_x + yU_y = 0,$$
$$U(0, y) = y^3.$$

From the previous example we know that the general solution is given by

$$U(x, y) = f(ye^{-x}).$$

Thus, one has that

$$U(0, y) = f(y) = f(C).$$

But one also has that

$$U(0, y) = y^3 = C^3$$

Hence, $f(C) = C^3$ and the required solution is given by

$$U(x, y) = \left(ye^{-x}\right)^3 = y^3 e^{-3x}.$$

Exercise. Check by direct computation that $U(x, y) = y^3 e^{-3x}$ is, indeed, the required solution.

Example 2.3.3. Find the general solution to

$$(1 + x^2)U_x + U_y = 0.$$

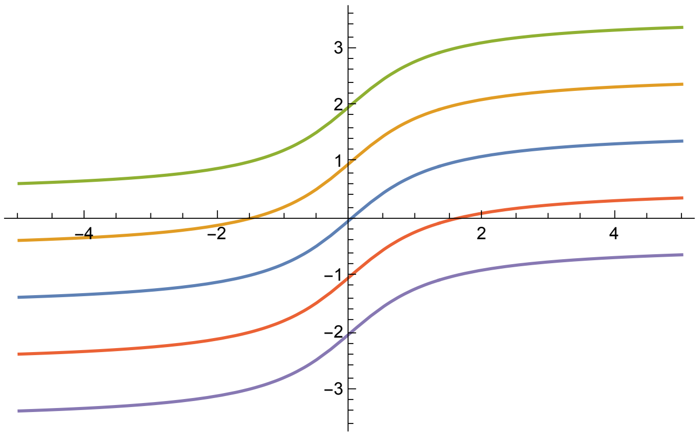
In this case the equation for the characteristic curves is given by

$$\frac{dy}{dx} = \frac{1}{1 + x^2}.$$

The solution to this ode is given by (why?):

$$y(x) = \arctan x + C, \qquad C \quad \text{a constant.}$$

A plot of the characteristics for various values of C is given below.



Again one can check that they actually cover the whole plane.
Now, from the general theory one has that

$$U(x, \arctan x + C)$$

is constant along the characteristics —of course, one can also verify it by direct computation. Hence,

$$U(x, \arctan x + C) = f(C) = \text{constant for given } C.$$

On the other hand one has that

$$C = y - \arctan x$$

so that

$$U(x, y) = U(0, y - \arctan x) = f(y - \arctan x),$$

with f a function of a single argument. This is the general solution of the equation.

Example 2.3.4. Find the general solution to

$$U_x + 2xy^2U_y = 0.$$

In this case the equation for the characteristics is given by

$$\frac{dy}{dx} = 2xy^2.$$

It follows that

$$\int \frac{dy}{y^2} = \int 2x dx + C.$$

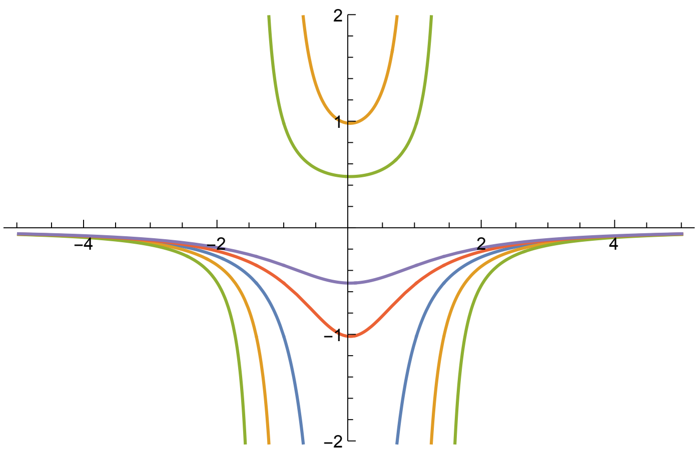
Integrating one gets

$$-\frac{1}{y} = x^2 + C,$$

so that after some reorganisation one ends up with

$$y = \frac{1}{C - x^2}.$$

A plot of the characteristic curves for various choices of C are given below.



Note that the curves do not seem to fill the plane so that the solution may not exist for all (x, y). Again, from general theory we know that $U(x, y)$ is constant along these curves. That is,

$$U(x, y(x)) = f(C).$$

Observing that in this case

$$C = x^2 + \frac{1}{y}$$

one concludes that the required general solution is given by

$$U(x, y) = f\left(x^2 + \frac{1}{y}\right).$$

Example 2.3.5. Find the solution to the boundary value problem

$$\begin{aligned} &\sqrt{1-x^2}U_x + U_y = 0, \\ &U(0, y) = y. \end{aligned}$$

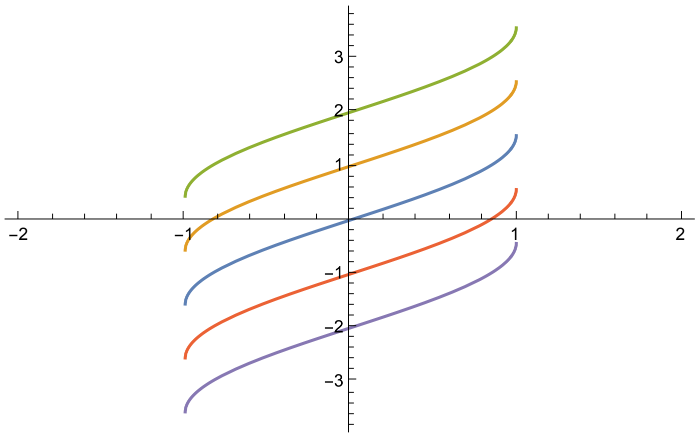
In this case the ode for the characteristic curves is given by

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

The general solution to this ode is

$$y(x) = \arcsin x + C$$

—why? A plot of the curves for various values of C is given below:



Observe, again, that the curves do not cover the whole plane. Now, by the general theory (or direct computation)

$$\frac{d}{dx}U(x, y(x)) = 0,$$

so that

$$U(x, y(x)) = f(C).$$

Hence, the general solution to the equation is given by

$$U(x, y) = f(y - \arcsin x).$$

Evaluating at $x = 0$ one finds that $U(0, y) = f(y)$. Thus, comparing with the boundary condition one concludes that $f(y) = y$. Hence, the solution we look for is

$$U(x, y) = y - \arcsin x.$$

Example 2.3.6. Find the general solution to the equation

$$U_t + xU_x = \sin t.$$

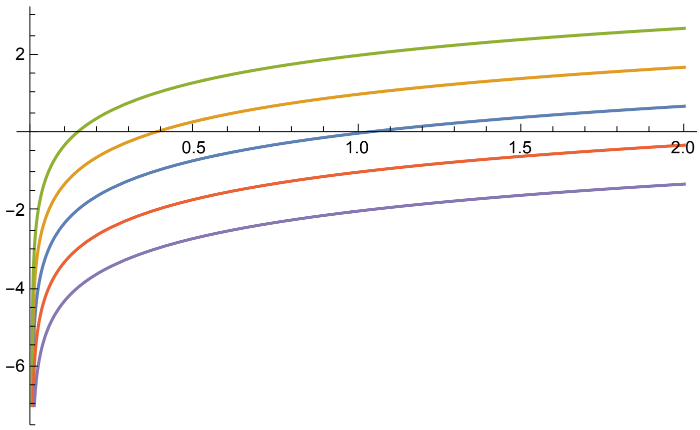
This is an example of an inhomogeneous equation. The ode for the characteristics is in this case given by

$$\frac{dt}{dx} = \frac{1}{x}.$$

The general solution to this equation is given by

$$t(x) = \ln x + C.$$

It will be convenient to rewrite the latter in a slightly different form: $t = \ln x + \ln C$, so that $t = \ln Cx$. A plot of the curves for various values of C is given below:



From the general theory (or direct computation) one further obtains the ode

$$\frac{d}{dx}U(x, t(x)) = \frac{\sin t}{x}$$

Expressing t in terms of x using the equation for the characteristic curves one finally finds that

$$\frac{dU}{dx} = \frac{\sin \ln(Cx)}{x}.$$

Using a the substitution $z = \ln Cx$ one has that

$$\int \frac{\sin \ln(Cx)}{x} dx = \int \sin z dz = -cosz = -\cos \ln Cx,$$

so that

$$U = -\cos \ln Cx + f(C).$$

Eliminating C using $C = e^t/x$ one concludes that

$$U(x, t) = -\cos t + f\left(\frac{e^t}{x}\right).$$

Example 2.3.7. Find the general solution to the equation

$$xU_x + yU_y = kU, \qquad k \quad \text{a constant.}$$

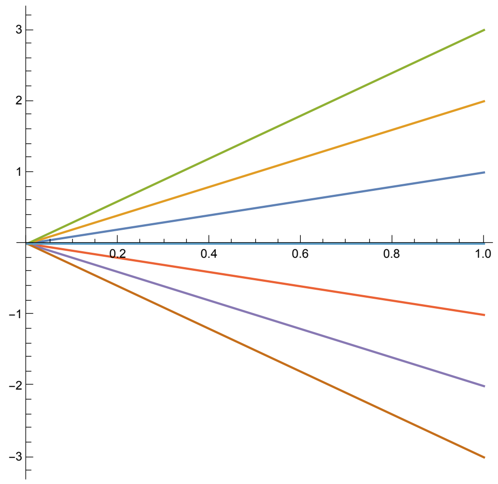
This equation is known as an Euler equation. The characteristic equation is then given by

$$\frac{dy}{dx} = \frac{y}{x},$$

which has general solution given by

$$y(x) = Cx,$$

with C a constant —why? A plot of the curves is shown below —observe that they intersect at the origin.



From the general theory (or direct computation) one has that

$$\frac{d}{dx}U(x, y(x)) = \frac{k}{x}U.$$

We can then integrate it as follows:

$$\int \frac{dU}{U} = k\int \frac{dx}{x} + f(C)$$

so that

$$U(x, y(x)) = f(C)x^k.$$

Now, using the equation $C = y/x$ to eliminate C one obtains the general solution

$$U(x, y) = f\left(\frac{y}{x}\right)x^k.$$

2.3.3 A second application: population models

First order pde's arise in the study of the age distribution in a population.

Populations (of persons, animals, plants...) obey a sort of **continuity equation** as in the case of traffic models. The key difference is that members in a population are born and die. So, the number of individuals in a population does not remain constant in time. Let $N(x, t)$

denote the number of individuals of age x at time t. We are interested in knowing what happens some time later, say at time $t+h$. Then

$$N(x+h,t+h) \ : \ \text{Number of people of age } x+h \text{ at time } t+h,$$

so that

$$N(x+h,t+h)-N(x,t) \ : \ -(\text{Number of deceases of age in the interval } [x,x+h] \text{ in the period } [t,t+h]).$$

At a given moment of time t the number of deceases of people of age x is proportional to the number of people at that particular age (roughly). We write this as $\mu(x,t)N(x,t)$, where $\mu(x,t) > 0$ is the **mortality rate**. The mortality rate depends on the age group and on time —e.g. there is more mortality in early age, and also more in Winter than in Summer. To obtain the total number of deceases of age in the interval $[x,x+h]$ in the period $[t,t+h]$ one needs to integrate:

$$\int_0^h \mu(x+s,t+s)N(x+s,t+s)ds.$$

Hence, we obtain that

$$N(x+h,t+h)-N(x,t) = -\int_0^h \mu(x+s,t+s)N(x+s,t+s)ds.$$

Now, divide by h to obtain

$$\frac{N(x+h,t+h)-N(x,t)}{h} = -\frac{1}{h}\int_0^h \mu(x+s,t+s)N(x+s,t+s)ds.$$

Taking the limit as $h \to 0$ one has that

$$\lim_{h\to 0}\frac{N(x+h,t+h)-N(x,t)}{h} = \left.\frac{dN}{dh}\right|_{h=0},$$

and that

$$\lim_{h\to 0} -\frac{1}{h}\int_0^h \mu(x+s,t+s)N(x+s,t+s)ds = -\mu(x,t)N(x,t),$$

where in the last equality one has used L'Hôpital's rule to evaluate the limit. Finally, using the chain rule we find that

$$\frac{dN}{dh} = \frac{\partial(x+h)}{\partial h}\frac{\partial N}{\partial(x+h)} + \frac{\partial(t+h)}{\partial h}\frac{\partial N}{\partial(t+h)},$$

so that

$$\left.\frac{dN}{dh}\right|_{h=0} = \frac{\partial N}{\partial x} + \frac{\partial N}{\partial t}.$$

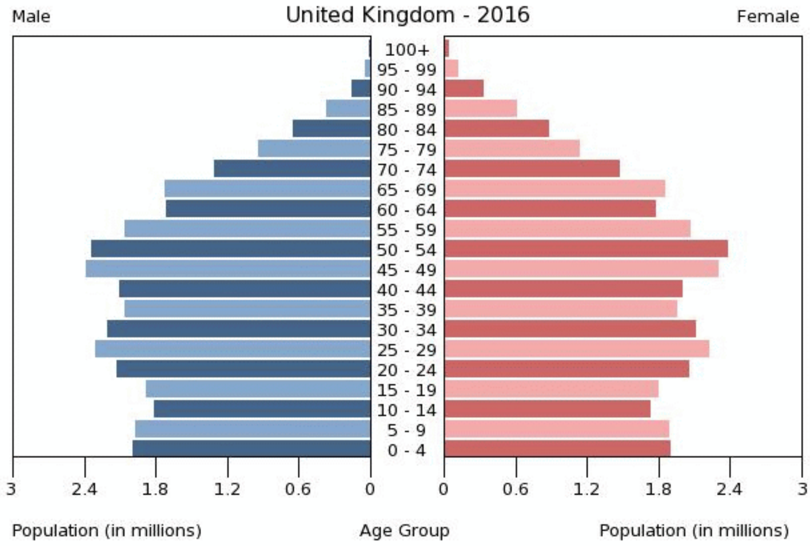
Then in a simple model (ignoring, say immigration and emigration) one has that

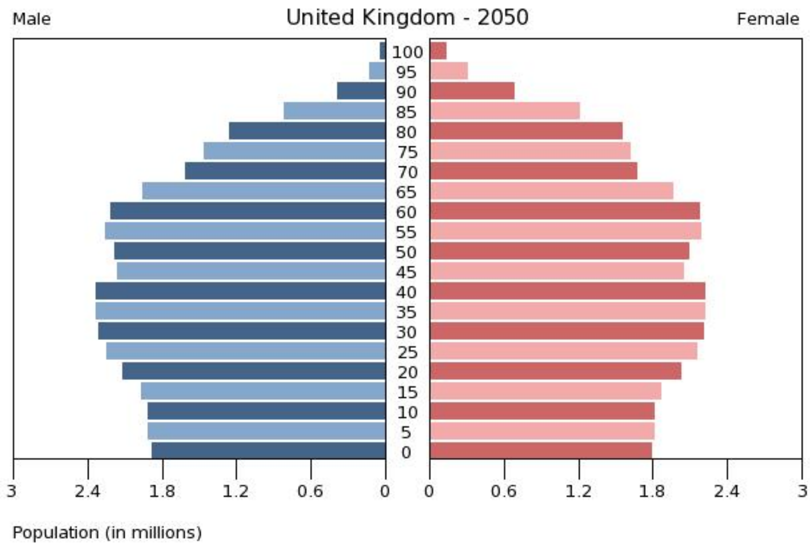
$$N_x + N_t = -\mu(x,t)N(x,t).$$

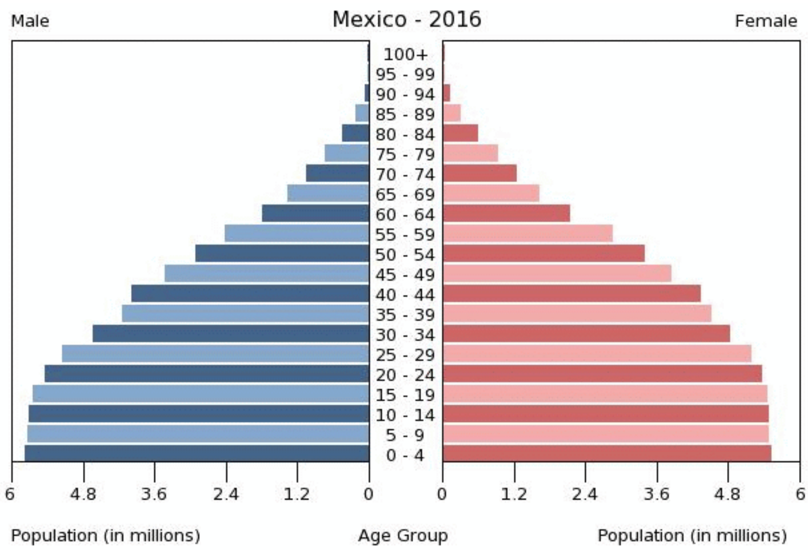
In principle, one knows

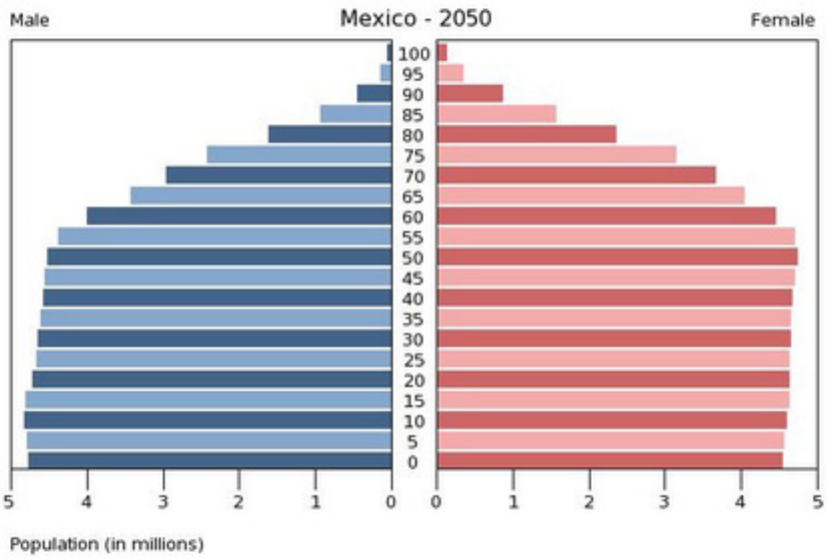
$$N(x,0) = f(x)$$

—that is, the initial distribution of population. Some examples are given below:


Male
United Kingdom - 2016
Female
100+
95 - 99
90 - 94
85 - 89
80 - 84
75 - 79
70 - 74
65 - 69
60 - 64
55 - 59
50 - 54
45 - 49
40 - 44
35 - 39
30 - 34
25 - 29
20 - 24
15 - 19
10 - 14
5 - 9
0 - 4
3 2.4 1.8 1.2 0.6 0
0 0.6 1.2 1.8 2.4 3
Population (in millions)
Age Group
Population (in millions)


Male
United Kingdom - 2050
Female
100
95
90
85
80
75
70
65
60
55
50
45
40
35
30
25
20
15
10
5
0
3 2.4 1.8 1.2 0.6 0
0 0.6 1.2 1.8 2.4 3
Population (in millions)


Male
Mexico - 2016
Female
100+
95 - 99
90 - 94
85 - 89
80 - 84
75 - 79
70 - 74
65 - 69
60 - 64
55 - 59
50 - 54
45 - 49
40 - 44
35 - 39
30 - 34
25 - 29
20 - 24
15 - 19
10 - 14
5 - 9
0 - 4
6 4.8 3.6 2.4 1.2 0
0 1.2 2.4 3.6 4.8 6
Population (in millions)
Age Group
Population (in millions)

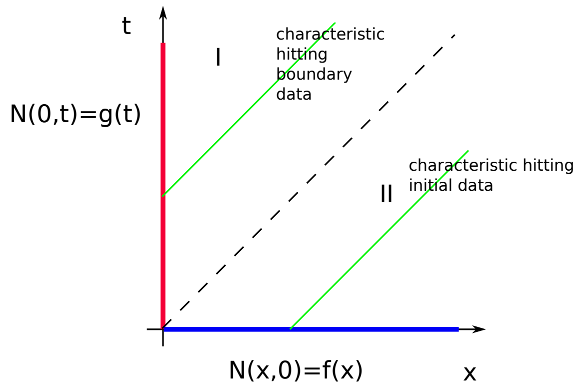


The function $f(x)$ is roughly obtained by adding the male/female figures together. One also requires boundary data $N(0,t) = g(t)$ which corresponds to the number of births at time t.

In this model one can readily see that the characteristics are of the form

$$t = x + C,$$

that is, they are lines. A plot illustrating what happens is shown below:



Observe that region I receives only information about the boundary, while II receives information about the initial condition. Also, one can see that if $g(t) = 0$ (no births), the population eventually dies.

Remark 2.3.8. There is a complication in the model: in a realistic population model, the number of births depends on $N(x,t)$ —that is, the boundary data! A good model describing this is

$$N(0,t) \equiv g(t) = \int_0^\infty \lambda(x,t) N(x,t)\,dx,$$

where $\lambda(x,t)$ is the so-called fecundity rate. This complicates the model considerably and we will not dwell into it further.

Progress Check

1. What is a characteristic curve? How do you determine them for a linear first order equation?
2. What is the ode satisfied by the solution to a first order pde along a characteristic curve?
3. What happens when the characteristic curves do not cover the whole plane?
4. What happens when two (or more) characteristic curves intersect?

Chapter 3

Second order pde's with constant coefficients

In this section we briefly look at the classification of second order partial differential equations with constant coefficients.

3.1 Introduction

The most general second order partial differential equation with constant coefficients is given by

$$aU_{xx} + 2bU_{xy} + cU_{yy} + dU_x + eU_y + fU = h(x,y) \tag{3.1}$$

with

$$a, \quad b, \quad c, \quad d, \quad e, \quad f,$$

are constants and $h(x,y)$ is an arbitrary function. The terms with the highest order derivatives, namely

$$aU_{xx} + 2bU_{xy} + cU_{yy} \tag{3.2}$$

are called the **principal part**. It determines the character of the solutions of the equation. In the following, to avoid messy computations, we consider only the principal part —i.e. we set d, e and f to zero. Particular cases of equation (3.1) are

$$\begin{aligned} &U_{xx} - U_{tt} = 0 \qquad \text{(wave equation)}, \\ &U_{xx} + U_{yy} = 0 \qquad \text{(Laplace equation)}, \\ &U_{xx} - U_t = 0 \qquad \text{(heat equation)}. \end{aligned}$$

The solutions to each of these equations have a completely different behaviour. In the following, we will see that, in a sense, these are the only possibilities.

