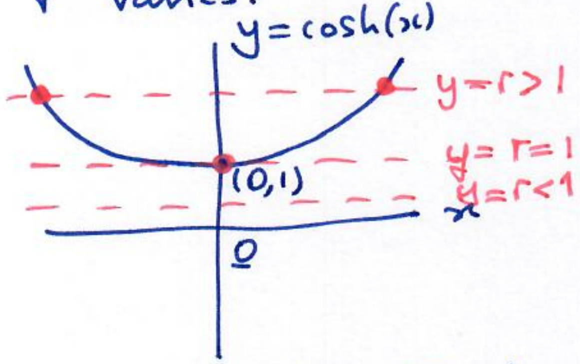
COURSEWORK 2 BIFURCATIONS 3.1.(2, 4) 3.2.(3,4) 3.4.(4,8)
(STROGATZ, p80)

3.1.2. To find the fixed point structure we need to find solns of $r - \cosh x = 0$. Plotting the graphs $y = r$ & $y = \cosh(x)$ will show different intersections as r varies.

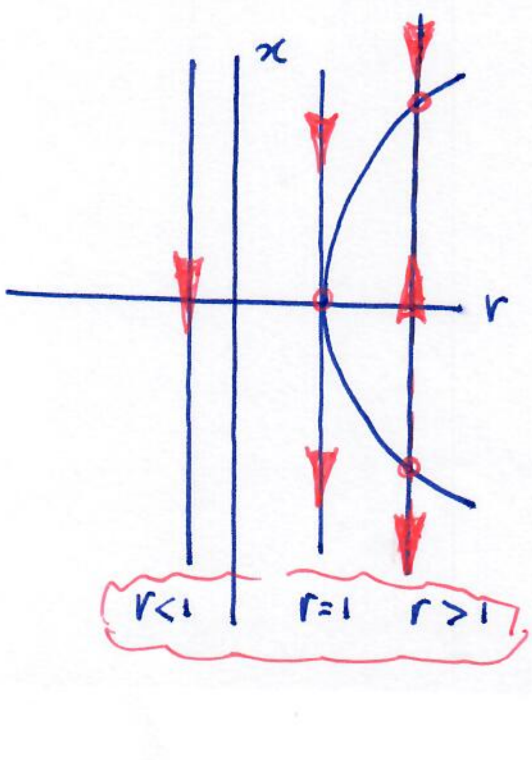


Cosh(x) has a minimum at $x = 0$ when $\cosh(0) = 1$.
So $r > 1$ gives two solutions
$r = 1$ gives one solution
$r < 1$ gives no solution

This fixed point structure certainly allows a saddle node bifurcation as a possibility. Considering $\dot{x} = r - \cosh(x)$ with $\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$ gives

$$\dot{x} = r - 1 - \frac{x^2}{2} + O(x^4) = (r-1) - \frac{x^2}{2}.$$

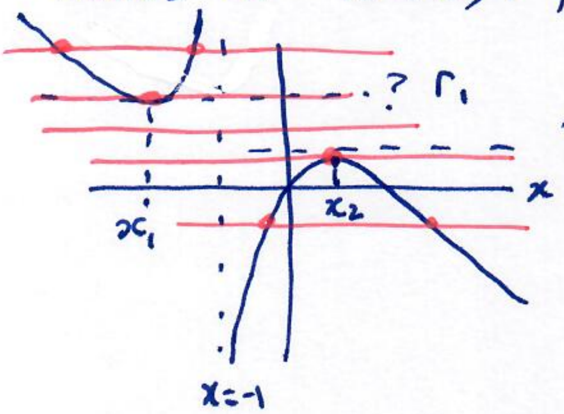
This is the normal form for a saddle-node at $r - 1 = 0$, i.e. $r = 1$ & $x = 0$.



3.1.4 $\dot{x} = r + \frac{1}{2}x - x/(1+x)$. Again we need to consider the intersections of $y = r$ & $y = \frac{x}{(1+x)} - \frac{1}{2}x$

Note $\frac{x}{1+x} - \frac{1}{2}x = \frac{2x - x(1+x)}{2(1+x)} = \frac{x - x^2}{2(1+x)}$

Zeroes at $x = 0, 1$, Asymptote at $x = -1$



Max, min of $g(x) = \frac{x}{1+x} - \frac{1}{2}x$ are given by $g'(x) = 0$

$$\frac{(1+x).1 - x.1}{(1+x)^2} - \frac{1}{2} = 0$$

$$\Rightarrow \frac{1}{(1+x)^2} = \frac{1}{2} \Rightarrow x = -1 \pm \sqrt{2}$$

$$r_1 = \frac{-1-\sqrt{2}}{-\sqrt{2}} - \frac{1}{2}(-1-\sqrt{2}) = \frac{1}{\sqrt{2}} + 1 + \frac{1}{2} + \frac{1}{\sqrt{2}} = \sqrt{2} + \frac{3}{2} \qquad x_1 = -1-\sqrt{2}$$

$$r_2 = \frac{-1+\sqrt{2}}{\sqrt{2}} - \frac{1}{2}(-1+\sqrt{2}) = \frac{-1}{\sqrt{2}} + 1 + \frac{1}{2} - \frac{1}{\sqrt{2}} = -\sqrt{2} + \frac{3}{2} \qquad x_2 = -1+\sqrt{2}$$

