Coursework 5

6.3.1. $\dot{x} = x - y$, $\dot{y} = x^2 - 4$. Phase portrait sketch?

fixed points: $x = y$, $x^2 = 4 \Rightarrow$ $x = +2, y = +2$; $x = -2, y = -2$

$\therefore$ $\underline{x}_1^* = (2,2)$, $\underline{x}_2^* = (-2,-2)$

Linearisation $Df(\underline{x}) = \begin{bmatrix} 1 & -1 \\ 2x & 0 \end{bmatrix}$.

$\underline{x}_1^*$: $Df(\underline{x}_1^*) = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}$; Eigenvalues $(\lambda - 1)\lambda + 4 = 0$

$\Rightarrow \lambda = \frac{1 \pm \sqrt{1-16}}{2} \Rightarrow \lambda = \frac{1 \pm i\sqrt{15}}{2}$

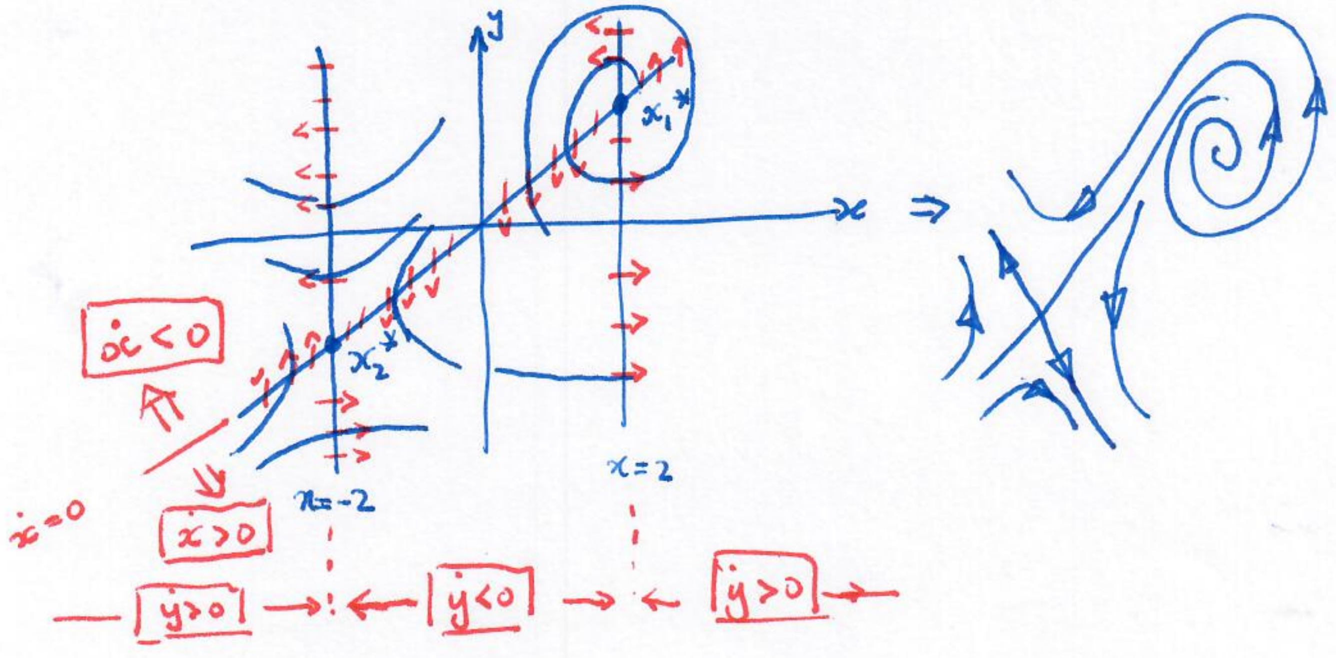
$\therefore$ x_1^* is an unstable spiral.

x_2^*: $Df(x_2^*) = \begin{bmatrix} 1 & -1 \\ -4 & 0 \end{bmatrix}$; Eigenvalues $(\lambda - 1)\lambda - 4 = 0$

$\Rightarrow \lambda = \frac{1 \pm \sqrt{1+16}}{2} \Rightarrow \lambda = \frac{1 \pm \sqrt{17}}{2}$,

x_2^* is a saddle.