3.2 Quadratic forms

The basic observation is the following: compare the principal part (3.2) with the **quadratic form**

$$ax^2 + 2bxy + cy^2.$$

We know from basic geometry that the solutions to the equation defined by this quadratic form represents a conic section —i.e. a hyperbola, a parabola or an ellipse. The type of conic section depends on the coefficients in the quadratic form. More precisely, completing squares one has that

$$ax^2 + 2bxy + cy^2 = a\left(\left(x + \frac{b}{a}y\right)^2 + \left(\frac{ac - b^2}{a^2}\right)y^2\right).$$

One then has the following classification:

$$\begin{aligned} b^2 - ac > 0 & \quad \text{hyperbola,} \\ b^2 - ac = 0 & \quad \text{parabola,} \\ b^2 - ac < 0 & \quad \text{ellipse.} \end{aligned}$$

One can do something similar with the principal part (3.2). One can readily check that

$$aU_{xx} + 2bU_{xy} + cU_{yy} = a\left(\left(\frac{\partial}{\partial x} + \frac{b}{a}\frac{\partial}{\partial y}\right)^2 + \left(\frac{ac - b^2}{a^2}\right)\frac{\partial}{\partial y^2}\right)U.$$

Accordingly, one classifies the pde's according to the same criteria as for the quadratic forms —more precisely, one says that (3.1) is

$$\begin{aligned} b^2 - ac > 0 & \quad \text{hyperbolic pde,} \\ b^2 - ac = 0 & \quad \text{parabolic pde,} \\ b^2 - ac < 0 & \quad \text{elliptic pde.} \end{aligned}$$

One can readily check that

wave equation	hyperbolic,
Laplace equation	elliptic,
heat equation	parabolic.

3.3 A change of variables

Consider now new coordinates (x', y') given by

$$\begin{aligned} x' &= x, \\ y' &= -\frac{b}{a}x + y, \end{aligned}$$

so that

$$\begin{aligned} x &= x', \\ y &= y' + \frac{b}{a}x. \end{aligned}$$

Using the chain rule for partial derivatives one finds that

$$\begin{aligned} \frac{\partial}{\partial y'} &= \frac{\partial}{\partial y}, \\ \frac{\partial}{\partial x'} &= \frac{\partial}{\partial x} + \frac{b}{a}\frac{\partial}{\partial y}. \end{aligned}$$

Substituting the above into the principal part (3.2) a calculation readily gives

$$aU_{xx} + 2bU_{xy} + cU_{yy} = a\left(U_{x'x'} + \left(\frac{ac-b^2}{a^2}\right)U_{y'y'}\right).$$

Now, if $ac - b^2 < 0$ one can write

$$\begin{aligned} U_{x'x'} + \left(\frac{ac-b^2}{a^2}\right)U_{y'y'} &= U_{x'x'} - \frac{|ac-b^2|}{|a|^2}U_{y'y'} \\ &= \left(\frac{\partial}{\partial x'} + \frac{\sqrt{|ac-b^2|}}{|a|}\frac{\partial}{\partial y'}\right)\left(\frac{\partial}{\partial x'} - \frac{\sqrt{|ac-b^2|}}{|a|}\frac{\partial}{\partial y'}\right)U. \end{aligned}$$

In fact, one can eliminate the factor $\sqrt{|ac-b^2|}/|a|$ by a further change of variables.

Note. The classification also works if the coefficients depend on the coordinates. In that case the character of the equation can change from point to point. As an example one has the equation

$$U_{xx} + xU_{yy} = 0.$$

Progress Check

1. How do you classify second order pde's with constant coefficients?
2. What is the advantage of using an appropriate change of coordinates when solving a second order pde?

Chapter 4

The wave equation in $1+1$ dimensions

In this chapter we study solutions to the wave equation in $1+1$ dimensions

$$U_{tt} - c^2 U_{xx} = 0 \tag{4.1}$$

with c a constant (**wave speed**) and $x \in I \subseteq \mathbb{R}$, $t > 0$ —i.e. I is an interval which can be finite, semi-infinite or infinite. The equation is supplemented by **initial conditions**

$$U(x,0) = f(x), \qquad U_t(x,0) = g(x),$$

and, possibly, also **boundary conditions** if $I \neq \mathbb{R}$.

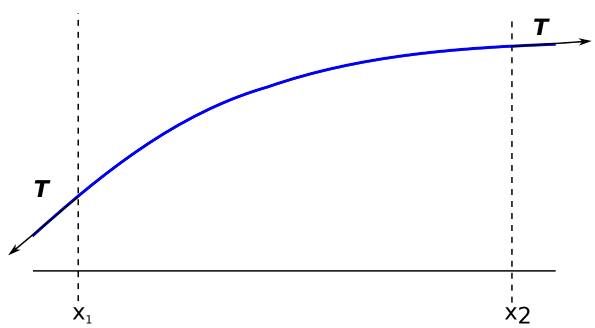
Note. In $3+1$ dimensions the wave equation takes the form

$$U_{tt} - c^2(U_{xx} + U_{yy} + U_{zz}) = 0.$$

The wave equation arises in problems describing the vibration of strings and membranes. More generally, the equations describe sound waves, electromagnetic waves, seismic waves, gravitational waves, propagation of epidemics, movement of populations, ...

4.1 The vibrating string

Consider, in the following, a **flexible, elastic, homogeneous string** of length L undergoing small transverse vibrations. Assume that the motion is restricted to a plane, and let $U(x,t)$ be the displacement from equilibrium position at time t and position x.



If the string is perfectly flexible, then the force (**tension**) responsible for the displacement is directed tangentially along the string and is constant in time since the string is homogeneous.

The position of the string at a point x is then given by $\big(x, U(x,t)\big)$ and the slope of the string at x is that of the tangent. The tangent vector at a point x is given by

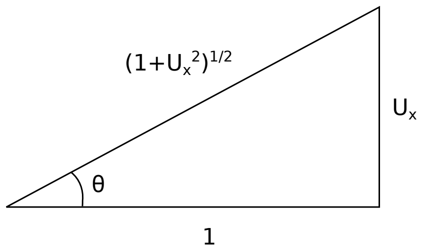
$$\frac{d}{dx}\big(x, U(x,t)\big) = \big(1, U_x(x,t)\big).$$

The key to obtaining an equation for $U(x,t)$ is **Newton's second law**

$$\vec{F} = m\vec{a}.$$

Now, from the diagram one has that

$$\cos\theta = \frac{1}{\sqrt{1+U_x^2}}, \qquad \sin\theta = \frac{U_x}{\sqrt{1+U_x^2}}.$$



The tension $\vec{F}$ is then given by

$$\vec{F} = F(x)\big(\cos\theta, \sin\theta\big) = \frac{F}{\sqrt{1+U_x^2}}\big(1, U_x\big),$$

where $F = F(x)$ is the norm of $\vec{F}$ and is assumed to be independent of time —see above. From the discussion above we have that the position of an element of string is then given by

$$\vec{x} = \big(x, U(x,t)\big)$$

so that its velocity and acceleration are given, respectively, by

$$\dot{\vec{x}} = \big(0, U_t\big), \qquad \ddot{\vec{x}} = \big(0, U_{tt}\big),$$

where the overdot $\dot{}$ denotes differentiation with respect to t. We can now compute the force along a segment of string $[x_1, x_2]$ using Newton's law:

$$\frac{F(x)}{\sqrt{1+U_x^2}}\big(1, U_x\big)\Big|_{x_1}^{x_2} = \int_{x_1}^{x_2} \rho\cdot\big(0, U_{tt}(s,t)\big)\mathrm{d}s,$$

where ρ is the density of the string (mass/unit length) which we assume to be constant. As this is a vector expression it implies two equations for the x and y components. The x component gives the equation

$$\frac{F(x)}{\sqrt{1+U_x^2}}\Big|_{x_1}^{x_2} = 0, \tag{4.2}$$

while the y component gives

$$\frac{F(x)}{\sqrt{1+U_x^2}}U_x\Big|_{x_1}^{x_2} = \int_{x_1}^{x_2} \rho U_{tt}(s,t)ds. \tag{4.3}$$

Now, as U_x is assumed to be small, then using Taylor series one has that

$$\sqrt{1+U_x^2} \approx 1 + \frac{1}{2}U_x^2 + \cdots \approx 1.$$

Using this approximation it follows from (4.2) that $F(x)$ is constant —i.e. independent of x. Equation (4.3) then gives

$$F\Big(U_x(x_2,t) - U_x(x_1,t)\Big) = \int_{x_1}^{x_2} \rho U_{tt}(s,t)ds.$$

Now, the fundamental theorem of calculus then gives that

$$F\big(U_x(x_2,t) - U_x(x_1,t)\big) = F\int_{x_1}^{x_2} U_{xx}(s,t)ds.$$

Hence,

$$F\int_{x_1}^{x_2} U_{xx}(x,t)ds = \int_{x_\star}^{x} \rho U_{tt}(s,t)ds.$$

As the points x_1 and x_2 are arbitrary the integrands must be equal so that

$$\frac{F}{\rho}U_{xx} = U_{tt}.$$

We write the latter as

$$U_{tt} = c^2 U_{xx}, \qquad c \equiv \sqrt{\frac{F}{\rho}}. \tag{4.4}$$

This is the (homogeneous) **wave equation**. The constant c is called the **wave speed**.

4.2 The wave equation on the real line

As a first approach to the wave equation we study the wave equation (4.4) on the real line so that there are no boundary conditions —physically, this means that we consider an infinitely long vibrating string. This is a useful **idealisation**.

4.2.1 Computing the general solution

Consider the change of variables

$$u = x - ct, \qquad v = x + ct. \tag{4.5}$$

Using the chain rule one has that

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v},$$
$$\frac{\partial}{\partial t} = \frac{\partial u}{\partial t}\frac{\partial}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial}{\partial v} = c\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right).$$

The second derivatives are computed as

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \left(\frac{\partial}{\partial x}\right)^2 = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right) \\
&= \frac{\partial^2}{\partial u^2} + 2\frac{\partial^2}{\partial u \partial v} + \frac{\partial^2}{\partial v^2}, \\
\frac{\partial^2}{\partial t^2} &= \left(\frac{\partial}{\partial t}\right)^2 = c^2\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right)\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right) \\
&= c^2\left(\frac{\partial^2}{\partial v^2} - 2\frac{\partial^2}{\partial u \partial v} + \frac{\partial^2}{\partial u^2}\right).
\end{aligned}$$

Thus, one has that

$$\begin{aligned}
U_{tt} - c^2 U_{xx} &= c^2\left(\frac{\partial^2}{\partial v^2} - 2\frac{\partial^2}{\partial u \partial v} + \frac{\partial^2}{\partial u^2}\right)U - c^2\left(\frac{\partial^2}{\partial u^2} + 2\frac{\partial^2}{\partial u \partial v} + \frac{\partial^2}{\partial v^2}\right)U \\
&= -4c^2\frac{\partial^2 U}{\partial u \partial v}.
\end{aligned}$$

Hence, we have transformed the original wave equation (4.4) into

$$\frac{\partial^2 U}{\partial u \partial v} = 0. \tag{4.6}$$

To solve equation (4.6) we notice that

$$\frac{\partial^2 U}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial U}{\partial v}\right),$$

so that integrating with respect to u one has

$$\int \frac{\partial}{\partial u}\left(\frac{\partial U}{\partial v}\right) du = \frac{\partial U}{\partial v} = f(v),$$

with $f(v)$ an arbitrary function of v. Integrating now with respect to v one gets

$$U(u,v) = \int f(v)dv + G(u),$$

with $G(u)$ an arbitrary function of u. Now, observe that the integral $\int f(v)dv$ is an arbitrary function of v so that one can write

$$U(u,v) = F(v) + G(u)$$

where $F(v)$ is another arbitrary function of v. Writing the latter in terms of the coordinates (x,y) one finds that

$$U(x,t) = G(x - ct) + F(x + ct). \tag{4.7}$$

This is the general solution to the wave equation (4.4).

4.2.2 Solution in terms of initial conditions

Now, suppose one has the initial conditions

$$U(x,0) = f(x), \qquad U_t(x,0) = g(x).$$

One needs to two initial conditions as the equation is second order. At $t = 0$ the general solution (4.7) gives

$$U(x,0) = G(x) + F(x) = f(x). \tag{4.8}$$

Moreover, a direct computation using the chain rule gives

$$U_t(x,t) = cF'(x+ct) - cG'(x-ct),$$

so that

$$U_t(x,0) = cF'(x) - cG'(x) = g(x) \tag{4.9}$$

Differentiating (4.8) with respect to x one obtains the system of equations

$$\begin{aligned} f'(x) &= G'(x) + F'(x), \\ g(x) &= cF'(x) - cG'(x). \end{aligned}$$

Adding and subtracting these equations one finds that

$$\begin{aligned} F'(x) &= \frac{1}{2c}\big(g(x) + cf'(x)\big), \\ G'(x) &= \frac{1}{2c}\big(cf'(x) - g(x)\big). \end{aligned}$$

Integrating the first of these equations with respect to x one finds that

$$\begin{aligned} F(x) - F(0) &= \int_0^x \frac{1}{2c}\big(g(s) + cf'(s)\big)ds \\ &= \frac{1}{2}\big(f(x) - f(0)\big) + \frac{1}{2c}\int_0^x g(s)ds, \end{aligned}$$

where in the second line we have used the **Fundamental theorem of Calculus**. Moreover, using this last expression one has that

$$\begin{aligned} G(x) &= f(x) - F(x) \\ &= f(x) - \frac{1}{2}f(x) + \frac{1}{2}f(0) - \frac{1}{2c}\int_0^x g(s)ds - F(0) \\ &= \frac{1}{2}f(x) + \frac{1}{2}f(0) - \frac{1}{2c}\int_0^x g(s)ds - F(0). \end{aligned}$$

It follows then that

$$\begin{aligned} U(x,t) &= G(x-ct) + F(x+ct) \\ &= \frac{1}{2}f(x-ct) + \frac{1}{2}f(0) - \frac{1}{2c}\int_0^{x-ct} g(s)ds - F(0) \\ &\quad + \frac{1}{2}f(x+ct) - \frac{1}{2}f(0) + \frac{1}{2c}\int_0^{x+ct} g(s)ds + F(0). \end{aligned}$$

Simplifying and rearranging one obtains the expression

$$U(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds, \tag{4.10}$$

which is known as **D'Alembert's solution**.

Note. Observe that by prescribing initial conditions one obtains a unique solution.

4.2.3 Where does the change of variables (4.5) come from?

To explain the change of variables (4.5) one observes that the wave equation can be rewritten as

$$U_{tt} - c^2 U_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)U = 0.$$

Letting

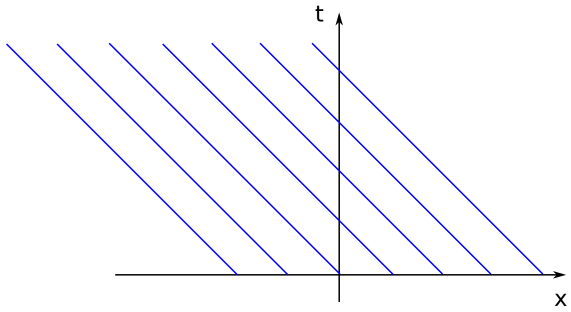
$$W \equiv \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)U,$$

then

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)W = \frac{\partial W}{\partial t} - c\frac{\partial W}{\partial x} = W_t - cW_x = 0.$$

Thus, W satisfies a first order pde with constant coefficients —we have already studied the solutions to this equation. The characteristics are lines with negative slope $dt/dx = -1/c$ (negative slope) so that

$$x + ct = \text{constant}.$$

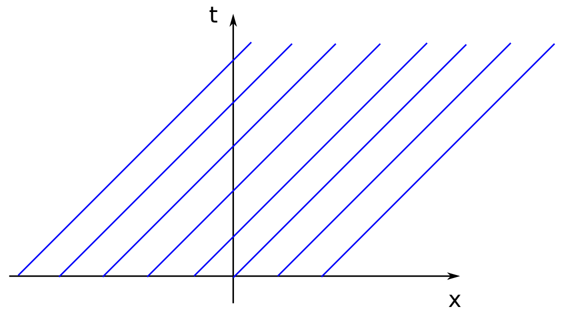


Once we know W one has to solve the equation

$$U_t + cU_x = W$$

which is, again, a first order pde with constant coefficients —observe, however, that the equation is inhomogeneous. The slope of the characteristics is $dt/dx = 1/c$ (positive slope) so that

$$x - ct = \text{constant}.$$



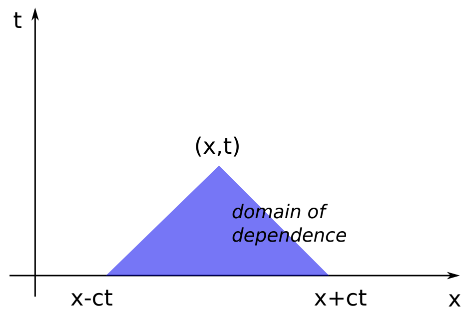
> **Note.** Thus, the wave equation has two sets of characteristics —that is, there is information travelling in two directions: to the left and to the right.

4.2.4 Interpretation of D'Alembert's solution

Formula (4.10) can be read as saying

$$U(x,t) = \big(\text{average of } U(x,0) \text{ on } x-ct \text{ and } x+ct\big)$$
$$+\big(\text{average of } U_t(x,0) \text{ over the interval } [x-ct, x+ct]\big).$$

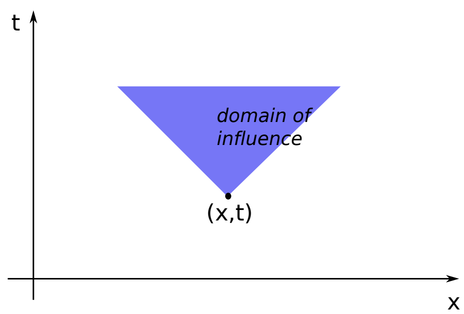
Hence, $U(x,t)$ only depends on the initial conditions on the interval $[x-ct, x+ct]$ —see the figure below.



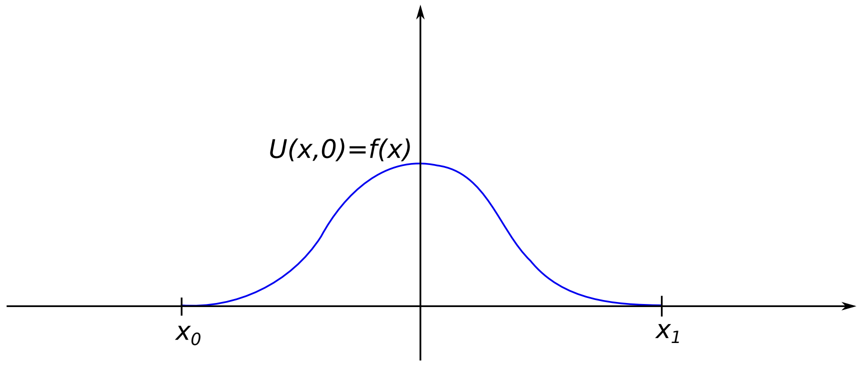
The region in the (x,t) diagram that have an influence in the value of $U(x,t)$ at (x,t) is called the **domain of dependence** of (x,t).

Note. This has connections with Relativity (MTH6132) —information cannot travel at infinite speed.

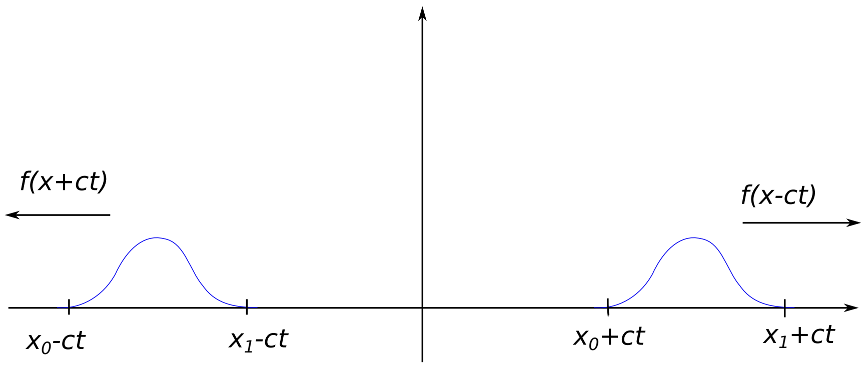
Conversely, given a point (x,t) (event) it influences the region shown below:



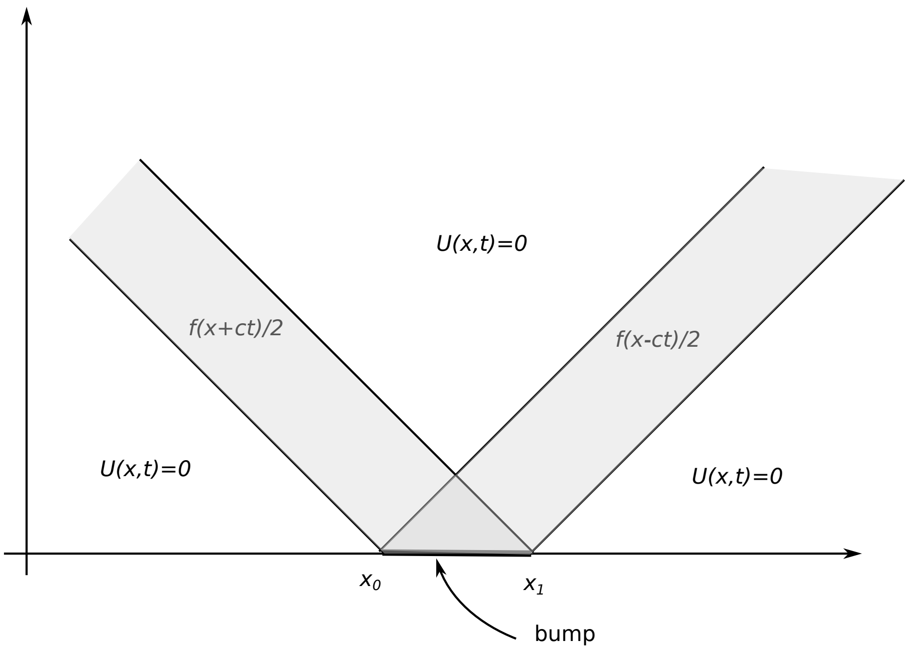
This region is called the **domain of influence**.
Also, assume that $g(x) = 0$ and that $f(x)$ has the shape of a bump:



Then, at later times the solution looks like:



That is, one has two bumps, half the size of the initial one moving in opposite directions. The above situation can be described in a diagram in the (x,t) plane as follows:



4.3 Conservation of energy

Consider the wave equation on the line:

$$U_{tt} = c^2 U_{xx}, \qquad x \in \mathbb{R}$$
$$U(x,0) = f(x), \qquad U_t(x,0) = g(x),$$

where $f(x),\ g(x) = 0$ for $|x| > R$, with R some big number. This means that $f(x)$ and $g(x)$ vanish for large $|x|$ —functions of this type are said to have **compact support**.

4.3.1 Derivation

Multiply now the wave equation by U_t on both sides:

$$U_t U_{tt} = c^2 U_{xx} U_t.$$

Observing that

$$U_t U_{tt} = \frac{1}{2}\frac{\partial}{\partial t}\left(U_t^2\right),$$

one has then that

$$\frac{1}{2}\frac{\partial}{\partial t}\left(U_t^2\right) - c^2 U_{xx} U_t = 0.$$

Integrating over the real line one then gets that

$$\begin{aligned}
0 &= \int_{\infty}^{-\infty} \left(\frac{1}{2}\frac{\partial}{\partial t}\left(U_t^2\right) - c^2 U_{xx} U_t\right) dx \\
&= \frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2} U_t^2 dx - c^2 \int_{-\infty}^{\infty} U_{xx} U_t dx \\
&= \frac{d}{dt}\left(\int_{-\infty}^{\infty} \frac{1}{2} U_t^2 dx\right) - \left(U_t U_x\Big|_{-\infty}^{\infty} - c^2 \int_{-\infty}^{\infty} U_x U_{xt} dx\right) \\
&= \frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2} U_t^2 dx + c^2 \int_{-\infty}^{\infty} U_x U_{xt} dx,
\end{aligned}$$

where to pass from the second to the third line we have used integration by parts and in the third line that $U(x,t) = 0$ if $|x| \to \infty$. Finally, observing that

$$U_x U_{xt} = \frac{1}{2}\frac{\partial}{\partial t}(U_x^2),$$

one concludes that

$$0 = \frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2} U_t^2 dx + c^2 \int_{-\infty}^{\infty} \frac{1}{2}\frac{\partial}{\partial t}(U_x^2) dx,$$

so that

$$\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} \left(U_t^2 + c^2 U_x^2\right) dx\right) = 0.$$

In other words, the quantity in brackets is constant in time. This calculation suggests the following definition:

Definition 4.3.1. The **energy** $E[U](t)$ of a solution to the wave equation is given by

$$E[U](t) \equiv \frac{1}{2}\int_{-\infty}^{\infty} \left(U_t^2 + c^2 U_x^2\right) dx.$$

Hence, the previous calculations show that

$$\frac{d}{dt}E[U](t) = 0,$$

that is, the energy is conserved —i.e. independent of t (**law of conservation of total energy**). The term $U_t^2/2$ is called the **kinetic energy** and $c^2U_x^2$ the **potential energy**.

4.3.2 An application: uniqueness of solutions

In this subsection we show how the total energy can be used to show that a solution to the initial value problem

$$\begin{aligned} &U_{tt} - c^2 U_{xx} = 0, \qquad x \in \mathbb{R} \\ &U(x,0) = f(x), \qquad U_t(x,0) = g(x) \end{aligned}$$

if it exists, then it must be unique.

Suppose one has 2 solutions U_1 and U_2 an let $W \equiv U_1 - U_2$. As the wave equation is linear one has that

$$\begin{aligned} &W_{tt} - c^2 W_{xx} = 0, \\ &W(x,0) = 0, \qquad W_t(x,0) = 0. \end{aligned}$$

The energy of W can be directly computed to be

$$\begin{aligned} E[W](t) &= E[W](0) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \big(W_t^2(x,0) + c^2 W_x^2(x,0)\big)dx, \\ &= 0. \end{aligned}$$

This means, in particular, that

$$\int_{-\infty}^{\infty} \big(W_t^2(x,t) + c^2 W_x^2(x,t)\big)dx = 0,$$

but $W_t^2 \geq 0$, $W_x^2 \geq 0$ so that, in order for the integral to vanish one actually needs

$$W_t(x,t) = 0, \qquad W_x(x,t) = 0.$$

Thus $W(x,t)$ is constant for all x, t. But $W(x,0) = 0$ so that $W(x,t) = 0$. Hence, $U_1 = U_2$ —that is, the solution is unique.

4.4 The wave equation on the half-line: reflection

In this section we analyse with more detail the phenomenon of reflection of waves on a wall. This problem naturally leads one to consider solutions of the wave equation on the half-line.The latter is modelled by the problem

$$U_{tt} - c^2 U_{xx} = 0,$$

with boundary condition

$$U(0,t) = 0, \qquad t > 0$$

and initial conditions

$$
\begin{aligned}
&U(x,0) = f(x), \\
&U_t(x,0) = g(x), \qquad x \geq 0.
\end{aligned}
$$

Boundary conditions like $U(0,t) = 0$ specifying the value of the solution on some boundary (in this case the t-axis) are known as **Dirichlet boundary conditions**. In this case it models a solid wall through which the wave cannot propagate.

To construct solutions to the above problem we will make use of D'Alembert's formula for solutions of the wave equation on the **whole** real line. Notice, however, that the initial conditions described by the functions f and g are only given on the half-line —i.e. for $x \geq 0$. To get around this problem we consider **odd extensions** of the functions f and g. More precisely, we define

$$
F(x) \equiv \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x < 0 \end{cases}
$$

and

$$
G(x) \equiv \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x < 0 \end{cases}
$$

—see the figure below for a depiction of the idea behind an odd extension of a function.

Now, with the help of the functions F and G we consider the problem on the the whole real line given by

$$
\begin{aligned}
&V_{tt} - c^2 V_{xx} = 0, \\
&V(x,0) = F(x), \\
&V_t(x,0) = G(x), \qquad x \in \mathbb{R}.
\end{aligned}
$$

The solution to the above problem is given then by D'Alembert's formula as

$$
V(x,t) = \frac{1}{2}\left(F(x+ct) + F(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds. \tag{4.11}
$$

To see what the relation between the above solution and the problem on the half-line is evaluate $V(x,t)$ as given above on the t-axis —i.e. at $x = 0$. One has that

$$
\begin{aligned}
V(0,t) &= \frac{1}{2}\left(F(ct) + F(-ct)\right) + \frac{1}{2c}\int_{-ct}^{ct} G(s)ds \\
&= 0,
\end{aligned}
$$

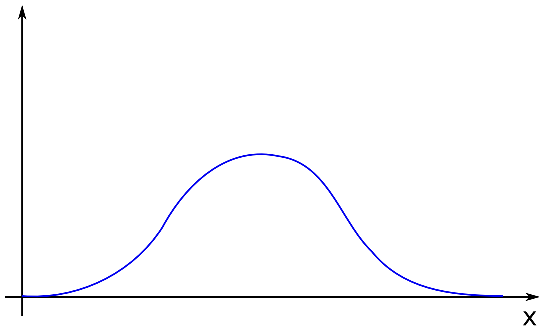
where it has been used that $F(-x) = -F(x)$ (because of the odd extension) and the fact that the integral of an odd function on a symmetric interval must vanish.

Accordingly $V(x,t)$ satisfies the boundary conditions for the problem on the half-line. Moreover, $V(x,t)$ satisfies the wave equation and the initial conditions for V and U coincide for $x \geq 0$. Thus, one has that

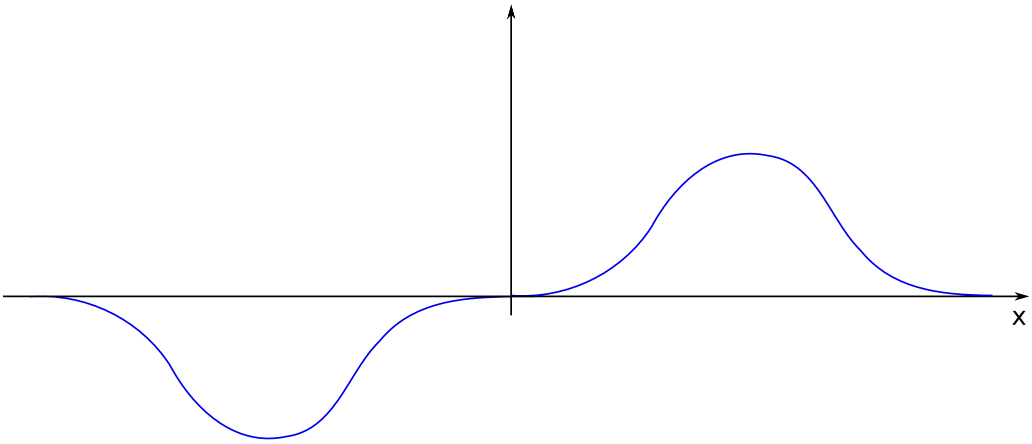
$$
U(x,t) = V(x,t), \qquad x \geq 0.
$$

This last result make use of the **uniqueness** of solutions to the wave equation —a topic not yet covered in the course!

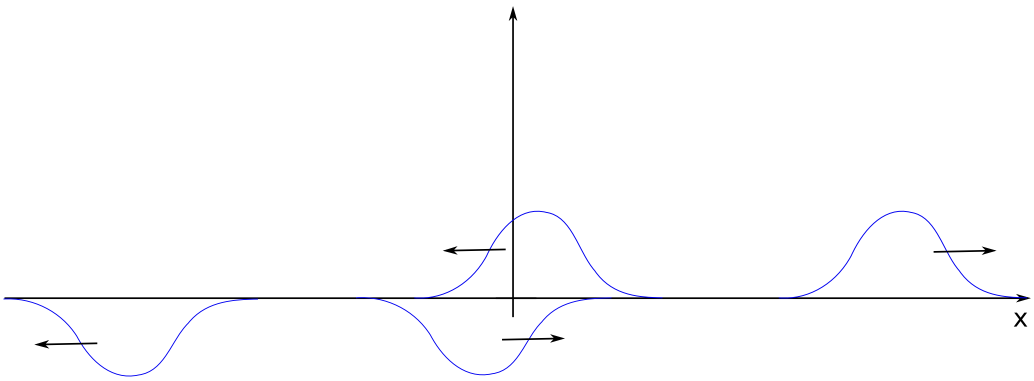
The claim is that formula (4.11) describes the phenomenon of reflection of waves. To see this better consider a situation for which $g(x) = 0$ and $f(x)$ has the form of a bump:



The odd extension of the bump has the form



As we have seen in the discussion of the interpretation of D'Alembert's formula, initial profiles split into two smaller profiles, half the size. In our particular case, each bump will split into two bumps half the size, one travelling to the right, the other to the left. The negative bump travelling to the right describes the reflected wave! When thinking about this problem we only need to concentrate on the solution for $x \geq 0$ and ignore the solution for $x < 0$.



4.5 Waves on an interval

In this section we will study the vibrations of a finite string with fixed ends. This problem will help to illustrate an important method to solve linear pde's —the so-called **method of separation of variables**. This method applies, more generally, to linear homogeneous pde's with homogeneous boundary conditions. The key to this method is the **principle of superposition**.

Assume that the string has ends given by $x = 0$ and $x = L$. The problem we want to study can be formulated mathematically as finding solutions to

$$U_{tt} - c^2 U_{xx} = 0 \tag{4.12}$$

with **boundary conditions** given by

$$U(0,t) = 0, \qquad U(L,t) = 0, \tag{4.13}$$

and **initial conditions** given by

$$U(x,0) = f(x), \qquad U_t(x,0) = g(x).$$

4.5.1 The method of separation of variables

In the method of separation of variables one looks for solutions to the wave equation (4.12) of the form

$$U(x,t) = X(x)T(t) \tag{4.14}$$

where X is a function of x and T of t only. Substitution of (4.14) into the wave equation (4.12) gives

$$X\ddot{T} = c^2 X'' T$$

where $\dot{}$ denotes differentiation with respect to t and $'$ differentiation with respect to x. Dividing by $c^2 XT$ one finds that

$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X}. \tag{4.15}$$

The key observation in the method of separation of variables is that the left hand side of equation (4.15) only depends on t while the right hand side only x. If these to sides are to be equal it means that they have to be both constant. Let us denote this constant by $-\lambda$ —the minus sign is conventional. The constant λ is called the **separation constant**.

Thus one has that

$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X} = -\lambda.$$

From the latter one obtains two ordinary differential equations —one for $T(t)$ and one for $X(x)$. Namely, one has that

$$\ddot{T} = -c^2\lambda T, \tag{4.16a}$$
$$X'' = -\lambda X. \tag{4.16b}$$

The eigenvalue problem

We begin by looking at equation (4.16b). The boundary conditions (4.13) imply that

$$X(0) = 0, \qquad X(L) = 0.$$

The combination

$$X'' = -\lambda X, \tag{4.17a}$$
$$X(0) = 0, \qquad X(L) = 0, \tag{4.17b}$$

is known as the **eigenvalue problem**. To solve it one needs to find all non-trivial solutions to (4.17a)-(4.17b). Observe that trivially $X(x) = 0$ is a solution.

Note. Eigenvalue problems arise in the context of Linear Algebra. In that subject matrices are linear operators. In Differential Equations the operator is the action of taking two derivatives of the function X.

An important property of the eigenvalue problem (4.17a)-(4.17b) is that $\lambda > 0$. To see this, rewrite (4.17a) as

$$X'' + \lambda X = 0.$$

Multiplying by X and integrating one obtains

$$\int_0^L (XX'' + \lambda X^2)dx = XX'\Big|_0^L - \int_0^L (X')^2 dx + \lambda \int_0^L X^2 dx = 0.$$

Where we have used integration by parts and the boundary conditions (4.17b) to eliminate the term $XX'|_0^L$. Hence,

$$0 < \int_0^L (X')^2 dx = \lambda \int_0^L X^2 dx.$$

The latter is only possible if $\lambda > 0$.

Given that $\lambda > 0$, the general solution to equation (4.17a) is given by

$$X(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x), \qquad c_1, \; c_2 \quad \text{constants}.$$

Now, using the boundary conditions one has

$$X(0) = c_1 = 0,$$
$$X(L) = c_2 \sin(\sqrt{\lambda} L) = 0.$$

So, either $c_2 = 0$ which forces $X(x) = 0$ or $\sqrt{\lambda} L = n\pi$ with $n = 1, 2, 3, \ldots$. Accordingly, one defines the eigenvalues

$$\lambda_n \equiv \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, \ldots.$$

The corresponding eigenfunction is given by

$$X_n(x) \equiv \sin\left(\frac{n\pi x}{L}\right).$$

The equation for $T(t)$

Now, we use the information from the eigenvalue problem to solve the equation

$$\ddot{T} = -c^2\lambda T.$$

As $\lambda > 0$ the general solution is given by

$$T(t) = d_1 \cos(\sqrt{\lambda}ct) + d_2 \sin(\sqrt{\lambda}ct).$$

The general solution

Combining the expressions obtained in the previous paragraphs, for a given n the solution consists of products

$$U_n(x,t) = a_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right)$$

with constants a_n, b_n. Now, recalling that the wave equation is linear, one has that the **principle of superposition** applies: the sum of two solutions is also a solution. Taking this to the extreme one has that

$$U(x,t) = \sum_{n=1}^{\infty} U(x,t).$$

For the time being we ignore the issues of convergence of the infinite sum. The initial conditions are satisfied if

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right),$$
$$g(x) = \sum_{n=1}^{\infty} b_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right).$$

These are examples of what is known as **Fourier sine series**. To compute the coefficients a_n one multiplies the first expression by $\sin\left(\frac{m\pi x}{L}\right)$, $m \in \mathbb{N}$ and integrates between 0 and L:

$$\sum_{n=1}^{\infty} a_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx.$$

Now, using integrating by parts one has that

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} L/2 & \text{for} \quad n = m \\ 0 & \text{for} \quad n \neq m \end{cases}.$$

One thus finds that

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

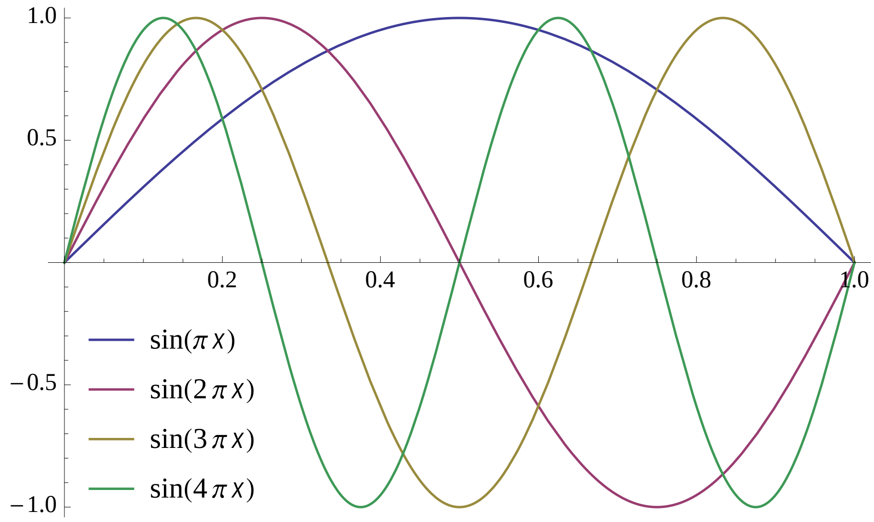
A similar computation with the expression for $g(x)$ gives

$$b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Interpretation of the solutions

The original interpretation of the solutions to the wave equation in an interval is due to Euler (1749).

A plot of the functions $X_n(x) = \sin(n\pi x/L)$ for various values of n is given below.



Collectively, they describe the shape of the wave —cf. the discussion below with Fourier series.

On the other hand, the functions $\sin(n\pi ct/L)$ and $\cos(n\pi ct/L)$ describe the behaviour in time. The behaviour described by these functions is called **standing waves**. The numbers $n\pi c/L$ are the **frequencies** of the waves with $c = (F/\rho)^{1/2}$. Thus, the natural frequencies of the string are

$$\frac{\pi}{L}\sqrt{\frac{F}{\rho}}, \quad \frac{2\pi}{L}\sqrt{\frac{F}{\rho}}, \quad \frac{3\pi}{L}\sqrt{\frac{F}{\rho}}, \ldots$$

The first harmonic **(fundamental note)** $\frac{\pi}{L}\sqrt{\frac{F}{\rho}}$ is the frequency used to tune a string instrument. Observe that this note depends on L, the length of the string. So an instrument with a longer string will have a lower fundamental note than one with a smaller string —that is, the pitch of the former is lower and that of the latter is higher; e.g. a cello and a violin. The multiples of the fundamental note are called the overtones.

Finally, observe that each standing wave is comprised of two travelling waves. More precisely, one has that

$$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) = \frac{1}{2}\cos\frac{n\pi}{L}(x-ct) - \frac{1}{2}\cos\frac{n\pi}{L}(x+ct).$$

The first term on the right hand side describes a wave travelling to the right while the second term describes a wave travelling to the left.

4.5.2 Fourier series

Definition 4.5.1. Let $f(x)$ denote a function on an interval $(-a, a)$. The **Fourier series** of $\hat{f}$ is defined as

$$\hat{f}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{a}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{a}\right)$$

where the coefficients a_0, a_n and b_n are given by

$$
\begin{aligned}
a_0 &\equiv \frac{1}{2a}\int_{-a}^{a} f(x)dx, \\
a_n &\equiv \frac{1}{a}\int_{-a}^{a} f(x)\cos\left(\frac{\pi n x}{a}\right)dx, \\
b_n &\equiv \frac{1}{a}\int_{-a}^{a} f(x)\sin\left(\frac{\pi n x}{a}\right)dx.
\end{aligned}
$$

The key result concerning Fourier series is the following (no proof provided):

Theorem 4.5.2. *If $f(x)$ is a piecewise smooth function on the interval $(-a,a)$ then the Fourier series of $f(x)$ converges pointwise to:*

(i) $f(x)$ if $f(x)$ is continuous on $x \in (-a,a)$;

(ii) $\frac{1}{2}(f(x_-)+f(x_+))$ if $f(x)$ has a jump at $x \in (-a,a)$.

Moreover, at the end points $-a$ and a the Fourier series $\hat{f}(x)$ converges to $\frac{1}{2}(f(-a)+f(a))$.

Note. One observes the following:

(i) the interval of definition is symmetric —namely, $[-a,a]$;

(ii) if $f(x)$ is an **odd function** on $[-a,a]$ then $a_n = 0$, $n \in \mathbb{N}$;

(iii) if $f(x)$ is an **even function** on $[-a,a]$ then $b_n = 0$, $n \in \mathbb{N}$.

Another important observation is the following: if $f(x)$ in defined on the interval $[0,a]$ we can always extend it to the interval $[-a,a]$ using either **odd** or **even** extensions. In particular, if $f(0) = 0$ then the odd extension of $f(x)$ is continuous and if $f'(0) = 0$ then the even extension of $f(x)$ is smooth.