Isoclines $\dot{x} = 0$ (H) $x = y$
$\dot{y} = 0$ (V) $x^2 = 4$



6.3.15 In formal notes & discussed in week 11 tutorial. lecture

(a) $\dot r = r(r-1)$ $\dot\theta = 1$ $r=0$ is an unstable spiral and $r=1$ is attracting ($\dot r>0$ for $r<1$ & $\dot r<0$ for $r>1$), $\dot\theta = 1$ ensures anticlockwise flow.

Therefore $r=1$ is a stable limit cycle

(b) $\dot r = r(1-r)$, $\dot\theta = 1-\cos\theta$.

$\dot r = 0$: $r=0$, $r=1$ / $\dot\theta = 0$: $\theta = 0$

Introduce local coordinate for r & θ at $r=1$, $\theta=0$

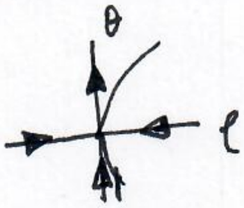
Let $\rho = r-1$ & $\theta = \theta$!

$\dot\rho = (1+\rho)(-\rho) = -\rho+\rho^2$, $\dot\theta = 1-\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\dots\right)$

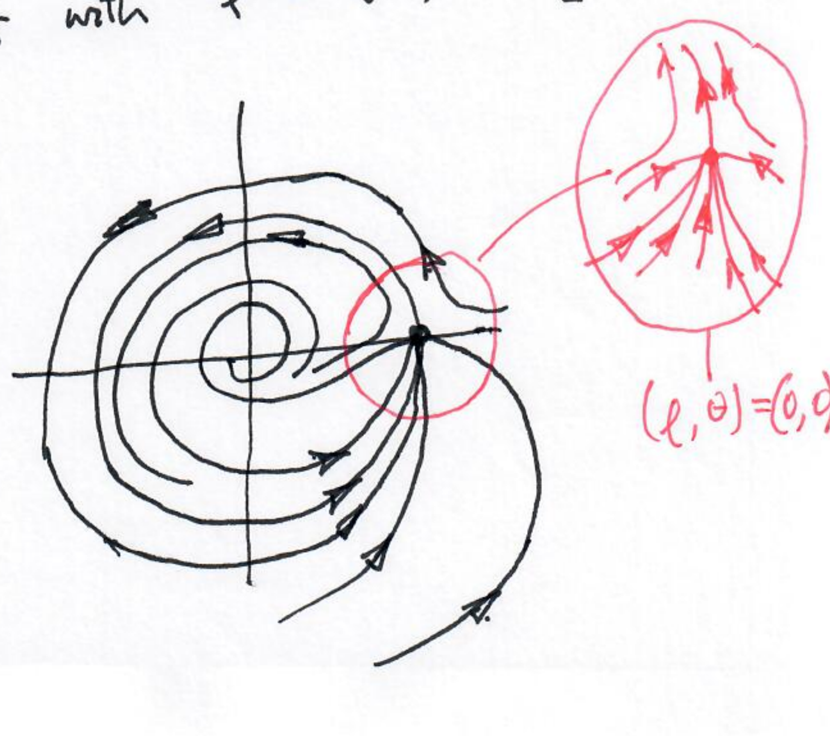
$= \frac{\theta^2}{2} + O(\theta^4)$.

So locally at $(\rho,\theta)=(0,0)$ we have $\dot\rho = -\rho$, $\dot\theta = \frac{\theta^2}{2}$

a saddle node fixed point with



Phase portrait is :



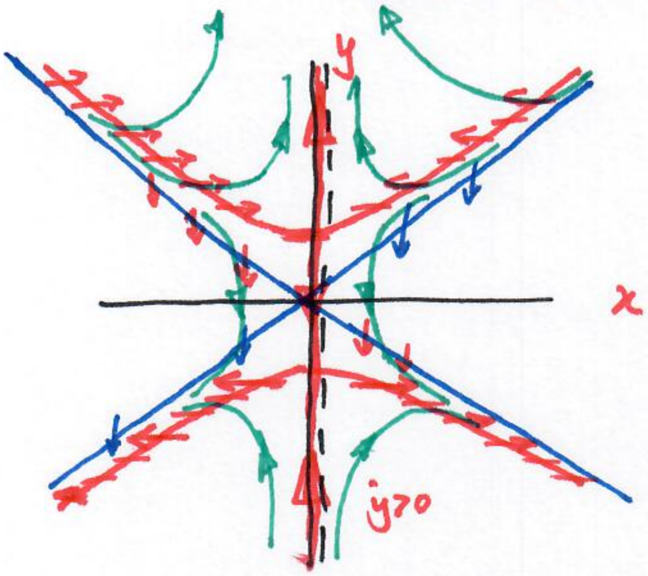
Note $\omega(\mathbb{R}-\{0\}) =$

6.3.16(a)