Some examples illustrate the above general discussion:

Example 4.5.3. Again, let $x \in (-1,1)$ and

$$f(x) = x.$$

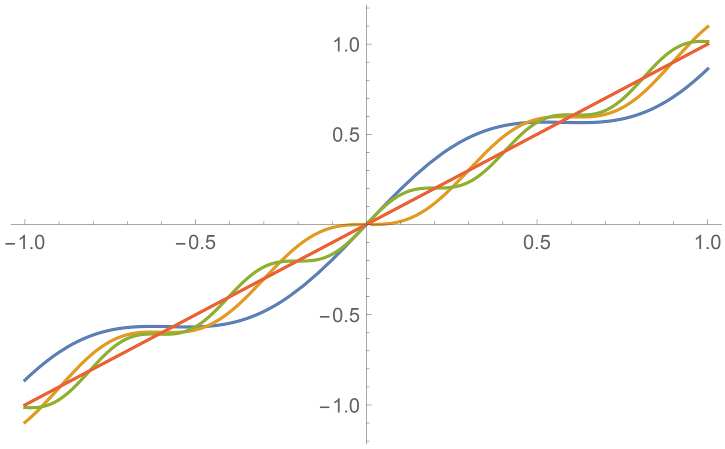
The function $f(x)$ is odd so that $a_n = 0$. For b_n one has that

$$
\begin{aligned}
b_n &= \int_{-1}^{1} x\sin(n\pi x)dx = 2\int_0^1 x\sin(n\pi x)dx \\
&= -\frac{2x}{n\pi}\cos(n\pi x)\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)dx = -\frac{2}{n\pi}(-1)^n + \frac{1}{(n\pi)^2}\sin(n\pi)\Big|_0^1 \\
&= \frac{2(-1)^{n+1}}{n\pi},
\end{aligned}
$$

where it has been used that $\cos(n\pi) = (-1)^n$. Hence the Fourier (sine) series is given by

$$\hat{f}(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x).$$

The way $\hat{f}(x)$ approximates $f(x)$ is illustrated in the following diagram:



The blue curve corresponds to a Fourier series with 5 terms, the yellow one with 10 and the green one with 15 terms.

Example 4.5.4. Let $x \in (-1, 1)$ and

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

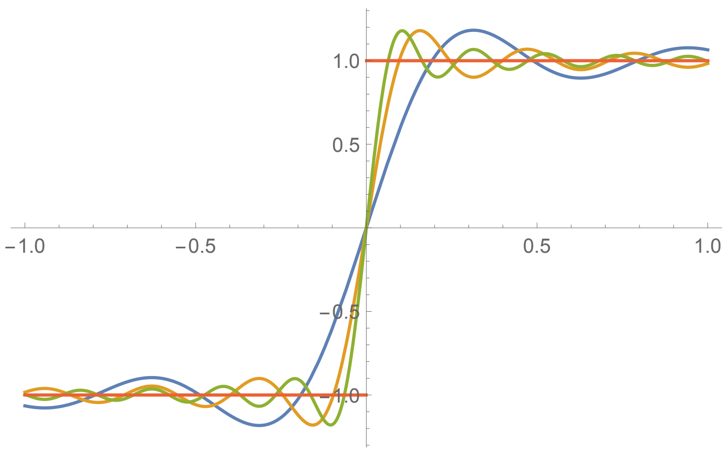
The function $f(x)$ is odd, so that $a_n = 0$, $n = 0, 1, 2, \ldots$. To compute b_n we exploit that $f(x)$ is odd so that

$$\begin{aligned} b_n &= \int_{-1}^{1} f(x) \sin(\pi n x) dx \\ &= 2\int_{0}^{1} f(x) \sin(\pi n x) dx = 2\int_{0}^{1} \sin(\pi n x) dx \\ &= -\frac{2}{n\pi} \cos(n\pi x)\Big|_{0}^{1} = -\frac{2}{n\pi}\big(\cos(n\pi) - \cos 0\big) \\ &= \frac{2}{n\pi}\big(1 + (-1)^{n+1}\big). \end{aligned}$$

Hence, one has the Fourier (sine) series

$$\hat{f}(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\big(1 + (-1)^{n+1}\big) \sin(n\pi x).$$

The way $\hat{f}(x)$ approximates $f(x)$ is illustrated in the following diagram:



The blue curve corresponds to a Fourier series with 10 terms, the yellow one with 20 and the green one with 30 terms. Observe that at $x = 0$ (the point where f is discontinuous) all the truncated Fourier series pass through the origin —this is consistent with the theory about convergence discussed previously, Theorem 4.5.2.

The previous two examples were for functions which are odd on $[-1, 1]$. Next is an example with a function which is even.

Example 4.5.5. Let $x \in [-1, 1]$ and

$$f(x) = \begin{cases} 1 + x & x < 0 \\ 1 - x & x > 0 \end{cases}$$

This function is even so that all the coefficients b_n vanish. For the a_n's one has

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)dx = \int_0^1 f(x)dx = \int_0^1 (1-x)dx = x\Big|_0^1 - \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}.$$

Similarly,

$$\begin{aligned} a_n &= \int_{-1}^{1} f(x)\cos(n\pi x)dx = 2\int_0^1 f(x)\cos(n\pi x)dx = 2\int_0^1 (1-x)\cos(n\pi x)dx \\ &= 2\int_0^1 \cos(n\pi x)dx - 2\int_0^1 x\cos(n\pi x)dx = \frac{2}{n\pi}\sin(n\pi x)\Big|_0^1 - \frac{2x}{n\pi}\sin(n\pi x)\Big|_0^1 - \frac{2}{(n\pi)^2}\cos(n\pi x)\Big|_0^1, \\ &= -\frac{2}{(n\pi)^2}\big(\cos(n\pi) - 1\big) = \frac{2}{n^2\pi^2}(1 - (-1)^n), \end{aligned}$$

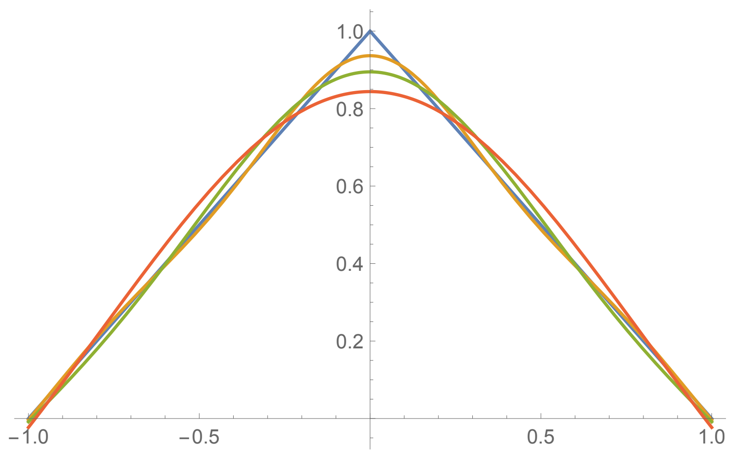
where in the second line above it has been used that

$$\int x\cos(n\pi x)dx = \frac{1}{n\pi}\sin(n\pi x) + \frac{1}{n^2\pi^2}\cos(n\pi x),$$

which can be readily obtained using integration by parts. Accordingly, the Fourier (cosines) series is given in this case by

$$\hat{f}(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big(1 - (-1)^n\big)\cos(n\pi x).$$

Plots of the series truncated at various orders can be seen in the next figure.



Finally, we conclud ethe discussion of Fourier series with an example of a non-symmetric function showing that it must, necessarily, contain both the sine and cosine series.

Example 4.5.6. Let Let $x \in [-1, 1]$ and

$$f(x) = \begin{cases} 0 & x < 0 \\ x & x > 0 \end{cases}$$

In order to compute the Fourier series observe that

$$\int x\cos(n\pi x)dx = \frac{x}{n\pi}\sin(n\pi x) + \frac{1}{n^2\pi^2}\cos(n\pi x),$$
$$\int x\sin(n\pi x)dx = -\frac{x}{n\pi}\cos(n\pi x) + \frac{1}{n^2\pi^2}\sin(n\pi x).$$

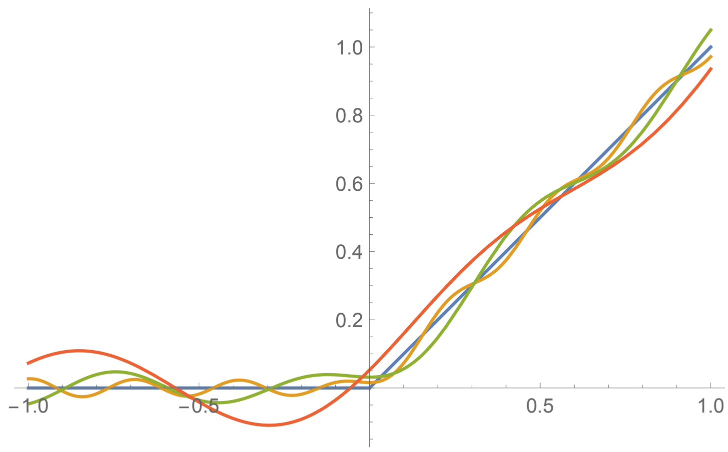
These formulae are readily obtained using integration by parts. Using these formulae one readily finds that

$$\begin{aligned} a_0 &= \frac{1}{2}\int_{-1}^{1} f(x)dx = \frac{1}{2}\int_{-1}^{1} xdx = \frac{1}{4}, \\ a_n &= \int_{-1}^{1} f(x)\cos(n\pi x)dx = \int_0^1 x\cos(n\pi x)dx \\ &= \frac{x}{n\pi}\sin(n\pi x)\Big|_0^1 + \frac{1}{n^2\pi^2}\cos(n\pi x)\Big|_0^1 = \frac{1}{n^2\pi^2}\big((-1)^n - 1\big), \\ b_n &= \int_{-1}^{1} f(x)\sin(n\pi x)dx = \int_0^1 x\sin(n\pi x)dx \\ &= -\frac{x}{n\pi}\cos(n\pi x)\Big|_0^1 + \frac{1}{n^2\pi^2}\sin(n\pi x)\Big|_0^1 = \frac{1}{n\pi}\big(1 - (-1)^n\big). \end{aligned}$$

It follows then that the Fourier series is given by

$$\hat{f}(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{\big(1-(-1)^n\big)}{n\pi}\left(\sin(n\pi x) - \frac{1}{n\pi}\cos(n\pi x)\right).$$

Observe that it contains both sine and cosine contributions. Plots of the series truncated at various orders can be seen in the next figure.



Progress Check

1. What sort of phenomena do the wave equation describe?

2. What sort of assumptions go into the derivation of the wave equation?

3. What is the main idea behind the derivation of the general solution of the wave equation?

4. What does D'Alembert's solution say? What is its interpretation?

5. What do the characteristics of the wave equation tell us?

   item What is the main idea behind the method of separation of variables?

6. What is the eigenvalue problem?

7. What is the interpretation of the solution to the wave equation on an interval obtained by the method of separation of variables?

8. What is a standing wave?

9. What is a Fourier series?

10. What is the relation between a function and its Fourier series?

11. How does the reflection of waves work?

12. What is an odd extension of function?

Chapter 5

Elliptic equations

In this part of the course we will study the properties of elliptic equations in two dimensions (spatial). More precisely, we will look at the **Laplace equation**

$$U_{xx} + U_{yy} = 0,$$

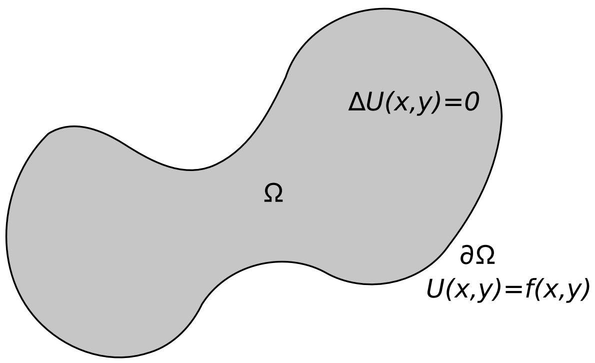
and the **Poisson equation**

$$U_{xx} + U_{yy} = f(x, y).$$

The Poisson equation is the inhomogeneous version of the Laplace equation.

5.1 Basic ideas

Typically we will be interested in the so-called **Dirichlet problem** in which we solve the Laplace equation on a domain $\Omega \subset \mathbb{R}^2$ given that the value of U on the **boundary** $\partial\Omega$ of Ω is known.



Notation. In what follows we write

$$\Delta U = U_{xx} + U_{yy}.$$

The physicists notation is

$$\nabla^2 U = U_{xx} + U_{yy}.$$

The operator Δ (∇^2) is called the **Laplacian**. The reason for the physicists notation is that the Laplacian is the divergence of the gradient of a function $\Delta U = \nabla \cdot \nabla U$.

The Laplace and Poisson equations arise from applications in physics (electrostatics, Newtonian gravity), fluid flows (steady state), soap films, elastic membranes, and also in pure mathematics (complex variables). As examples consider the wave equation in $1+2$ dimensions

$$U_{tt} = c^2(U_{xx} + U_{yy})$$

and the $1+2$ heat equation

$$U_t = \varkappa(U_{xx} + U_{yy}).$$

For both of these equations it is of interest to look for solutions which are independent of time —i.e. $U_t = 0$. These solutions describe the **asymptotic behaviour** —i.e. at late times. This is a statement that is hard to show and that is at the forefront of modern pde research.

Harmonic functions

Definition 5.1.1. A function having second partial derivatives on a domain $\Omega \subset \mathbb{R}^2$ is called harmonic if $\Delta U = 0$ for all $(x, y) \in \Omega$.

Example 5.1.2.

(i) the function $U(x, y) = x + y$ is harmonic for all $\Omega \subset \mathbb{R}^2$;

(ii) similarly for the function $U(x, y) = x^2 - y^2$;

(iii) the function $U(x, y) = \ln(x^2 + y^2)$ for any domain Ω not containing the origin as the function $U(x, y)$ is not defined there.

Relation to complex variables

Let $f(z) = u(x, y) + \mathrm{i}v(x, y)$ be an analytic function with $z = x + \mathrm{i}y$. To verify that the function $f(z)$ is analytic on a domain Ω one can make use of the **Cauchy-Riemann** equations:

$$v_y = u_x, \tag{5.1a}$$
$$v_x = -u_y. \tag{5.1b}$$

Applying $\partial/\partial y$ to equation (5.1a) one has that

$$v_{yy} = u_{xy} = -v_{xx}$$

where the second equality follows from (5.1b). Thus, one has that

$$v_{xx} + v_{yy} = 0,$$

that is, the imaginary part of analytic function is harmonic. A similar relation follows for the real part u.

Note. This observation indicates a very deep connection between pde's and complex variables!

5.2 Separation of variables for the Laplace equation

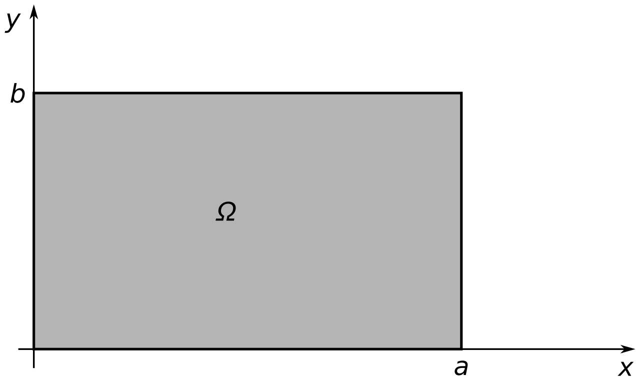
Before studying the general properties of the Laplace and Poisson equations, let us consider some explicit solutions using separation of variables.

5.2.1 Separation of variables in Cartesian coordinates

Consider the rectangular domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq a,\ 0 \leq y \leq b\}$$

as shown in the figure below:



We want to solve

$$\Delta U = 0, \qquad \text{on} \quad \Omega.$$

We make use of the method of separation of variables. To this end, given the geometric structure of the problem we look for solutions of the form

$$U(x, y) = X(x)Y(y).$$

Substituting the latter into

$$U_{xx} + U_{yy} = 0$$

one readily gets that

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Rearranging one finds that

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left hand side of this equality only depends on x while the left hand side only depends on y. Thus, both sides have to be equal to a constant. That is, one has

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = k,$$

with k the **separation constant**. It follows then that one has the following ode's to solve:

$$X''(x) = kX(x), \tag{5.2a}$$
$$Y''(y) = -kY(y). \tag{5.2b}$$

The type of solutions depends on the sign of k. For example, if $k < 0$, then the solutions to (5.2a) are trigonometric functions while if $k > 0$ they are exponentials.

Note. To determine k one needs **boundary conditions**.

Example 5.2.1. Suppose that

$$U(0, y) = 0, \qquad U(a, y) = 0.$$

It follows from the above that

$$X(0) = X(a) = 0.$$

Accordingly, one needs to have $X(x)$ as periodic solutions —that is, one requires trigonometric functions and necessarily $k < 0$. We thus write

$$k = -\mu^2, \qquad \mu \quad \text{a constant.}$$

The solution to (5.2a) is then given by

$$X(x) = A\sin(\mu x) + B\cos(\mu x), \qquad A,\ B \quad \text{constants.}$$

Now, we have to match the boundary conditions. For this we observe that

$$\begin{aligned} &X(0) = B = 0, \\ &X(a) = A\sin(\mu a) + B\cos(\mu a) = A\sin(\mu a) = 0, \end{aligned}$$

where in the second line we have used that $B = 0$ (from the first line). Thus, in order to have $X(a) = 0$ one requires

$$\mu a = n\pi, \qquad n \in \mathbb{N}.$$

Hence, the required solution of (5.2a) is

$$X(x) = \sin\left(\frac{n\pi x}{a}\right).$$

We are now in the position of solving (5.2b) which takes the form

$$Y''(y) = \mu^2 Y(y).$$

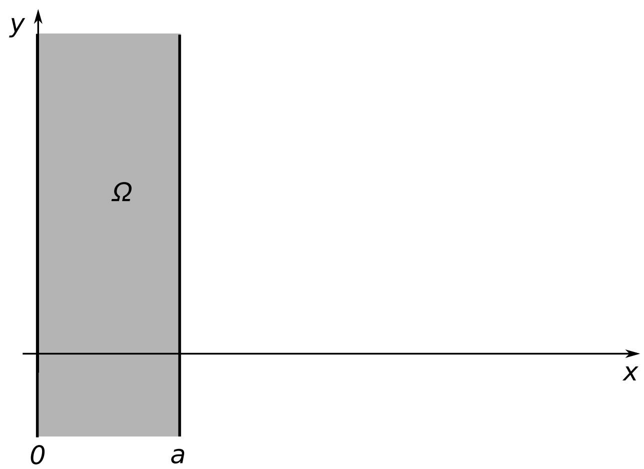
The solution can be expressed in terms of hyperbolic functions or, alternatively, trigonometric functions:

$$Y(y) = A_n \sinh\left(\frac{n\pi y}{a}\right) + B_n \cosh\left(\frac{n\pi y}{a}\right), \tag{5.3a}$$

$$Y(y) = C_n e^{\frac{n\pi y}{a}} + D_n e^{-\frac{n\pi y}{a}}. \tag{5.3b}$$

The expression (5.3a) is conventionally used when the y-domain is finite while (5.3b) is used when it is infinite.

Note. Observe that if one requires $U(x, y) \to 0$ as $y \to \infty$ one then necessarily has that $C_n = 0$ —see figure below.



Putting everything together one ends up, for given n, with solutions of the form

$$\begin{aligned} U_n(x,y) &= \sin\left(\frac{n\pi x}{a}\right)\left(A_n \sinh\left(\frac{n\pi y}{a}\right) + B_n \cosh\left(\frac{n\pi y}{a}\right)\right), \\ &= \sin\left(\frac{n\pi x}{a}\right)\left(C_n e^{\frac{n\pi y}{a}} + D_n e^{-\frac{n\pi y}{a}}\right). \end{aligned}$$

Now, recall that the equation $\Delta U = 0$ is linear —thus, the principle of superposition holds. The general solution is then a linear combination of all possible $U_n(x,y)$'s:

$$U(x,y) = \sum_{n=1}^{\infty} U_n(x,y).$$

The problem on a rectangle

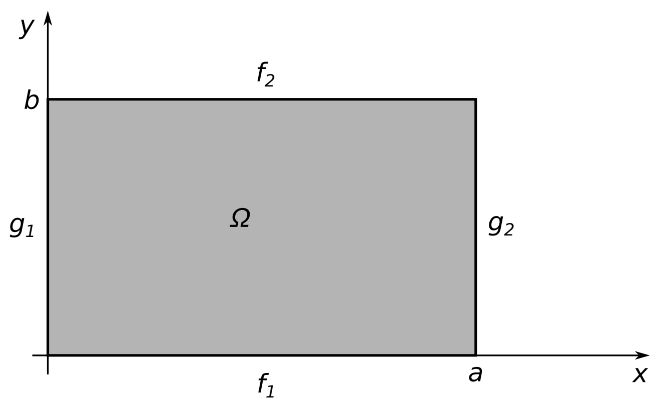
A more elaborated problem is:

$$U_{xx} + U_{yy} = 0,$$

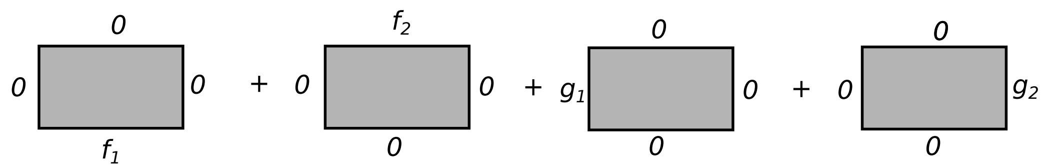
with boundary conditions

$$\begin{aligned} U(0,y) &= g_1(y), \\ U(a,y) &= g_2(y), \\ U(x,0) &= f_1(x), \\ U(x,b) &= f_2(x). \end{aligned}$$

A schematic depiction of the situation is given in the picture below:



Observe that the pde is homogeneous but the boundary conditions are not. To solve the problem above we exploit the linearity of the equation and the boundary conditions and break the original problem into 4 problems, each one with non-homogeneous boundary conditions:



In the following we concentrate, for conciseness on

$$\begin{aligned}
&\Delta U = 0,\\
&U(0,y) = U(a,y) = 0,\\
&U(x,0) = f_1(x),\\
&U(x,b) = 0.
\end{aligned}$$

From the discussion in the previous subsection we already know that

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right), \qquad n \in \mathbb{N}.$$

We also have that the general solution to $Y'' = \mu^2 Y$ is given by

$$Y(y) = B_n \cosh(\mu y) + A_n \sinh(\mu y).$$

However, one also needs that $Y(b) = 0$ so use the solution

$$Y(y) = B_n \cosh \mu(y-b) + A_n \sinh \mu(y-b), \tag{5.4}$$

which can be readily verified to solve $Y'' = \mu^2 Y$ (**exercise!**).