$\dot{x} = x^2 - xy + a, \quad \dot{y} = y^2 - x^2 - 1$ $\Big\} = \underline{f}(x,y)$. For $a = 0$

(a) Null-clines $\dot{y} = 0 : y^2 = x^2 + 1$, note $y \approx \pm x$ for large $|x|$

$\dot{x} = 0 : x = 0, \; x = y.$



$\dot{y} < 0$ for $y^2 < x^2 + 1$

$\dot{y} > 0$ for $y^2 > x^2 + 1$

Fixed points at

$x = 0, \; y = \pm 1$

Linearization

$$D\underline{f}(x) = \begin{bmatrix} 2x - y & -x \\ -2x & 2y \end{bmatrix}$$

$D f(0,1) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ - saddle $\lambda_1 = -1, \lambda_2 = 2$
& eigenvector $\underline{v}_1 = [1,0]$, $v_2 = [0,1]$

$D f(0,-1) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ - saddle $\lambda_1 = 1, \lambda_2 = -2$
& eigenvectors $\underline{v}_1 = [1,0]$, $v_2 = [0,1]$

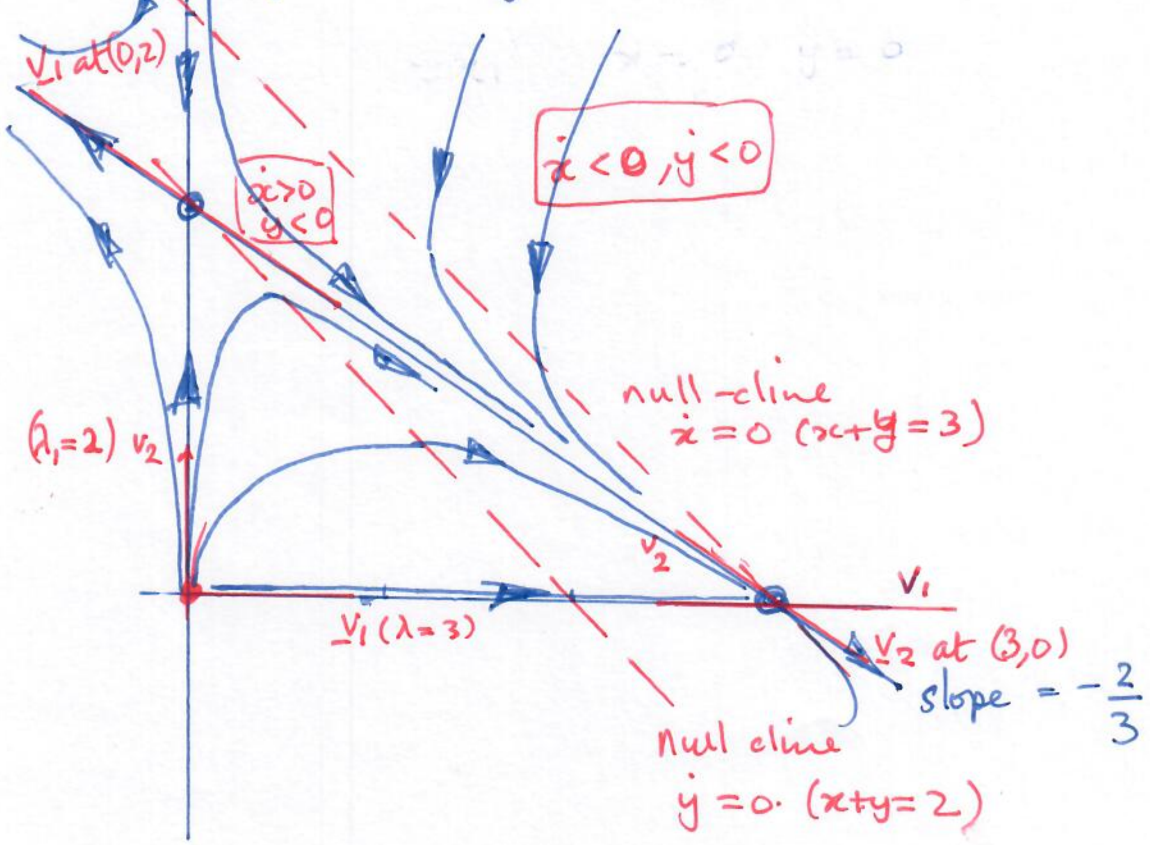
Using information, there are some sample trajectories in GREEN.