Note. That (5.4) is also a solution to $Y'' = \mu^2 Y$ is, ultimately, a consequence of the fact that the Laplace equation is **translationally invariant**.

The boundary condition $Y(b) = 0$ readily implies that $B_n = 0$. Hence, one has that

$$Y(y) = A_n \sinh \frac{n\pi}{a}(y-b).$$

Thus, the full solution for fixed $n \in \mathbb{N}$ is

$$U_n(x,y) = A_n \sin\left(\frac{n\pi x}{a}\right) \sinh \frac{n\pi}{a}(y-b),$$

while the general solution is a sum of all the possibilities:

$$U(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh \frac{n\pi}{a}(y-b).$$

Finally, one needs to implement the boundary condition at $y = 0$. For this we observe that

$$\begin{aligned}
U(x,0) &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh \frac{n\pi}{a}(-b)\\
&= -\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{nb\pi}{a}\right)\\
&= f_1(x).
\end{aligned}$$

Using the orthogonality property of the sine function one can find that

$$A_n = -\frac{\int_0^a f_1(x) \sin\left(\frac{n\pi x}{a}\right) dx}{\frac{a}{2} \sinh\left(\frac{nb\pi}{a}\right)}.$$

Separation of variables in polar coordinates

The method of separation of variables can be used to find solutions to the Laplace equation in settings with circular symmetry —i.e. a disk or an annulus.

Given the polar coordinates (r,θ) given by

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

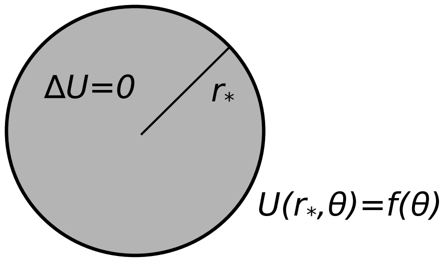
the Laplacian can be expressed as

$$\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}.$$

Consider the boundary value problem for the Laplace equation in which the value of the solution is given on a circumference of a disk of radius r_* —namely,

$$\Delta U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0,$$
$$U(r_*,\theta) = f(\theta),$$

see the figure below:



Our task is to find the solution $U(r,\theta)$ in the interior of the circumference (**disk**). Following the general strategy of the method of separation of variables we look for solutions of the form

$$U(r,\theta) = R(r)\Theta(\theta).$$

Plugging into the Laplace equation in polar coordinates one obtains the expression

$$\Theta R'' + \frac{1}{r}\Theta R' + \frac{1}{r^2}R\Theta'' = 0.$$

Dividing the above expression by $R\Theta/r^2$ and rearranging one finds that

$$\frac{r^2R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta}.$$

The left hand side of the above expression depends only on r while the right hand side only on θ. Thus, each must be equal to some separation constant k —namely:

$$\frac{r^2R''}{R} + \frac{rR'}{R} = k, \qquad -\frac{\Theta''}{\Theta} = k,$$

or

$$
\begin{aligned}
&r^2R'' + rR' - kR = 0,\\
&\Theta'' + k\Theta = 0.
\end{aligned}
$$

The Θ-equation. This equation is used to set the value of k. Observe that we need periodic solutions so $k > 0$. In the following we write $k = m^2$. Then

$$\Theta(\theta) = A\cos m\theta + B\sin m\theta.$$

To enforce periodicity we require that

$$
\begin{aligned}
U(r,\theta) &= U(r,\theta+2\pi),\\
U_\theta(r,\theta) &= U_\theta(r,\theta+2\pi).
\end{aligned}
$$

Observing that

$$\cos m(\theta+2\pi) = \cos(m\theta + 2\pi m) = \cos m\theta$$

if $m \in \mathbb{N}$ (and similarly for $\sin m\theta$) then $m \in \mathbb{N}$.

The R-equation. Following the previous discussion one has that the equation for $R(r)$ takes the form

$$r^2R'' + rR' - m^2R = 0.$$

We look for solutions to this equations of the form

$$R(r) = r^\alpha,$$

for some constant α. It follows then that

$$\alpha(\alpha-1)r^\alpha + \alpha r^\alpha - m^2r^\alpha = 0$$

so that

$$(\alpha^2 - m^2)r^\alpha = 0.$$

Hence, $\alpha^2 = m^2$ —that is,

$$\alpha = \pm m.$$

So the general solution for the R equation is

$$R(r) = C_m r^m + \frac{D_m}{r^m}.$$

For $m = 0$ one needs to do more work as there must be two independent solutions. In that case one has the equation

$$r^2R'' + rR' = 0.$$

If $r \neq 0$ the latter implies

$$r\frac{dR'}{dr} = -R',$$

which can be read as an equation for R'. Integrating one obtains

$$R'(r) = \frac{D_0}{r},$$

from where a further integration gives

$$R(r) = C_0 + D_0 \ln r.$$

The general solution. Combining the whole of the previous discussion one finds that the general solution to the Laplace equation in polar coordinates is given by

$$U(r,\theta) = \Big(C_0 + D_0 \ln r\Big) + \sum_{m=1}^{\infty} \Big(C_m r^m + \frac{D_m}{r^m}\Big)(A_m \cos m\theta + B_m \sin m\theta). \tag{5.5}$$

Example 5.2.2. Consider solutions such that $U(r_*,\theta) = f(\theta)$ and $U(r,\theta)$ well defined at the origin. Observe that the general solution as given by (5.5) are singular at $r=0$. To avoid this behaviour set $D_0 = 0$ and $D_m = 0$. Hence,

$$U(r,\theta) = a_0 + \sum_{m=1}^{\infty} r^m (a_m \cos m\theta + b_m \sin m\theta), \tag{5.6}$$

where

$$a_m \equiv A_m C_m \qquad b_m \equiv B_m C_m.$$

The boundary condition at $r = r_*$ gives then

$$f(\theta) = a_0 + \sum_{m=1}^{\infty} r_*^m (a_m \cos m\theta + b_m \sin m\theta).$$

This is an example of a Fourier series! The Fourier coefficients can be computed (using the standard method) to be

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta, \\ a_n &= \frac{1}{\pi r_*^n}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta, \\ b_n &= \frac{1}{\pi r_*^n}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta. \end{aligned}$$

Example 5.2.3. Let $r_* = 1$ and $f(\theta) = \sin\theta$. One can then evaluate equation (5.6) at $r = 1$to yield

$$U(1,\theta) = a_0 + \sum_{m=1}^{\infty} a_m \cos m\theta + \sum_{m=1}^{\infty} b_m \sin m\theta.$$

As, on the other hand,

$$U(1,\theta) = \sin\theta$$

and the sines and cosines are independent, then by direct inspection one finds that

$$\begin{aligned} a_0 &= 0, \qquad a_m = 0, \\ b_1 &= 1, \qquad b_m = 0, \quad m \neq 1. \end{aligned}$$

So, in this case, the solution takes the simple form

$$U(r,\theta) = r\sin\theta.$$

Example 5.2.4. Let $r_* = 1$ and $f(\theta) = \cos^2\theta$. Recall the identity

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta.$$

So, in this case we have that

$$\frac{1}{2} + \frac{1}{2}\cos 2\theta = a_0 + \sum_{m=1}^{\infty} a_m \cos m\theta + \sum_{m=1}^{\infty} b_m \sin m\theta,$$

from where direct inspection yields

$$\begin{aligned} a_0 &= \frac{1}{2}, \qquad a_2 = \frac{1}{2}, \qquad a_m = 0, \quad m \neq 0,\, 2, \\ b_m &= 0. \end{aligned}$$

Thus, the solution is given by

$$U(r,\theta) = \frac{1}{2} + \frac{1}{2}r^2\cos 2\theta.$$

Example 5.2.5. Now, suppose that the boundary conditions are such that on half of the circle the function takes the constant value U_1 and in the lower part it takes the value U_2. More precisely, one has that

$$f(\theta) = \begin{cases} U_1 & 0 < \theta < \pi \\ U_2 & \pi < \theta < 2\pi. \end{cases}$$

Assume, further for simplicity that $r_* = 1$.

From the theory of Fourier series we have that

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta = \frac{U_1}{2\pi}\int_0^{\pi} d\theta + \frac{U_2}{2\pi}\int_{\pi}^{2\pi} d\theta = \frac{U_1+U_2}{2} \\ a_m &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos m\theta d\theta = \frac{U_1}{\pi}\int_0^{\pi}\cos m\theta d\theta + \frac{U_2}{\pi}\int_{\pi}^{2\pi}\cos m\theta d\theta = 0. \end{aligned}$$

However, one also has that

$$\begin{aligned} b_m &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin m\theta d\theta = \frac{U_1}{\pi}\int_0^{\pi}\sin m\theta d\theta + \frac{U_2}{\pi}\int_{\pi}^{2\pi}\sin m\theta d\theta \\ &= -\frac{U_1}{\pi m}\big((-1)^m - 1\big) - \frac{U_2}{\pi m}\big(1-(-1)^m\big) = \frac{(U_1-U_2)\big(1-(-1)^m\big)}{\pi m}. \end{aligned}$$

Hence, the solution to the Laplace equation is given by

$$U(r,\theta) = \frac{U_1+U_2}{2} + \frac{U_1-U_2}{\pi}\sum_{m=1}^{\infty}\frac{r^m}{m}\big(1-(-1)^m\big)\sin m\theta.$$

Observe that the solution only contains terms with m odd. In Exercise 6 of Coursework 7, you will be asked to write the above series in closed form.

Example 5.2.6. Consider now $\Delta U = 0$ in $1 < r < 2$ —this type of region is called an annulus (ring). Boundary conditions are then given by

$$\begin{aligned} U(1,\theta) &= f(\theta), \\ U(2,\theta) &= g(\theta). \end{aligned}$$

In this case one can keep the general solution as the origin is excluded.

5.3 Poisson's formula

In a previous example we have obtained the solution to Dirichlet's problem on a disk in the form of the infinite series

$$U(r,\theta) = a_0 + \sum_{m=1}^{\infty} r_*^m (a_m \cos m\theta + b_m \sin m\theta),$$

with

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta, \\ a_n &= \frac{1}{\pi r_*^n}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta, \\ b_n &= \frac{1}{\pi r_*^n}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta. \end{aligned}$$

Remarkably, the previous solution can be written in closed form —i.e. in a way it does not involve an infinite series.

5.3.1 Some useful facts

We recall some useful fact that will be used in the following calculation.

Writing trigonometric functions in terms of exponentials. One has that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where $i = \sqrt{-1}$. From the above expression it follows that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Geometric series. Recall that for $|x| < 1$ one has that

$$1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

So, in particular

$$x + x^2 + \cdots = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}. \tag{5.7}$$

5.3.2 Expressing the series solution in closed form

Substituting the expressions for the Fourier coefficients into the series solution (5.6) one obtains

$$\begin{aligned} U(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta')d\theta' + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{r^m}{r_*^m}\left(\int_0^{2\pi} f(\theta')\cos m\theta' d\theta' \cos m\theta + \int_0^{2\pi} f(\theta')\sin m\theta' d\theta' \sin m\theta\right) \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta')d\theta' + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{r^m}{r_*^m}\int_0^{2\pi} f(\theta')\big(\cos m\theta'\cos m\theta + \sin m\theta' \sin m\theta\big)d\theta'. \end{aligned}$$

Recalling that

$$\cos m(\theta-\theta') = \cos m\theta' \cos m\theta + \sin m\theta' \sin m\theta,$$

one gets then

$$\begin{aligned}
U(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta')d\theta' + \frac{1}{\pi}\int_0^{2\pi} f(\theta')\sum_{m=1}^{\infty}\left(\frac{r}{r_*}\right)^m \cos m(\theta-\theta')d\theta' \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\theta')\left(1+2\sum_{m=1}^{\infty}\left(\frac{r}{r_*}\right)^m \cos m(\theta-\theta')\right)d\theta'.
\end{aligned}$$

Now, rewriting $\cos m(\theta-\theta')$ in terms of integrals one finds that

$$\begin{aligned}
1+2\sum_{m=1}^{\infty}\left(\frac{r}{r_*}\right)^m \cos m(\theta-\theta') &= 1+\sum_{m=1}^{\infty}\left(\frac{r}{r_*}\right)^m\left(e^{im(\theta-\theta')}+e^{-im(\theta-\theta')}\right) \\
&= 1+\sum_{m=1}^{\infty}\left(\frac{re^{i(\theta-\theta')}}{r_*}\right)^m+\sum_{m=1}^{\infty}\left(\frac{re^{-i(\theta-\theta')}}{r_*}\right)^m.
\end{aligned}$$

The last two terms are geometric series like in (5.7) series with x given by the expressions in brackets. Accordingly, we can write

$$\begin{aligned}
1+2\sum_{m=1}^{\infty}\left(\frac{r}{r_*}\right)^m \cos m(\theta-\theta') &= 1+\frac{(r/r_*)e^{i(\theta-\theta')}}{1-(r/r_*)e^{i(\theta-\theta')}}+\frac{(r/r_*)e^{-i(\theta-\theta')}}{1-(r/r_*)e^{-i(\theta-\theta')}} \\
&= 1+\frac{re^{i(\theta-\theta')}}{r_*-re^{i(\theta-\theta')}}+\frac{re^{-i(\theta-\theta')}}{r_*-re^{-i(\theta-\theta')}} \\
&= \frac{r_*^2-r^2}{r_*^2-2rr_*\cos(\theta-\theta')+r^2}.
\end{aligned}$$

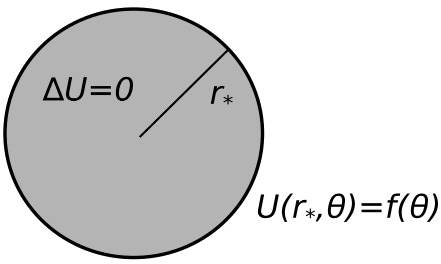
Thus, one has that

$$U(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{f(\theta')(r_*^2-r^2)}{r_*^2-2rr_*\cos(\theta-\theta')+r^2}d\theta',$$

or after some rearrangements

$$\boxed{U(r,\theta) = \frac{(r_*^2-r^2)}{2\pi}\int_0^{2\pi}\frac{f(\theta')d\theta'}{r_*^2-2rr_*\cos(\theta-\theta')+r^2}. \qquad (5.8)}$$

The latter is known as **Poisson's formula**. It expresses the solution to the Dirichlet problem on a disk as an integral of the boundary data over the boundary of the disk.



Note. The term

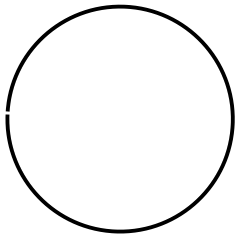
$$r_*^2 - 2rr_* \cos(\theta - \theta') + r^2$$

is, essentially the **cosine's law** of trigonometry and gives the distance between a point with polar coordinates (r, θ) in the interior of the disk where we want to know the value of U and the points (r_*, θ') on the boundary of the disk (over which one is integrating).

5.4 The mean value property

Several important properties of harmonic functions follow directly from Poisson's formula (5.8). In particular, one has the following:

Proposition 5.4.1 (the first mean value property)**.** *Let U be a harmonic function on a disk Ω. Then the value of U at the centre of the disk is equal to the average of U on its circumference.*



Proof. Without loss of generality set the centre of the disk at the origin of the polar coordinates. Then, setting $r = 0$ in Poisson's formula (5.8) one obtains

$$U(0) = \frac{r_*^2}{2\pi} \int_0^{2\pi} \frac{f(\theta')}{r_*^2} d\theta'.$$

The latter can be rewritten as

$$U(0) = \frac{1}{2\pi r_*} \int_0^{2\pi} f(\theta') r_* d\theta'.$$

This is nothing but the average of $f(\theta)$ over the circumference —observe that $2\pi r_\star$ is the value of the circumference while $r_* d\theta'$ is the infinitesimal arc-length. □

Note. The first mean value property allows one to determine the value of a harmonic function at the centre of the disk without actually having to solve the Laplace equation!

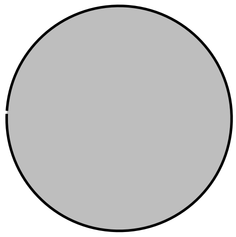
Example 5.4.2. For the problem in Example 5.2.5 a quick calculation gives that

$$\begin{aligned} U(0) &= \frac{1}{2\pi} \int_0^{\pi} U_1 d\theta + \frac{1}{\pi} \int_{\pi}^{2\pi} U_2 d\theta \\ &= \frac{1}{2}(U_1 + U_2). \end{aligned}$$

That is the value at the centre is the average of the two different (constant) values at the boundary —this is an intuitive observation.

There is a stronger version of the mean value property:

Proposition 5.4.3 (the second mean value property)**.** *Let U be a harmonic function on a disk Ω. Then the value of U at the centre of Ω equals the average on the disk.*



Proof. Let $r \leq r_*$. The first mean value property then gives that

$$U(0) = \frac{1}{2\pi}\int_0^{2\pi} U(r,\theta)d\theta.$$

Multiplying by $2\pi r$ and integrating from 0 to r_* gives

$$\int_0^{r_*} 2\pi r U(0)dr = \int_0^{r_*}\int_0^{2\pi} U(r,\theta)rd\theta dr$$

However,

$$\int_0^{r_*} 2\pi r U(0)dr = 2\pi U(0)\int_0^{r_*} rdr = \pi r_*^2 U(0).$$

Hence,

$$U(0) = \frac{1}{\pi r_*^2}\int_0^{r_*}\int_0^{2\pi} U(r,\theta)rd\theta dr.$$

The above expression gives the value of u over the disk. In particular, πr_*^2 gives the area of the disk. □

Remark 5.4.4. One can write the second mean value property in the more concise form

$$U(0) = \frac{1}{\pi r_*^2}\int_{B_{r*}(0)} UdV,$$

where $\mathcal{B}_{r_*}(0)$ denotes the ball (disk) of radius r_* centred at the origin.

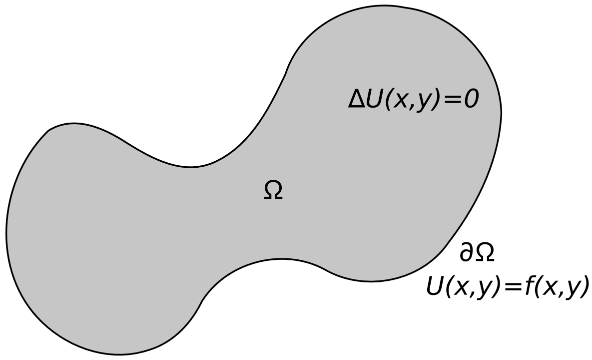
5.5 The maximum principle

In this section we will discuss the important properties of the maximum and minimum of harmonic functions. These properties have important application —mainly to discuss the uniqueness of solutions to the Laplace and Poisson equation.

We recall some **technical** concepts which will be used in the following discussion:

Open domain. An open domain (i.e. set) is one for which at every point in the set it is possible to have a sufficiently small ball (centred at the point in question) which is contained within the set. In particular, an open domain does not include its boundary.

Connected domain. A connected domain is one which consists only of one piece. More precisely, given two arbitrary points in a connected set, it is always possible to find a curve connecting the two points which is completely contained in the set.



The first result of this section is the following:

Proposition 5.5.1. *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and U be a harmonic function defined on Ω. Assume U achieves its maximum at a point $(x_\star, y_\star) \in \Omega$. Then $U(x,y)$ is constant for all $(x,y) \in \Omega$.*

Proof. Since $(x_\star, y_\star) \in \Omega$ and Ω is open, we can find $r > 0$ such that $\mathcal{B}_r(x_\star, y_\star) \subset \Omega$. By the mean value property we have that

$$U(x_\star, y_\star) = \frac{1}{\pi r^2} \int_{\mathcal{B}_r(x_\star, y_\star)} U(\underline{x}) d\underline{x}.$$

Since $U(x_\star, y_\star) \geq U(x,y)$ for all $(x,y) \in \Omega$ (it is a maximum!), then the only way to satisfy the mean value property is to have

$$U(x,y) = U(x_\star, y_\star) \qquad \text{for all } (x,y) \in \mathcal{B}_r(x_\star, y_\star).$$

Now, take any point $(x_n, y_n) \in \Omega$. We want to show that $U(x_n, y_n) = U(x_\star, y_\star)$. For this, we connect $(x_\star, y_\star)$ and (x_n, y_n) with a continuous curve that is covered by intersecting balls $\mathcal{B}_{r_0}(x_i, y_i)$, $2r_0 < r$, in such a way that

$$|(x_{i+1}, y_{i+1}) - (x_i, y_i)| < r_0, \qquad \text{for} \quad i = 0, \ldots, n-1.$$

By the first step in the proof one already knows that $U(x_1, y_1) = U(x_\star, y_\star)$. So, repeating the argument we obtain that

$$U(x_\star, y_\star) = U(x_i, y_i) \qquad \text{for} \quad i = 1, \ldots, n.$$

As the domain is connected then any point in it can be joined to $(x_\star, y_\star)$ by means of a curve completely contained in Ω. Thus, The argument used before shows that $U(x,y)$ must be constant throughout Ω. □

Changing U to $-U$ in the previous argument one obtains the following:

Corollary 5.5.2. *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and U be a harmonic function defined on Ω. Assume U achieves its minimum at a point $(x_\star, y_\star) \in \Omega$. Then $U(x,y)$ is constant for all $(x,y) \in \Omega$.*

Combining the above results one obtains the main result of this section:

Theorem 5.5.3 (the maximum/minimum principle)**.** *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and U be a harmonic function defined on Ω. Then U attains its maximum and minimum values on the boundary $\partial\Omega$.*

Note. In particular, if U is constant on $\partial\Omega$, then it is also constant on Ω.

5.5.1 Application to uniqueness

The maximum principle is key to showing uniqueness of solution to the Laplace and Poisson equation.