6.4.1 $\dot{x} = x(3-x-y)$, $\dot{y} = y(2-x-y)$

ClK 7.11

Null-clines $\dot{x} = 0$: $x = 0$, $x+y=3$

$\dot{y} = 0$: $y = 0$, $x+y=2$



Fixed points x_1^* , $x=0$, $y=0$

x_2^* , $x=0$, $y=2$

x_3^* , $x=3$, $y=0$

$$A = Df(x^*) = \begin{bmatrix} 3-2x-y & -x \\ -y & 2-x-2y \end{bmatrix}$$

x_1^* $(x=0, y=0)$:

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 3$, $\lambda_2 = 2$

Eigenvectors: $\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

x_2^* $(x=0, y=2)$

$$A = \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -2$

Eigenvectors: $\underline{v}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

x_3^* $(x=3, y=0)$

$$A = \begin{bmatrix} -3 & -3 \\ 0 & -1 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = -3$ $\lambda_2 = -1$

Eigenvectors: $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ v_2

AWK7.12

In[5]:=

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StreamPlot[{x (3 - x - y), y (2 - x - y)}, {x, -4, 4}, {y, -4, 4}]
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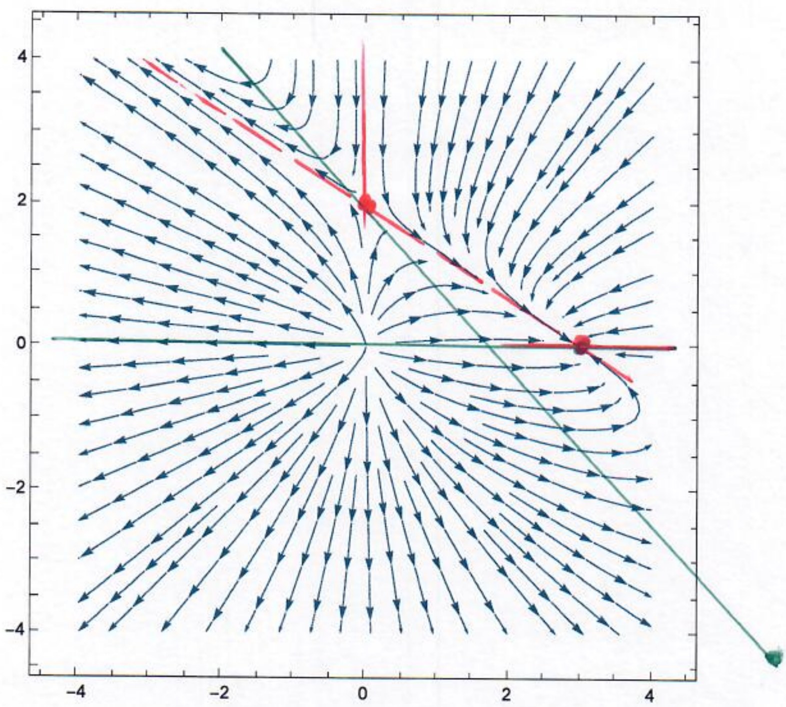
Out[5]=



red line eigendirections

$\dot{y} = 0$

In[4]:=

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∫PhasePortrait[{x (3 - x - y), (2 - x - y) y}, {x, -4, 4}, {y, -4, 4}] ⅆx
```

Out[4]=

```
∫PhasePortrait[{x (3 - x - y), (2 - x - y) y}, {x, -4, 4}, {y, -4, 4}] ⅆx
```

6.5.2 $\qquad \ddot{x} = x - x^2$

Let $\dot{x} = y$, then $\dot{y} = \ddot{x} = x - x^2$

Consider $\dot{x} = y$, $\dot{y} = x - x^2$; $\underline{f}(x,y) = (y, x - x^2)$

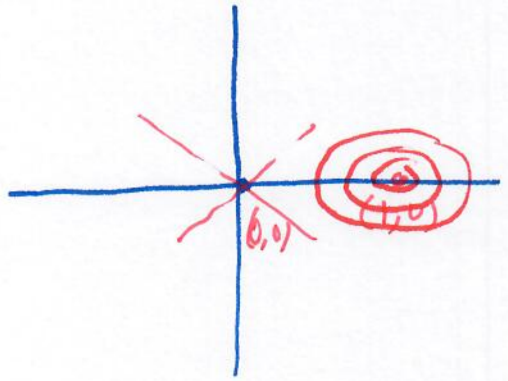
$$dt = \frac{dx}{y} = \frac{dy}{x - x^2} \Rightarrow \int (x^2 - x)\, dx + \int y\, dy = \text{const}$$

$$\Rightarrow + \frac{x^3}{3} - \frac{x^2}{2} + \frac{y^2}{2} = C, \text{ constant.}$$

Note fixed points at $\quad \dot{x} = 0: \; \cancel{x}y = 0$

$\qquad\qquad\qquad\qquad\quad\; \dot{y} = 0: \; x - x^2 = 0 \Rightarrow x = 1, 0$

$\therefore \quad x_1^* = (0,0) \;\; \& \;\; x_2^* = (1,0)$ are fixed pts.



Linearisation $D\underline{f}(0,0) = \begin{bmatrix} 0 & 1 \\ 1-2x & 0 \end{bmatrix}_{(0,0)}$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$\therefore$ Eigenvalues: $\lambda = \pm 1$

Eigendirection $+1$: $[1, 1]$

$\qquad\qquad\quad\;\; -1$ $[1, -1]$.

$$D\underline{f}(1,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \text{Eigenvalues } \lambda = \begin{matrix} +i \\ -i \end{matrix} \text{ - linear centre.}$$

Also non-linear centre because

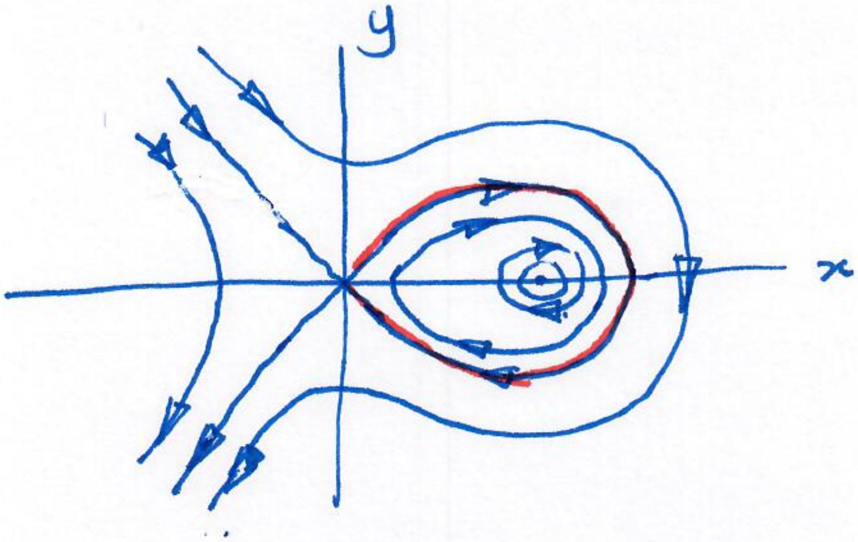
$$z = \frac{x^3}{2} - \frac{x^2}{2} + \frac{y^2}{2} \quad \text{has critical points at}$$

$x_1^* = (0,0) \;\; \& \;\; x_2^* = (1,0)$

Note at $(0,0)$ $\quad f_{xx} = -1$, $f_{yy} = +1$ $\quad \left.\begin{matrix} \text{max in } x \\ \text{min in } y \end{matrix}\right\} \Rightarrow$ saddle.

(checks with above analysis).

at $(1,0)$ $\quad f_{xx} = 2$, $f_{yy} = 1$ $\quad \therefore \quad \left.\begin{matrix} \text{min in } x \\ \text{min in } y \end{matrix}\right\}$ centre (non-linear!)



These are the level curves of $\frac{y^2}{2} - \frac{x^2}{2} + \frac{x^3}{3}$.

The homoclinic orbit (red) emanates from the saddle point at $(x, y) = (0,0)$

So the integral curve containing the saddle curve is ~~(x,y)≠(0,0)~~.

$$\boxed{\frac{y^2}{2} - \frac{x^2}{2} + \frac{x^3}{3} = 0.}$$ with $x > 0$.

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StreamPlot[{x^2 - x*y, y^2 - x^2 - 1}, {x, -2, 2}, {y, -2, 2}]
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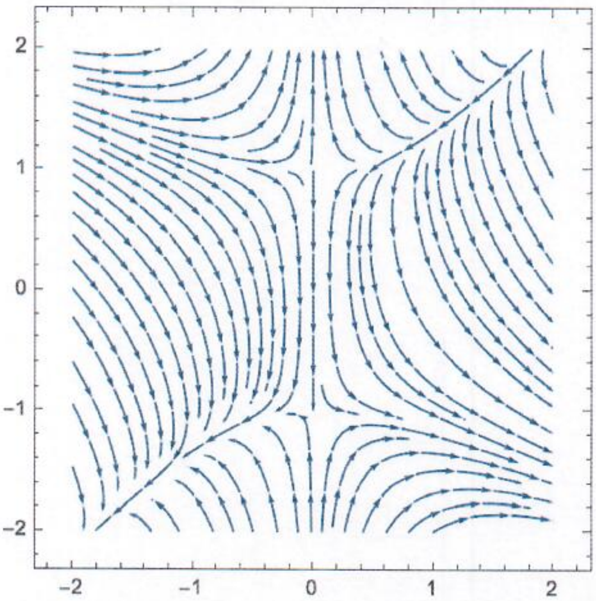
ut[5]=



$a = 0$

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In[8]:= StreamPlot[{0.1 + x^2 - x*y, y^2 - x^2 - 1}, {x, -2, 2}, {y, -2, 2}]
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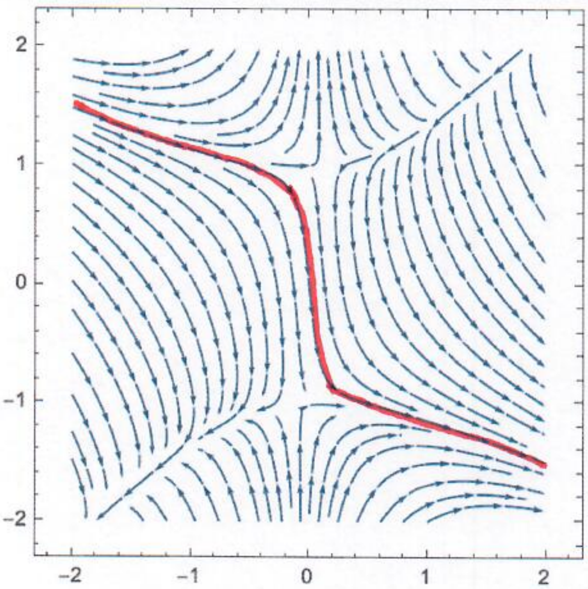
ut[8]=



$a > 0.$