Proposition 5.5.4. *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and U be a harmonic function defined on Ω. Then U is uniquely defined by its values on the boundary $\partial\Omega$.*

Proof. Assume there is a harmonic function V defined on Ω such that $V(x, y) = U(x, y)$ for all $(x, y) \in \partial\Omega$. Clearly also

$$W(x, y) \equiv U(x, y) - V(x, y)$$

is harmonic. Moreover, $W(x, y) = 0$ for all $(x, y) \in \partial\Omega$. From the maximum principle we have that

$$0 \leq W(x, y) \leq 0 \qquad \text{for all} \quad (x, y) \in \Omega.$$

Therefore $W(x, y) = 0$ and $U(x, y) = V(x, y)$. $\square$

Progress Check

1. How does the method of separation of variables for the Laplace equation on a rectangle work?
2. What is the strategy to solve the Laplace equation on a rectangle with arbitrary boundary conditions? Why this strategy works?
3. What is the strategy to solve the Laplace equation in polar coordinates by the method of separation of variables.
4. What is the argument used to determine the values of the separation constant?
5. What is the reason that logarithmic and $1/r^m$ solutions are discarded in the solution to the Laplace equation on a disk?
6. What is an annular region?
7. What is the difference between the solution to the Laplace equation on a disk and an annular region?
8. What does Poisson's formula says/does/give?
9. Describe in few works the procedure to obtain Poisson's formula.
10. What is the mean value property? How it is related to Poisson's formula? What does it allow to do?
11. What is the difference between the first and the second mean value property?
12. What does the maximum/minimum principle for the Laplace equation says?
13. How is the maximum/minimum principle related to the mean value properties?
14. What does the uniqueness resolut for the Dirichlet problem for the Laplace equation says?

Chapter 6

The heat equation

In this chapter we study the $1+1$-dimensional heat equation —this is the paradigmatic example of parabolic equations:

$$U_t - \varkappa U_{xx} = 0,$$

with $\varkappa$ the so-called **diffusivity constant**. In $3+1$ dimensions the equation is given by

$$U_t - \varkappa\big(U_{xx} + U_{yy} + U_{zz}\big) = 0.$$

Thus, **time independent** solutions (i.e. with $U_t = 0$) satisfy the Laplace equation

$$\Delta U = 0.$$

We will interested in the following:

(i) **The boundary value problem**. Here one prescribes U at $t = 0$ and on $x = a$, $x = b$.

(ii) **The heat equation on the whole line.** In this case there are no boundary conditions and one only prescribes U at $t = 0$.

The heat equation has a wide range of applications in the study of heat propagation, diffusion of substances in a medium, finance, geometry...

6.1 General remarks

Consider the $1+1$ heat equation in the form

$$U_t = \varkappa U_{xx}, \qquad \varkappa > 0.$$

Geometrically, given a function $U(x,t)$, the second derivative U_{xx} is the rate of change of slope (at fixed time) —that is, it determines whether the graph of U (for fixed t) is concave or convex. On the other hand, U_t is the rate of change of $U(x,t)$ at some fixed point. Thus, one has that

$$\begin{aligned} &U_t > 0 && \text{if the graph of U(x,t) (for fixed } t\text{) is convex,}\\ &U_t = 0 && \text{if the graph is a straight line,}\\ &U_t < 0 && \text{if the graph is concave.} \end{aligned}$$

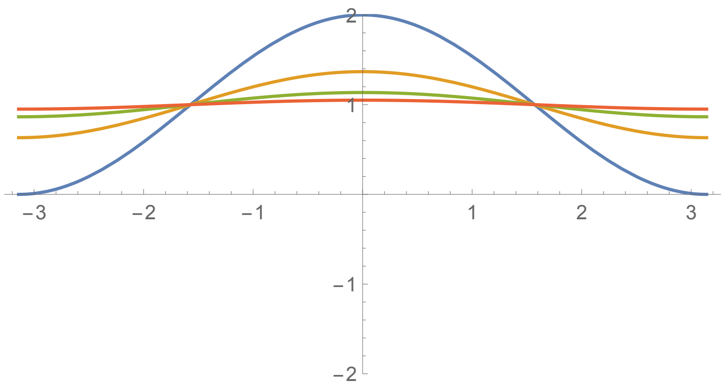
Thus, at all points x where $U_{xx} > 0$ we have that $U(x,t)$ increases in time , and at points where $U_{xx} < 0$ we have that $U(x,t)$ is decreasing in time.

Note. The previous discussion shows that the effect of the heat equation is to **smooth out** bumps.

Example 6.1.1. Consider the function

$$U(x,t) = 1 + e^{-\varkappa t}\cos x.$$

It can be checked to satisfy the heat equation. Plots of this function for various times are given below.



The plots show an initial central concentration spreading out an becoming more and more uniform as t increases. Observe, in particular how U increases where $U_{xx} > 0$ and decreases where $U_{xx} < 0$. Changing the value of $\varkappa$ affects the rate of smoothing: larger $\varkappa$ means faster smoothing and viceversa.

6.2 Derivation of the $1+1$ heat/diffusion equation

Consider a motionless liquid in an insulated pipe and dye diffusing through the liquid.Let $U(x,t)$ denote the concentration (mass/length) of substance at position x of pipe at time t. Then, the total amount of dye in the section of pipe at time t is given by

$$M(t) = \int_{x_1}^{x_2} U(x,t)dx.$$

Thus, the change in the amount of dye is given by

$$\begin{aligned}\frac{dM}{dt} &= \frac{\partial}{\partial t}\int_{x_1}^{x_2} U(x,t)dx \\ &= \int_{x_1}^{x_2} U_t(x,t)dx.\end{aligned}$$

The key observation is that diffusion is characterised by **Fick's law**: dye moves from regions of higher concentration to regions of lower concentration; the rate of motion is proportional to the gradient of the concentration.

Thus, one has that

$$\begin{aligned}\frac{dM}{dt} &= \text{(flow in)} - \text{(flow out)}\\ &= \varkappa U_x(x_2,t) - \varkappa U_x(x_1,t).\end{aligned}$$

Hence,

$$\int_{x_1}^{x_2} U_t(x,t)dx = \varkappa U_x(x_2,t) - \varkappa U_x(x_1,t).$$

Accordingly, using the Fundamental Theorem of Calculus one obtains

$$\int_{x_1}^{x_2} U_t(x,t)dx = \varkappa \int_{x_1}^{x_2} U_{xx}(x,t)dx.$$

As the points x_2 and x_1 are arbitrary, then one is led to

$$U_t - \varkappa U_{xx} = 0.$$

Note. Heat transmission follows a similar law (**Fourier's heat law**). In that case $U(x,t)$ describes the temperature.

6.3 Boundary conditions

Recall that when solving first order ode's one needs one condition on the unknown (initial condition) to determine fully the solution. Since we want to predict the distribution of concentration/temperature $U(x,t)$ for all $t > 0$ and the heat equation has only one derivative in time, then at every x we need to prescribe one initial condition for $U(x,t)$ at $t = 0$ —that is

$$U(x,0) = f(x).$$

On the other hand, since $U_t = \varkappa U_{xx}$ contains U_{xx} and $x \in (a,b)$, we need to prescribe boundary conditions at the end points a and b at each time. This is consistent with the general principle for ode's that to solve second order boundary value problems one needs two boundary conditions (one at each point). These boundary conditions are determined by physical modelling and might contain U and U_x. The most common types are:

(i) **Dirichlet boundary conditions.** Here one prescribes

$$U(a,t) = h(t), \qquad U(b,t) = g(t).$$

These boundary conditions correspond to the temperature/concentration at the endpoints.

(ii) **Neumann boundary conditions.** Here one prescribes

$$U_x(a,t) = h(t), \qquad U_x(b,t) = g(t).$$

In this case one prescribes a flux of U rather than U itself. In particular, if

$$U_x(a,t) = U_x(b,t) = 0,$$

the endpoints are insulated —i.e. no flux.

(iii) **Mixed boundary conditions.** One can also have situations as

$$U_x(a,t) = h(t), \qquad U(b,t) = g(t),$$

or

$$U(a,t) = h(t), \qquad U_x(b,t) = g(t).$$

(iv) **Periodic boundary conditions.** One can also have

$$U(-a,t) = U(a,t)$$

or

$$U_x(-a,t) = U_x(a,t).$$

6.4 The heat equation on an interval

In this section we will see how the method of separation of variables can be used to obtain solutions to the heat equation on an interval. More precisely, we consider the following problem:

$$\begin{aligned} &U_t = \varkappa U_{xx}, \qquad x \in [0,L], \quad t > 0, \\ &U(x,0) = f(x), \\ &U(0,t) = 0, \qquad U(L,t) = 0. \end{aligned}$$

The boundary conditions describe, for example, a metallic wire whose ends are set (by means of some device) at a temperature of 0 degrees.

6.4.1 Separation of variables

Following the general strategy of the method we consider solutions of the form

$$U(x,t) = X(x)T(t).$$

Substitution into the heat equation gives

$$X(x)\dot{T}(t) = \varkappa X''(x)T(t).$$

Hence, dividing by XT we find that

$$\frac{\dot{T}(t)}{\varkappa T(t)} = \frac{X''(x)}{X(x)}.$$

We observe that the left hand side of this last expression only depends on x. The right hand side depends only on t. Thus, for the equality to hold one needs both sides to be constant. That is, one has that

$$\frac{\dot{T}(t)}{\varkappa T(t)} = \frac{X''(x)}{X(x)} = k,$$

with k the so-called separation constant. Thus, we end up with the following ordinary differential equations:

$$\dot{T} = \varkappa k T, \tag{6.1a}$$

$$X'' = kX. \tag{6.1b}$$

Moreover, from the boundary conditions one has that

$$X(0)T(t) = 0, \qquad X(L)T(t) = 0,$$

so that

$$X(0) = X(L) = 0. \tag{6.2}$$

6.4.2 Solving the equation for $T(t)$

Equation (6.1a) has (independently of the sign of k) the solution

$$T(t) = Ce^{k\varkappa t}, \qquad C \quad \text{a constant.}$$

6.4.3 Solving the equation for $X(x)$

Combining equation (6.1b) with the boundary conditions (6.2) one obtains the eigenvalue problem

$$X'' = kX,$$
$$X(0) = X(L) = 0.$$

The boundary condition suggest a periodic solution. In order to get this type of solution one needs $k < 0$.

Remark 6.4.1. It can be readily shown, using an argument previously used for the wave equation, that indeed $k < 0$. More precisely, from the ordinary differential equation it follows that

$$\int_0^L XX'' dx = k \int_0^L X^2 dx.$$

Using integration by parts in the integral on the right hand side one obtains

$$XX'\Big|_0^L - \int_0^L X'^2 dx = k \int_0^L X^2 dx.$$

Using the boundary conditions the latter reduces to

$$-\int_0^L X'^2 dx = k \int_0^L X^2 dx$$

As the integrand in both integrals are non-negative, one then needs that $k \leq 0$. Actually, one needs $k < 0$ to avoid the (trivial) solution $X(x) = 0$.

Once we have established that $k < 0$ one can write, for convenience

$$k = -\lambda^2.$$

Thus, the general solution to equation (6.1b) is given by

$$X(x) = A \cos \lambda x + B \sin \lambda x.$$

Now, we make use of the boundary conditions. First we observe that

$$X(0) = A \cos 0 + B \sin 0.$$

Thus, from (6.2) it follows that

$$A = 0.$$

Using now $X(L) = 0$ one finds that

$$B \sin \lambda L = 0.$$

Clearly one needs $B \neq 0$ to get a non-trivial solution. Thus

$$\lambda = \frac{\pi n}{L}, \qquad n = 1, 2, \ldots$$

Hence, the solution to the eigenvalue problem is given (ignoring the constant B) by

$$X_n(x) = \sin\left(\frac{\pi n x}{L}\right), \qquad k = -\frac{\pi^2 n^2}{L^2}.$$

6.4.4 General solution

The calculations from the previous sections can be combined to obtain the family of solutions to the heat equation

$$U_n(x,t) = e^{-\frac{\pi^2 n^2}{L^2}\varkappa t} \sin\left(\frac{\pi n x}{L}\right).$$

The general solution is then applied using the principle of superposition:

$$U(x,t) = \sum_{n=1}^{\infty} a_n e^{-\frac{\pi^2 n^2}{L^2}\varkappa t} \sin\left(\frac{\pi n x}{L}\right), \tag{6.3}$$

with a_n constants that are fixed through the initial conditions.

6.4.5 Initial conditions

The condition $U(x,0) = f(x)$ with $0 < x < L$ fixes the solution. Evaluating (6.3) at $t = 0$ one has

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi n x}{L}\right).$$

This is a Fourier sine series —we have already found these series a couple of times before. The coefficients a_n are then determined via the Fourier coefficients —thus,

$$a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{\pi n x}{L}\right) dx.$$

6.4.6 Examples

We now look at some concrete examples of the discussion in the previous paragraphs.

Example 6.4.2. Let the initial conditions be given by

$$f(x) = \sin\left(\frac{\pi x}{L}\right).$$

It follows then that

$$U(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = \sin\left(\frac{\pi x}{L}\right).$$

Comparing the two sides of the last equality, and given that the sine functions in the infinite series are independent of each other one finds that

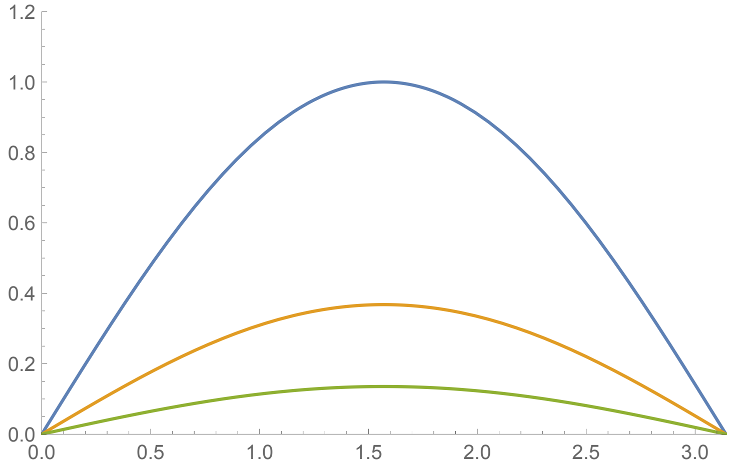
$$a_1 = 1, \qquad a_n = 0, \quad n \geq 2.$$

Thus, the particular solution to the heat equation is given by

$$U(x,t) = e^{-\pi^2 \varkappa t/L^2} \sin\left(\frac{\pi x}{L}\right).$$

A plot of the solution for various values of t is given below. Observe that

$$U(x,t) \to 0, \qquad \text{as} \quad t \to \infty.$$



Example 6.4.3. Let the initial conditions be given by

$$f(x) = 1, \qquad x \in [0, L].$$

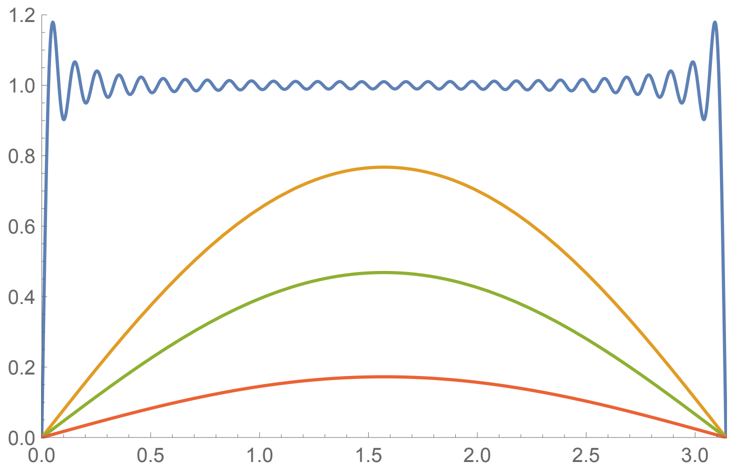
In this case we have to explicitly compute the Fourier coefficients —this is because the constant function does not appear in the series. One has that

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) dx = -\frac{2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L \\ &= -\frac{2}{n\pi}\left((-1)^n - 1\right) \\ &= \begin{cases} 0 & n \text{ even} \\ \dfrac{4}{n\pi} & n \text{ odd} \end{cases} \end{aligned}$$

Hence, one can write

$$U(x,t) = \frac{4}{\pi}\sum_{n \text{ odd}}^{\infty} e^{-n^2\pi^2\varkappa t/L^2}\sin\left(\frac{n\pi x}{L}\right).$$

A plot of the solution for various values of $t > 0$ is given below:



Observe that the solution for $t > 0$ instantly drops to 0 at the ends. Observe that for $n \geq 3$ one has that

$$e^{-\pi^2\varkappa t/L^2} >> e^{-9\pi^2\varkappa t/L^2}.$$

Thus, one has that

$$U(x,t) \approx \frac{4}{\pi} e^{-\pi^2 \varkappa t/L^2} \sin\left(\frac{\pi x}{L}\right).$$

In other words, the first term in the infinite series dominates.

Example 6.4.4. Let $L = 1$ and

$$f(x) = \begin{cases} 1 & 0 < x < 1/2 \\ 0 & 1/2 < x < 1 \end{cases}.$$

Again, we need to compute explicitly the Fourier coefficients. In this case we have

$$a_n = 2\int_0^{1/2} \sin(n\pi x)dx = -\frac{2}{n\pi}\cos(n\pi x)\Big|_0^{1/2} = -\frac{2}{n\pi}\left(\cos(\frac{n\pi}{2}) - 1\right).$$

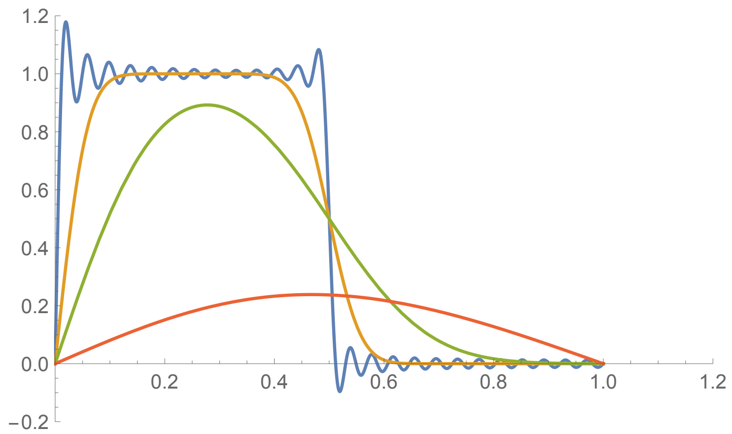
Hence,

$$U(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi/2}{n}\right) e^{-n^2\pi^2\varkappa t}\sin n\pi x.$$

Observe that

$$1 - \cos\frac{n\pi}{2} = 1,\ 2,\ 1,\ 0,\ 1,\ 2, \ldots.$$

A plot of the solution for various $t > 0$ is given below:



Observe that initially one has a step. The solution immediately becomes smooth. It gets more symmetric and sinusoidal as time increases.

6.5 The heat equation on the real line

In this section we will see how to solve the problem

$$\begin{aligned} &U_t = \varkappa U_{xx}, \qquad x \in \mathbb{R}, \quad t > 0, \\ &U(x,0) = f(x). \end{aligned}$$

That is, we want to solve the heat equation on the real line given that we know the initial form of U.

In order to solve this problem we will need some further assumptions on the solution $U(x,t)$ and the initial data $f(x)$. In particular, we want $U(x,t)$ to be absolutely integrable —that is,

$$\int_{-\infty}^{\infty} |U(x,t)|dx < \infty. \tag{6.4}$$

Also, we require that U and U_x go to zero at infinity —that is,

$$U(x,t),\ U_x(x,t) \longrightarrow 0, \qquad x \to \pm\infty.$$

Note. Observe that to have condition (6.4) one needs $U(x,t)$ to go to zero at infinity.

We also require f to be absolutely integrable:

$$\int_{-\infty}^{\infty} f(x)dx < \infty.$$

Example 6.5.1. Functions which are absolutely integrable are special —i.e. not all functions are absolutely integrable. Some examples are:

(i) $f(x) = \sin x$. One then has that

$$\int_{-\infty}^{\infty} |\sin x| dx = \infty.$$

That is, $\sin x$ is not integrable.

(ii) Let

$$f(x) = \frac{1}{1+x^2}.$$

One has that

$$\int_{-\infty}^{\infty} \left|\frac{1}{1+x^2}\right| dx = \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \arctan x\Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi < \infty.$$

Thus $f(x) = 1/(1+x^2)$ is absolutely integrable.

(iii) Let $f(x) = e^{-x^2}$. From Calculus we know that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} < \infty,$$

so, again, absolutely integrable.

6.5.1 Invariance properties of the heat equation

An important property of the heat equation involves the behaviour of its solutions with respect to scalings of the coordinates. More precisely,

Lemma 6.5.2. *If $U(x,t)$ solves the heat equation then also $V(x,t) \equiv U(ax, a^2t)$ also solves the heat equation.*

Proof. Let $v = ax$, $w = a^2t$. Then, using the chain rule one finds that

$$\begin{aligned} U_t(v,w) &= \frac{\partial w}{\partial t} U_w(v,w) = a^2 U_w(v,w), \\ U_x(v,w) &= \frac{\partial v}{\partial x} U_v(v,w) = aU_v(v,w), \\ U_{xx}(v,w) &= a^2 U_{vv}(v,w). \end{aligned}$$

Hence,

$$U_t(ax, a^2t) - \varkappa U_{xx}(ax, a^2t) = a^2\Big(U_w(v,w) - \varkappa U_{vv}(v,w)\Big) = 0.$$

□

Note. Observe that

$$\frac{v^2}{w} = \frac{a^2x^2}{a^2t} = \frac{x^2}{t}.$$

Thus, this hints that the ratio x^2/t is important for the heat equation. Thus, it makes sense to look for solutions of the form

$$U(x,t) = F\left(\frac{x^2}{t}\right).$$

In the following we will look for solutions with a slightly different form.

6.5.2 Invariant solutions

In this section we consider solutions to the heat equation of the form

$$U(x,t) = \frac{1}{t^{\alpha/2}} F\left(\frac{x}{\sqrt{t}}\right) \tag{6.5}$$

with α a constant to be determined. In view of the scaling property of solutions to the heat equation, the $U(x,t)$ as given by (6.5) satisfies the property

$$U(x,t) = \frac{1}{t^{\alpha/2}} U\left(\frac{x}{\sqrt{t}}, 1\right).$$

That is, solutions of the form (6.5) have **invariance properties**.

We now compute the partial derivatives of $U(x,t)$ as given by (6.5). For convenience, let

$$z \equiv \frac{x}{\sqrt{t}}.$$

Using the chain rule one finds that

$$\begin{aligned}
U_x(x,t) &= \frac{1}{t^{\alpha/2+1/2}} F'(z), \\
U_{xx}(x,t) &= \frac{1}{t^{\alpha/2+1}} F''(z), \\
U_t(x,t) &= -\frac{\alpha}{2t^{\alpha/2+1}} F(z) - \frac{z}{2t^{\alpha/2+1}} F'(z).
\end{aligned}$$

Thus, the heat equation gives

$$-\frac{\alpha}{2t^{\alpha/2+1}} F(z) - \frac{z}{2t^{\alpha/2+1}} F'(z) = \frac{\varkappa}{t^{\alpha/2+1}} F''(z).$$

So, if $t \neq 0$, the latter can be simplified to

$$\varkappa F''(z) + \frac{z}{2} F'(z) + \frac{\alpha}{2} F(z) = 0, \tag{6.6}$$

that is, we have obtained an ode for $F(z)$. In order to solve it, we need to fix the value of α. This requires making use of the extra requirements on $U(x,t)$ like absolute integrability —condition (6.4). In this relation it is noticed the following:

Lemma 6.5.3. *Let $U(x,t)$ be a solution to the heat equation which is absolutely integrable and satisfying $U_x(x,t) \to 0$ as $x \to \pm\infty$. Then*

$$\int_{-\infty}^{\infty} U(x,t)dx$$

is constant for $t \geq 0$.

Proof. To see this integrate the heat equation over the real line:

$$\int_{-\infty}^{\infty} U_t(x,t)dx = \varkappa \int_{-\infty}^{\infty} U_{xx}(x,t)dx.$$

This can be rewritten, using the Fundamental Theorem of Calculus as

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^{\infty} U(x,t)dx &= \varkappa U_x(x,t)\Big|_{-\infty}^{\infty} \\ &= 0. \end{aligned}$$

The last equality follows from the requirement that U_x goes to zero at infinity. Thus, the integral

$$\int_{-\infty}^{\infty} U(x,t)dx$$

does not depend on t —that is, it is constant. □

Note. Without loss of generality we can set

$$\int_{-\infty}^{\infty} U(x,t)dt = 1. \tag{6.7}$$

Recalling that $U(x,t) = t^{-\alpha/2}F(x/\sqrt{t})$ it follows then from equation (6.7) that

$$\begin{aligned} 1 &= \frac{1}{t^{\alpha/2}}\int_{-\infty}^{\infty} F\left(\frac{x}{\sqrt{t}}\right)dx \\ &= \frac{\sqrt{t}}{t^{\alpha/2}}\int_{-\infty}^{\infty} F(z)dz, \end{aligned}$$

where in the last line we have used the change of coordinates $z = x/\sqrt{t}$. Thus, setting

$$\alpha = 1$$

we obtain

$$\int_{-\infty}^{\infty} F(z)dz = 1. \tag{6.8}$$

With this choice of the constant α equation (6.6) reduces to

$$\varkappa F'' + \frac{z}{2}F' + \frac{1}{2}F = 0.$$

One can readily check that

$$\begin{aligned} \varkappa F'' + \frac{z}{2}F' + \frac{1}{2}F &= \varkappa F'' + \frac{1}{2}(zF)' \\ &= \left(\varkappa F' + \frac{z}{2}F\right)' = 0. \end{aligned}$$

Hence, integrating one obtains

$$\varkappa F' + \frac{z}{2} F = C = \text{constant}.$$

To determine the constant C it is observed that in order for equation (6.8) to make sense one needs that

$$F(z),\ F'(z) \longrightarrow 0 \quad \text{as} \quad z \longrightarrow \pm\infty.$$

Thus, in fact, one has that

$$C = 0$$

and the differential equation reduces to

$$\varkappa F' + \frac{z}{2} F = 0.$$

This is an equation that can be readily solved by means separation. Writing it in the form

$$\frac{dF}{dz} = -\frac{z}{2\varkappa} F$$

one then has that

$$\int \frac{dF}{F} = -\frac{1}{2\varkappa} \int z dz + \tilde{C},$$

with $\tilde{C}$ an integration constant. Writing, for convenience, the integrating constant as $\ln \tilde{C}$ one obtains

$$\ln F = -\frac{1}{4\varkappa} z^2 + \ln \tilde{C},$$

so that

$$F(z) = \tilde{C} e^{-z^2/4\varkappa}.$$

To determine the integration constant we recall, again, the normalisation condition (6.8). It follows then that

$$\begin{aligned} 1 &= \tilde{C} \int_{-\infty}^{\infty} e^{-z^2/4\varkappa} dz \\ &= \tilde{C} \int_{-\infty}^{\infty} e^{-y^2} 2\sqrt{\varkappa} dy \\ &= 2\tilde{C}\sqrt{\varkappa\pi}, \end{aligned}$$

where in the second line we have used the substitution $y = z/2\sqrt{\varkappa}$. Hence,

$$\tilde{C} = \frac{1}{\sqrt{4\varkappa\pi}},$$

so that

$$F(z) = \frac{1}{\sqrt{4\varkappa\pi}} e^{-z^2/4\varkappa}.$$

Recalling the ansatz (6.5) one finally finds that

$$U(x,t) = \frac{e^{-\frac{x^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi t}}.$$

This solution is known as the **heat kernel** or **fundamental solution** of the heat equation.

6.5.3 Properties of the heat kernel

Given its importance, in the following we denote the heat kernel by

$$K(x,t) = \frac{e^{-\frac{x^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi t}}.$$

We note the following properties:

(i) By construction $K(x,t)$ satisfies the heat equation. That is,

$$K_t = \varkappa K_{xx}, \qquad x \in \mathsf{R}, \quad t > 0.$$

(ii) The heat kernel is an even function —that is, $K(x,t) = K(-x,t)$.

(iii) $K(x,t)$ is a smooth function (i.e. C^∞) for $x \in \mathbb{R}$, $t > 0$.

(iv) One has that

$$\int_{-\infty}^{\infty} K(x,t)dx = 1, \qquad t \geq 0.$$

(v) For $x \neq 0$ one has that

$$K(x,t) \to 0, \qquad \text{as} \quad t \to 0^+,$$

while for $x = 0$ one has that

$$K(0,t) \to \infty, \qquad \text{as} \quad t \to 0^+.$$

Remark 6.5.4. Properties **(i)**-**(iv)** above, follow from the construction of the heat Kernel given in the previous section. Only property **(v)** requires further work. If $x \neq 0$ then to compute the limit it is enough to consider

$$\frac{e^{-1/t}}{\sqrt{t}} = \frac{1}{\sqrt{t}e^{1/t}} \longrightarrow 0 \qquad \text{as} \quad t \to 0^+,$$

given that $e^{1/t} \to \infty$ and recalling that the exponential grows faster than any power of t so it dominates $\sqrt{t}$. For $x = 0$ one has that

$$K(0,t) = \frac{e^0}{\sqrt{4\pi\varkappa t}} = \frac{1}{\sqrt{4\pi\varkappa t}} \longrightarrow \infty \qquad \text{as} \quad t \to 0^+.$$

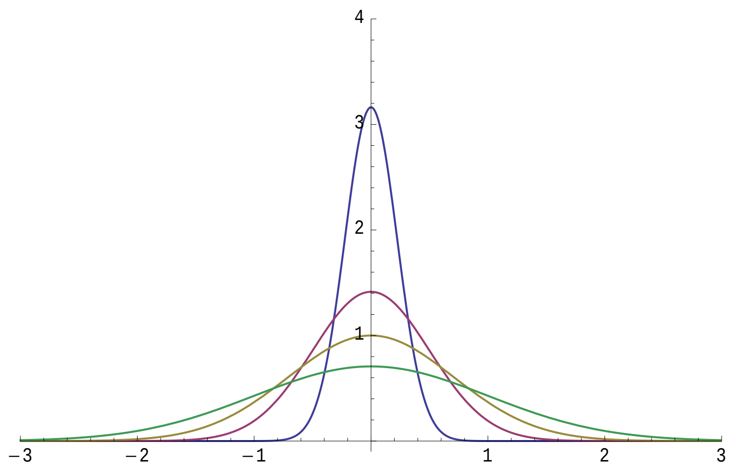
Note. Property **(v)** together with **(iv)** show that $K(x,0)$ is a very special object —in fact, it turns out that $K(x,0)$ cannot be a function. It is a more general type of object known as **generalised function** or **distribution**.

6.5.4 Dirac's delta function

In order to better understand the properties of the heat Kernel at $t = 0$ consider the sequence of functions

$$\{f_\lambda(x)\} = \left\{ \frac{e^{-\frac{x^2}{\lambda^2}}}{\lambda\sqrt{\pi}} \right\}, \qquad \lambda \in \mathbb{R}^+.$$

Graphs of the functions f_λ for various values of λ can be seen in the figure below:



Observe that as $\lambda \to 0$, the Gaussian bells become increasingly peaked. One can then check that:

(i) if $x \neq 0$ then $f_\lambda(x) \to 0$ as $\lambda \to 0$;

(ii) if $x = 0$ then $f_\lambda(0) \to \infty$ as $\lambda \to 0$;

(iii) moreover, one has that

$$\int_{-\infty}^{\infty} f_\lambda(x)dx = 1$$

for all λ so that, in particular, one has

$$\lim_{\lambda\to 0} \int_{-\infty}^{\infty} f_\lambda(x)dx = 1.$$

The limit of the family $\{f_\lambda\}$ is not a proper function. However, one can **formally** write

$$\delta(x) \equiv \lim_{\lambda\to 0} f_\lambda(x) = \lim_{\lambda\to 0} \frac{e^{-x^2/\lambda^2}}{\sqrt{\pi}}.$$

This is the so-called **Dirac's delta "function"**.

Note. There is a branch of mathematics known as **distribution theory** aimed at making sense of objects like Dirac's delta.

Definition 6.5.5. Dirac's delta, δ, is defined by the conditions:

(i) $\delta(x) = 0$ for $x \neq 0$;

(ii) $\delta(0) = \infty$;

(iii) for any $a < 0 < b$ one has
$$\int_a^b \delta(x) = 1.$$

Note. From the previous discussion it follows that
$$K(x,0) = \delta(x).$$

In terms of diffusion processes, $\delta(x)$ describes an infinitesimally small "drop" of ink concentrated at the origin. This "drop" then spreads with time.

6.5.5 The general solution to the heat equation on the real line

The heat kernel is the basic building block to obtain the general solution to the heat equation on the real line.

We begin by observing the following property:

Lemma 6.5.6. *If $U(x,t)$ is a solution to*
$$U_t = \varkappa U_{xx}$$
then
$$V(x,t) \equiv \int_{-\infty}^{\infty} U(x-y,t)g(y)dy \tag{6.9}$$
is also a solution for any function g —as long as the integral converges.

Proof. This follows by direct computation:
$$V_t(x,t) = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} U(x-y,t)g(y)dy = \int_{-\infty}^{\infty} U_t(x-y,t)g(y)dy,$$
$$V_{xx}(x,t) = \frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty} U(x-y,t)g(y)dy = \int_{-\infty}^{\infty} U_{xx}(x-y,t)g(y)dy.$$
Hence,
$$V_t(x,t) - \varkappa V_{xx}(x,t) = \int_{-\infty}^{\infty}\Big(U_t(x-y,t) - \varkappa U_{xx}(x-y,t)\Big)g(y)dy = 0.$$
□

Note. The operation given by (6.9) is called the **convolution** of U and g. This is sometimes denoted as
$$V(x,t) = (f * g)(x,t).$$

Now, consider the problem
$$U_t = \varkappa U_{xx}, \qquad x \in \mathbb{R}, \quad t > 0, \tag{6.10}$$
$$U(x,0) = f(x). \tag{6.11}$$

Claim: the (unique) solution to (6.10)-(6.11) is given by
$$U(x,t) = \int_{-\infty}^{\infty} K(x-y,t)f(y)dy,$$
with K denoting the heat kernel. Or, more explicitly,

$$U(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi t}} f(y)dy. \tag{6.12}$$

The latter is known as the **Fourier-Poisson formula**.

As a consequence of Lemma 6.5.6, and given that $K(x,t)$ satisfies the heat equation, then $U(x,t)$ as defined by (6.12) is a solution to the heat equation.

Note. To fully address the claim it is only necessary to verify that $U(x,0) = f(x)$.

Some auxiliary calculations

In the following it will be convenient to consider the function

$$Q(x,t) \equiv \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4\varkappa t}} e^{-s^2} ds, \qquad t > 0. \tag{6.13}$$

Observe that

$$\begin{aligned} Q_x(x,t) &= \frac{1}{\sqrt{\pi}} \frac{d}{dx}\left(\frac{x}{\sqrt{4\varkappa t}}\right) e^{-\frac{x^2}{4\varkappa t}} \\ &= \frac{e^{-\frac{x^2}{4\varkappa t}}}{\sqrt{4\pi\varkappa t}} = K(x,t). \end{aligned}$$

Thus, $Q(x,t)$ is the antiderivative (with respect to x) of $K(x,t)$.

Next, we consider the limit of $Q(x,t)$ as $\to 0^+$. There are 2 cases:

(i) $x > 0$. Here we have

$$\lim_{t\to 0^+} Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-s^2} ds = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = 1.$$

(ii) $x < 0$. Here one has

$$\lim_{t\to 0^+} Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-s^2} ds = \frac{1}{2} - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-s^2} ds = \frac{1}{2} - \frac{1}{2} = 0.$$

Hence, one concludes that

$$\lim_{t\to 0^+} Q(x,t) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \equiv H(x).$$

The function H defined above is called **Heaviside's step function**.

Note. As Q is the antiderivative of K it follows from the above discussion that

$$H'(x) = \delta(x).$$

That is, Dirac's delta is the derivative of Heaviside's step function.

Concluding the main computation

Using the properties of Q as discussed in the previous subsection one has that

$$\begin{aligned}
U(x,t) &= \int_{-\infty}^{\infty} K(x-y,t)f(y)dy \\
&= \int_{-\infty}^{\infty} Q_x(x-y,t)f(y)dy \\
&= -\int_{-\infty}^{\infty} Q_y(x-y,t)f(y)dy \\
&= -Q(x-y,t)f(y)\bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} Q(x-y,t)f'(y)dy,
\end{aligned}$$

where in the third line one makes use of the chain rule to change the x-derivative to a y-derivative and in the fourth line one employs integration by parts to pass the derivative from Q to f. Now, as $K(x-y,t)$ decays very fast to 0 as $|x-y| \to \infty$ it follows that

$$-Q(x-y,t)f(y)\bigg|_{-\infty}^{\infty} = 0.$$

Hence,

$$U(x,t) = \int_{-\infty}^{\infty} Q(x-y,t)f'(y)dy.$$

We make use of this expression to compute the limit $t \to 0^+$:

$$\begin{aligned}
U(x,0^+) &= \int_{-\infty}^{\infty} Q(x-y,0^+)f'(y)dy = \int_{-\infty}^{\infty} H(x-y)f'(y)dy \\
&= \int_{-\infty}^{x} f'(y)dy = f(y)\bigg|_{-\infty}^{x} = f(x),
\end{aligned}$$

where in the last line it has been assumed that $f(x) \to 0$ as $x \to -\infty$.

We summarise the previous discussion in the following:

Proposition 6.5.7. *For $t > 0$, the function*

$$U(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi t}} f(y)dy$$

is the (unique) solution to

$$\begin{aligned}
&U_t(x,t) = \varkappa U_{xx}(x,t), \qquad x \in \mathbb{R}, \quad t > 0, \\
&U(x,0) = f(x).
\end{aligned}$$

6.5.6 Some examples

In this section we discuss some examples of computation involving the Fourier-Poisson formula.

Example 6.5.8. Analyse the behaviour of the solution $U(x,t)$ given by the Fourier-Poisson formula in the case

$$f(x) = H(x) = \left\{ \begin{array}{ll} 0 & x < 0 \\ 1 & x > 0 \end{array} \right. .$$

In this case one has

$$\begin{aligned} U(x,t) &= \frac{1}{\sqrt{4\pi\varkappa t}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4\varkappa t} H(y)dy \\ &= \frac{1}{\sqrt{4\pi\varkappa t}}\int_{0}^{\infty} e^{-(x-y)^2/4\varkappa t} dy \end{aligned}$$

as $H(x) \neq 0$ only for $x > 0$. Letting now

$$s = \frac{x-y}{\sqrt{4\varkappa t}} \Longrightarrow dy = -\sqrt{4\varkappa t}ds,$$

one finds that

$$\begin{aligned} U(x,t) &= -\frac{\sqrt{4\varkappa t}}{\sqrt{4\pi\varkappa t}}\int_{x/\sqrt{4\varkappa t}}^{-\infty} e^{-s^2}ds \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x/\sqrt{4\varkappa t}} e^{-s^2}ds \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-s^2}ds + \frac{1}{\sqrt{\pi}}\int_{0}^{x/\sqrt{4\varkappa t}} e^{-s^2}ds \\ &= \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_{0}^{x/\sqrt{4\varkappa t}} e^{-s^2}ds. \end{aligned}$$

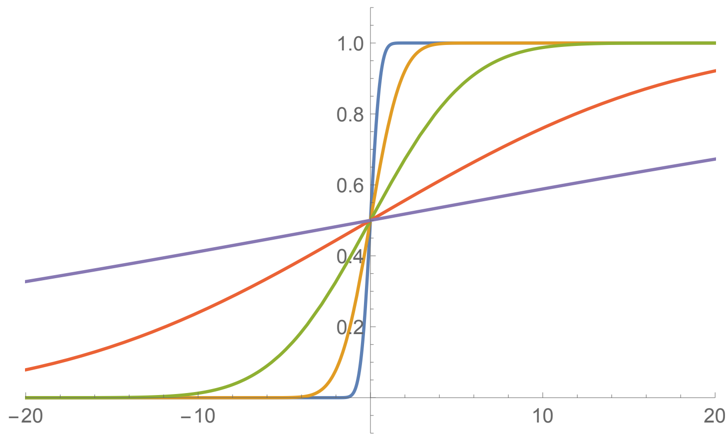
Thus, observe that, in fact

$$U(x,t) = Q(x,t).$$

We now investigate the behaviour of $U(x,t)$ for fixed x as $t \to \infty$:

$$\lim_{t\to\infty} U(x,t) = \frac{1}{2} + \lim_{t\to\infty}\frac{1}{\sqrt{\pi}}\int_{0}^{x/\sqrt{4\varkappa t}} e^{-s^2}ds = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_{0}^{0} e^{-s^2}ds = \frac{1}{2},$$

as $x/\sqrt{4\varkappa t} \to 0$ as $t \to \infty$.



Example 6.5.9. Evaluate the Fourier-Poisson formula in the case

$$f(x) = e^{-x}.$$

Substituting the above expression in the formula one obtains

$$U(x,t) = \frac{1}{\sqrt{4\varkappa\pi t}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4\varkappa t} e^{-y} dy.$$

The exponent in the integral can be rearranged as

$$\begin{aligned}-\frac{(x-y)^2}{4\varkappa t}-y&=-\frac{x^2-2xy+y^2+4\varkappa ty}{4\varkappa t}\\&=-\frac{(y+2\varkappa t-x)^2}{4\varkappa t}+\varkappa t-x.\end{aligned}$$

Hence,

$$\begin{aligned}U(x,t)&=\frac{1}{\sqrt{4\varkappa\pi t}}\int_{-\infty}^{\infty}e^{-\frac{(y+2\varkappa t-x)^2}{4\varkappa t}+\varkappa t-x}dy\\&=\frac{e^{\varkappa t-x}}{\sqrt{4\varkappa\pi t}}\int_{-\infty}^{\infty}e^{-\frac{(y+2\varkappa t-x)^2}{4\varkappa t}}dy.\end{aligned}$$

Letting

$$s=\frac{y+2\varkappa t-x}{\sqrt{4\varkappa t}}\Longrightarrow ds=\frac{dy}{\sqrt{4\varkappa t}},$$

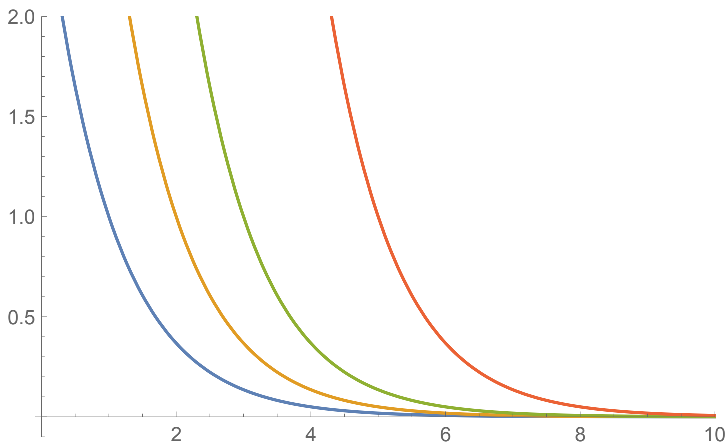
it follows then that

$$\begin{aligned}U(x,t)&=\frac{e^{\varkappa t-x}}{\sqrt{4\varkappa\pi t}}\int_{-\infty}^{\infty}e^{-s^2}\sqrt{4\varkappa t}ds\\&=\frac{e^{\varkappa t-x}}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-s^2}ds=e^{\varkappa t-x}.\end{aligned}$$

Observe, in particular, that

$$U(x,t)\to\infty,\qquad \text{as}\quad t\to\infty.$$

Thus, the solution does not decay but grows at every point x. Plots of this solution for various values of t are shown below.