```
[10]:= StreamPlot[{-0.1 + x^2 - x*y, y^2 - x^2 - 1}, {x, -2, 2}, {y, -2, 2}]
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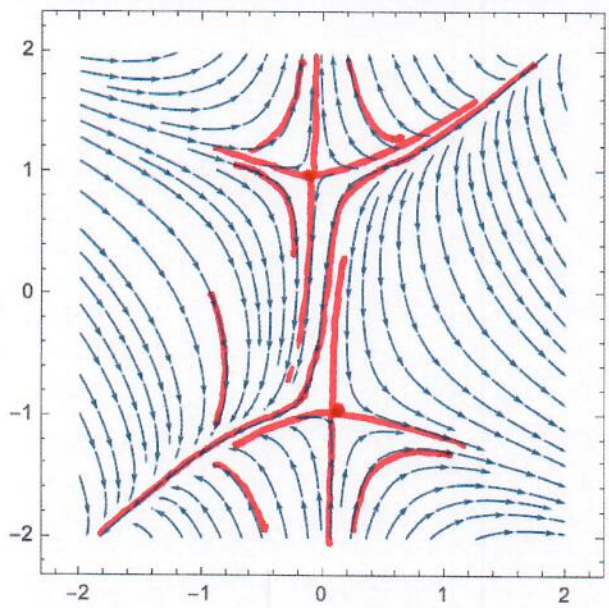
t[10]=



$a < 0.$

no saddle