We conclude the list of examples with one particular solution to the heat equation which evidences an important property of the heat equation:

Example 6.5.10. Compute the solution to the heat equation on the real line if the initial condition is given by

$$f(x)=\frac{e^{x^2/4\varkappa}}{\sqrt{4\pi\varkappa}}.$$

In this case substitution of the initial condition into the Fourier-Poisson formula gives

$$\begin{aligned} U(x,t) &= \frac{1}{\sqrt{4\pi\varkappa t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\varkappa t}} \frac{e^{\frac{y^2}{4\varkappa}}}{\sqrt{4\pi\varkappa}} dy \\ &= \frac{1}{4\pi\varkappa\sqrt{t}} \int_{-\infty}^{\infty} e^{\frac{y^2}{4\varkappa} - \frac{(x-y)^2}{4\varkappa t}} dy. \end{aligned}$$

The exponent in the integral can be manipulated by completing squares to get

$$\begin{aligned} \frac{y^2}{4\varkappa} - \frac{(x-y)^2}{4\varkappa t} &= \frac{1}{4\varkappa}\left(y^2 - \frac{x^2}{t} - \frac{y^2}{t} + \frac{2xy}{t}\right) \\ &= \frac{1}{4\varkappa}\left(\left(1 - \frac{1}{t}\right)y^2 + \frac{2xy}{t} - \frac{x^2}{t}\right) \\ &= \frac{1}{4\varkappa}\left(\left(\frac{t-1}{t}\right)\left(y^2 + \frac{2xy}{t-1} + \frac{x^2}{(t-1)^2}\right) - \frac{x^2}{t} - \frac{x^2}{t(t-1)}\right) \\ &= \frac{1}{4\varkappa}\left(\left(\frac{t-1}{t}\right)\left(y + \frac{x}{t-1}\right)^2 - \frac{x^2}{t-1}\right). \end{aligned}$$

Hence,

$$\begin{aligned} U(x,t) &= \frac{1}{4\pi\varkappa\sqrt{t}} \int_{-\infty}^{\infty} e^{\frac{1}{4\varkappa}\left(\left(\frac{t-1}{t}\right)\left(y+\frac{x}{t-1}\right)^2 - \frac{x^2}{t-1}\right)} dy \\ &= \frac{e^{-\frac{x^2}{4\varkappa(t-1)}}}{4\pi\varkappa\sqrt{t}} \int_{-\infty}^{\infty} e^{-\frac{1}{4\varkappa}\left(\frac{1-t}{t}\right)\left(y+\frac{x}{t-1}\right)^2} dy. \end{aligned}$$

Finally, letting

$$s = \sqrt{\frac{1-t}{4\varkappa t}}\left(y + \frac{x}{t-1}\right) \Longrightarrow ds = \sqrt{\frac{1-t}{4\varkappa t}} dy,$$

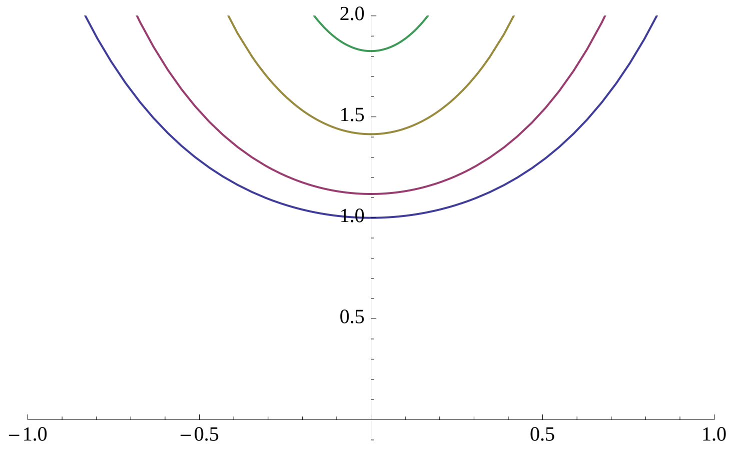
one concludes that

$$\begin{aligned} U(x,t) &= \frac{e^{-\frac{x^2}{4\varkappa(t-1)}}}{4\pi\varkappa\sqrt{t}} \sqrt{\frac{4\varkappa t}{1-t}} \int_{-\infty}^{\infty} e^{-s^2} ds \\ &= \frac{e^{\frac{x^2}{4\varkappa(1-t)}}}{\pi\sqrt{4\varkappa(1-t)}} \sqrt{\pi} \\ &= \frac{e^{\frac{x^2}{4\varkappa(1-t)}}}{\sqrt{4\varkappa\pi(1-t)}}. \end{aligned}$$

Observe that

$$U(x,t) \longrightarrow \infty \qquad \text{as} \quad t \to 1.$$

That is, the solution becomes singular in a finite amount of time! A plot of the solutions for various values of t is shown below:



6.6 The heat equation on the half-line

As an application of the Fourier-Poisson formula we now study the initial value problem for the heat equation on the half-line with Dirichlet boundary conditions. More precisely, we have

$$U_t = \varkappa U_{xx}, \qquad x \geq 0, \quad t > 0, \tag{6.14}$$
$$U(x,0) = f(x), \tag{6.15}$$
$$U(0,t) = 0 \qquad \text{(Dirichlet boundary condition)}. \tag{6.16}$$

We have studied a similar problem for the wave equation. These require the use of **odd extensions**. Recall that the odd extension of f is defined as

$$F(x) = \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x < 0 \end{cases}.$$

The point behind the use of odd extensions is that it allows one to formulate an **auxiliary initial value problem** on the whole real line —namely,

$$V_t = \varkappa V_{xx}, \qquad x \in \mathbb{R},$$
$$V(x,0) = F(x).$$

The solution to this problem is given in terms of the Fourier-Poisson formula

$$\begin{aligned} V(x,t) &= \frac{1}{\sqrt{4\pi\varkappa t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\varkappa t}} F(y)dy \\ &= \frac{1}{\sqrt{4\pi\varkappa t}} \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4\varkappa t}} F(y)dy + \frac{1}{\sqrt{4\pi\varkappa t}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4\varkappa t}} F(y)dy. \end{aligned}$$

Observe, however, that the integrand in the second equation is odd, so setting $y \mapsto -y$ (so that $dy \mapsto -dy$) in the first integral gives

$$\begin{aligned} V(x,t) &= -\frac{1}{\sqrt{4\pi\varkappa t}} \int_{\infty}^{0} e^{-\frac{(x+y)^2}{4\varkappa t}} F(-y)dy + \frac{1}{\sqrt{4\pi\varkappa t}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4\varkappa t}} F(y)dy \\ &= \frac{1}{\sqrt{4\pi\varkappa t}} \int_{\infty}^{0} e^{-\frac{(x+y)^2}{4\varkappa t}} F(y)dy + \frac{1}{\sqrt{4\pi\varkappa t}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4\varkappa t}} F(y)dy \\ &= \frac{1}{\sqrt{4\pi\varkappa t}} \int_{0}^{\infty} \left(e^{-\frac{(x-y)^2}{4\varkappa t}} - e^{-\frac{(x+y)^2}{4\varkappa t}} \right) F(y)dy. \end{aligned}$$

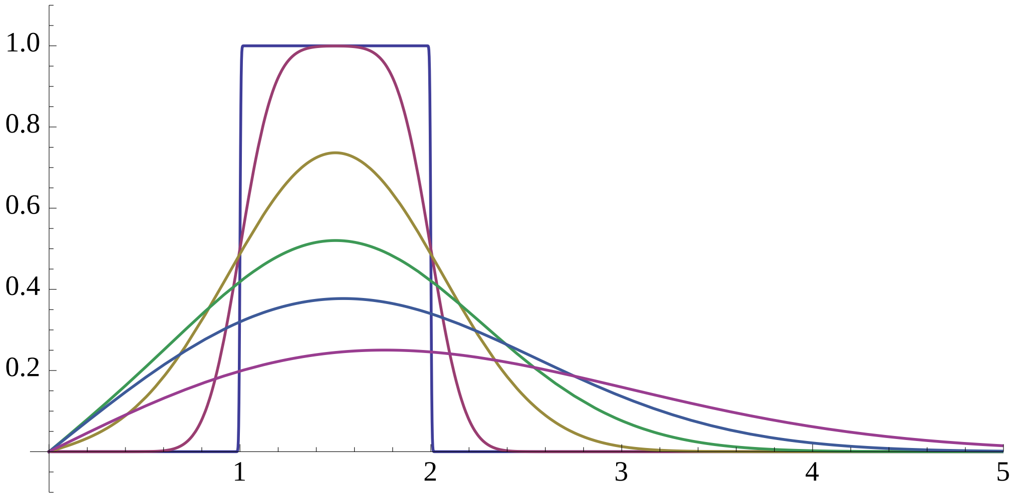
Now, recalling that $F(y) = f(y)$ for $x \in [0,\infty)$ we find that:

$$V(x,t) = \frac{1}{\sqrt{4\pi\varkappa t}} \int_0^\infty \left(e^{-\frac{(x-y)^2}{4\varkappa t}} - e^{-\frac{(x+y)^2}{4\varkappa t}} \right) f(y) dy. \tag{6.17}$$

Now, observe that by construction one readily has that

$$V(0,t) = \frac{1}{\sqrt{4\pi\varkappa t}} \int_0^\infty \left(e^{-\frac{y^2}{4\varkappa t}} - e^{-\frac{y^2}{4\varkappa t}} \right) f(y) dy = 0.$$

Thus, the function V as given by equation (6.17) satisfies the boundary conditions for $U(x,t)$. Thus, if the solution $U(x,t)$ to (6.14)-(6.16) is **unique** (something we have not proved!) then $U(x,t)$ and $V(x,t)$ must coincide for $x > 0$, $t > 0$. A plot of the solution (6.17) for various times for an initial function f with the shape of a top hat function is given below:



It is worth pointing out that

$$\tilde{K}(x,t) \equiv \frac{1}{\sqrt{4\pi\varkappa t}} \left(e^{-\frac{(x-y)^2}{4\varkappa t}} - e^{-\frac{(x+y)^2}{4\varkappa t}} \right)$$

for y fixed is also a solution to (6.14)-(6.16) with initial condition a Dirac delta centred at $x = y$. We call $\tilde{K}(x,t)$ the fundamental solution to the heat equation with Dirichlet boundary conditions.

Note. The case of Neumann boundary conditions (where $U_x(0,t) = 0$) can be studied in a similar manner using even extensions —see Coursework 10.

6.7 The maximum/minimum principle for the heat equation

We conclude our discussion of the heat equation with a discussion of some general properties of the heat equation.

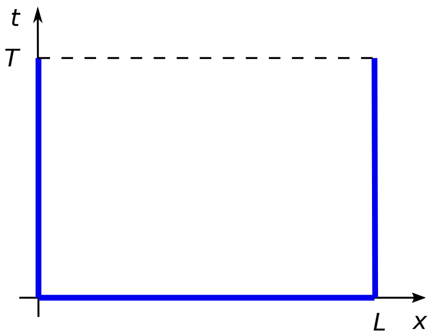
The heat equation satisfies a maximum/minimum principle which is similar to that of the Laplace equation. More precisely, one has the following:

Theorem 6.7.1. *If $U(x,t)$ satisfies the heat equation on the* **rectangle**

$$\Omega = \Big\{ 0 \le x \le L, \;\; 0 \le t \le T \Big\}$$

then the maximum value of $U(x,t)$ is either assumed initially (i.e. at $t = 0$) or on the boundaries $x = 0$ or $x = L$.

The region in which one has to look for the maxima/minima of the solution to the heat equation are highlighted in blue in the figure below:



Remark 6.7.2. The minimum of $U(x,t)$ on Ω satisfies a similar property.

Example 6.7.3. Find the maxima and minima of

$$U(x,t) = 1 + e^{-\varkappa t} \cos x$$

on the rectangle

$$\Omega = \Big\{ -\pi \le x \le \pi, \;\; 0 \le t \le T \Big\}$$

for some $T > 0$. This solution to the heat equation has already been discussed in Example 6.1.1.

Following the maximum/minimum principle for the heat equation one then has to look for the maximum/minimum values of U on the sides $t = 0$, $x = -\pi$ and $x = \pi$:

(i) **On $t = 0$.** One has that

$$U(x,0) = 1 + \cos x.$$

As $\cos x$ ranges between -1 and 1 if $x \in [-\pi, \pi]$ it follows that the maximum of $U(x,0)$ $x \in [-\pi,\pi]$ occurs at $x = 0$ where $U(x,0)$ takes the value 2 and the minima at $x = -\pi$, π where $U(x,0)$ takes the value 0.

(ii) **On $x = -\pi$.** Here one has

$$U(0,t) = 1 - e^{-\varkappa t}.$$

It can be verified that the derivative of this function is positive for $t \ge 0$. Thus, the maximum must happen at $t = T$ (where U takes the value $1 - e^{-\varkappa T} < 1$) and its minimum at $t = 0$ (where U takes the value 0).

(iii) **On $x = \pi$.** Here one has

$$U(0,t) = 1 - e^{-\varkappa t}.$$

Thus, again, the maximum must happen at $t = T$ (where U takes the value $1 - e^{-\varkappa T} < 1$) and its minimum at $t = 0$ (where U takes the value 0).

So, putting together the information above one has that the maximum of U occurs at $(0,0)$ where it takes the value 2 and the minima, which takes the value 0, occur at $(-\pi,0)$ and $(\pi,0)$.

In the following we give a sketch of the proof of the above Theorem —it illustrates the applicability of some ideas of Calculus.

Proof. The rectangle Ω is finite region (a **bounded set**) so that the function $U(x,t)$ should attain a maximum and a minimum somewhere.

If the maximum occurs in the interior of Ω at a point (x_*,t_*) one then has that

$$U_t(x_*,t_*) = U_x(x_*,t_*) = 0.$$

This is the standard condition of the vanishing of the gradient at an extremal point. In addition, as one has a maximum then the **second derivative test** has to hold —i.e. one has that

$$U_{xx}(x_*,t_*) < 0. \tag{6.18}$$

However, the function $U(x,t)$ satisfies the heat equation $U_t = \varkappa U_{xx}$. Thus, in addition one has that

$$U_{xx}(x_*,t_*) = 0.$$

The latter is in contradiction with (6.18). Thus, the maximum cannot occur in the interior of Ω. It can only be attained somewhere on the sides (boundary) of the rectangle.

Next, assume that the maximum is attained somewhere, (x_*,T) $x \in (0,L)$, in the middle of the top of the rectangle. The restriction of U to the top of the rectangle, $U(x,T)$ $x \in (0,L)$, is a function of x only. At the maximum one has that

$$U_x(x_*,T) = 0, \qquad U_{xx}(x_*,T) < 0.$$

Now, as (x_*,T) is a maximum over the whole of Ω one has that

$$U_t^- \equiv \lim_{t\to T} U_t(x_*,t) \geq 0.$$

Again, the heath equation gives

$$0 \geq U_t(x_*,T) = \varkappa U_{xx}(x_*,T) < 0.$$

This, again, is a contradiction. Hence, the maximum cannot be attained at the top of the rectangle —it can only be attained on the bottom or on the sides.

□

Remark 6.7.4. The argument above is a sketch of the actual proof. A full proof needs to exclude some **pathological situations** which we have overlooked for the sake of conciseness.

Remark 6.7.5. The argument cannot be used to exclude the bottom of the rectangle as the heat equation is only solved for $t > 0$.

Progress Check

1. What does the heat equation tends to do to an initial profile. In which regions does the solution grow? In which does it decrease?
2. What is the main difference between solving the heat equation and the wave equation by means of the method of separation of variables.
3. What sort of real situation does the heat equation on the real line describes?
4. What sort of invariance properties does the heat equation possess?
5. What is the heat Kernel?
6. Why is the heat Kernel special?
7. What are the main properties of the heat Kernel?
8. What is the Fourier-Poisson formula?
9. What are the defining properties of Dirac's delta "function"?
10. What is the Fourier-Poisson formula?
11. How does one define the convolution of two functions?
12. How does the temperature of an infinitely long metallic wire which initially has one half at 0 degrees and the other at 1 degree evolve with time? What is the final temperature?

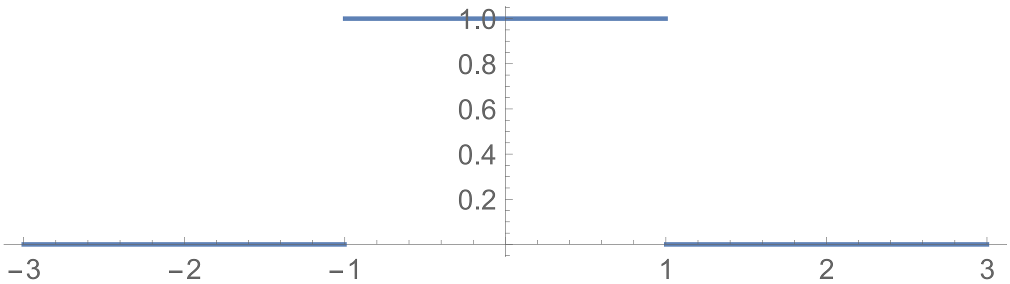
Chapter 7

Concluding remarks

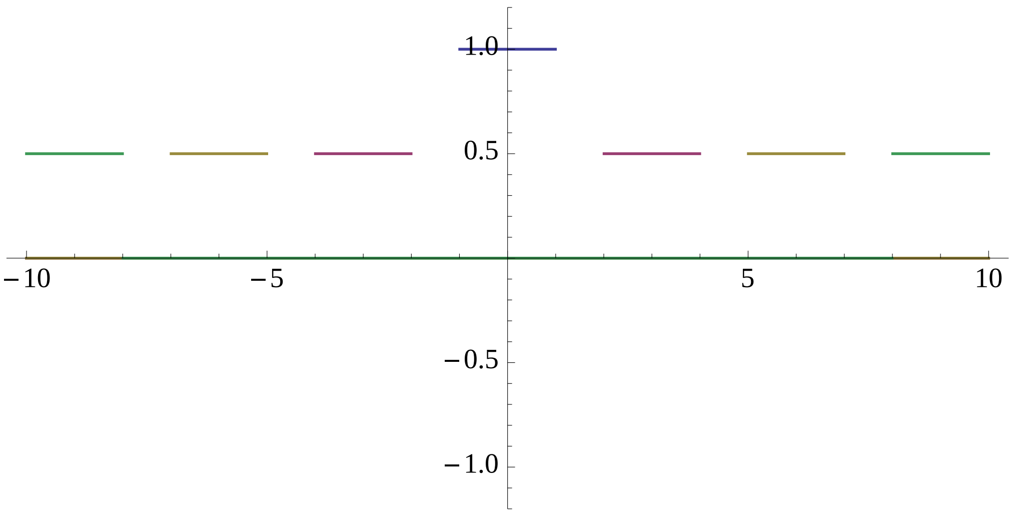
The conclude these lecture notes by contrasting the properties of the wave, heat and Laplace equations.

7.1 Comparison of the wave and the heat equation

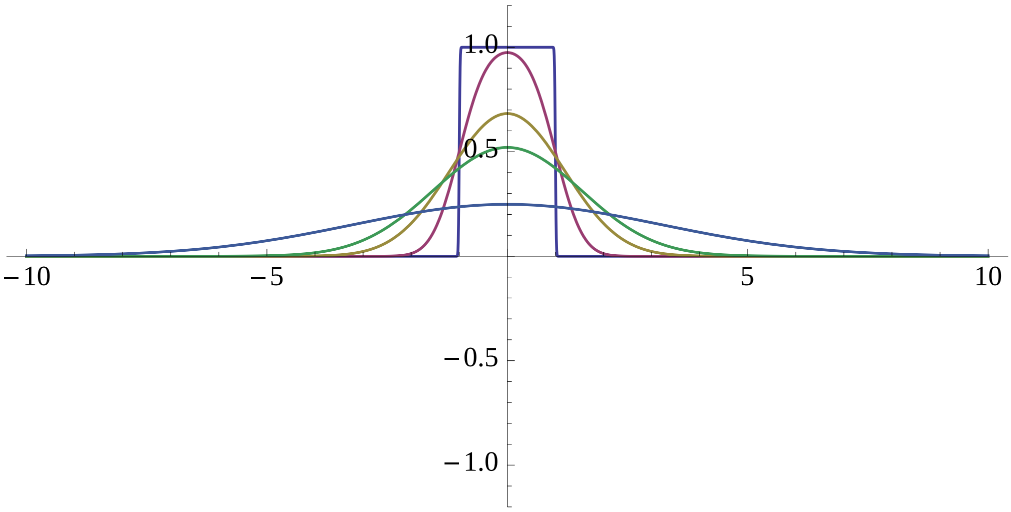
The wave and heat equation both describe evolutive processes. However, the properties of the solutions are very different. This can be readily seen by comparing the evolution of an initial profile on the form of a **top-hat function**:



Snapshots at various times of the above initial profile as evolved by the wave equation are given below:



Snapshots of the evolution of the initial top-hat profile according to the heat equation look like:



From the above one sees that while**the perturbation encoded in the top-hat function travels with with finite speed if evolved with the wave equation**, in the case of **the heat equation it travels at infinite speed** —i.e. points arbitrarily far way from the origin "feel" the presence of the top-hat function. Also, we observe that in the case of the wave equation the sharp edges of the top-hat are transported in the case of the heat equation they are lost immediately. Thus, one can say that **the wave equation transports information** while **the heat equation looses information**.

We collect the above and some more facts about the wave and heat equation in the form of a table:

Property	**Wave equation**	**Heat equation**
Speed of propagation	finite ($\leq c$)	infinite!
Singularities	propagated	lost immediately
Information	transported	lost
Maximum/minimum principle	no	yes
General solution	D'Alambert's formula	Fourier-Poisson formula
Initial data	$U(x,0)$, $U_t(x,0)$	$U(x,0)$

7.2 The Laplace equation

The Laplace equation does not describe evolution processes but, rather, it is associated to steady state situations —i.e. the end state of evolution.

Key properties of the Laplace equation are:

(i) Mean value property —the value of the solution at a point reflects its average in a neighbourhood.

(ii) Maximum/minimum principle.

(iii) The general solution on a disk is given by Poisson's integral formula.

(iv) The Laplace equation does not admits initial data —only boundary data.