Fixed points of $\dot{x} = r + \frac{1}{2}x - \frac{x}{(1+x)}$

2 for $r > r_1$ & $r < r_2$

1 for $r = r_1$ & $r = r_2$

0 for $r_2 < r < r_1$

To show we have a saddle node at $x_2 = -1+\sqrt{2}$, $r_2 = -\sqrt{2} + \frac{3}{2}$, let $y = (x + 1 - \sqrt{2})$, $\mu = (r + \sqrt{2} - \frac{3}{2})$

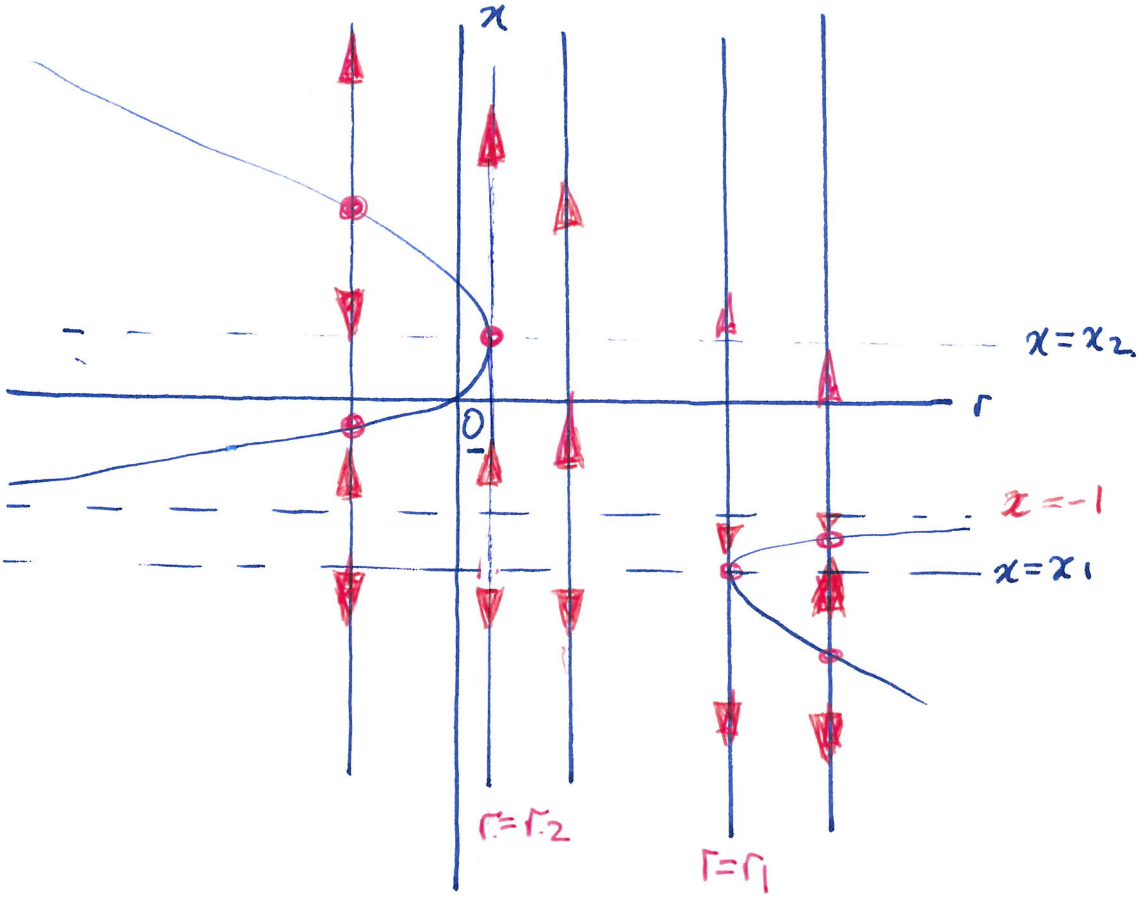
$$\dot{y} = \mu - \sqrt{2} + \frac{3}{2} + \frac{1}{2}(y - 1 + \sqrt{2}) - \frac{(y-1+\sqrt{2})}{(y+\sqrt{2})}$$

$$= \mu + 1 - \sqrt{2} + \frac{1}{2}y + \frac{1}{y+\sqrt{2}}(y - 1 + \sqrt{2})$$

$$= \mu + \frac{y^2}{2\sqrt{2}} + O(y^3) \quad \left[\text{using } \frac{1}{(y+\sqrt{2})} = \frac{1}{\sqrt{2}}\left(1 + \frac{y}{\sqrt{2}}\right)^{-1} = \frac{1}{\sqrt{2}}\left(1 - \frac{y}{\sqrt{2}} + \frac{y^2}{2} + \dots\right)\right]$$

∴ Saddle node. bifurcation

3.1.4



The diagram suggests saddle node bifurcations but we need to prove the existence

The procedure for the bifurcation at $(x, r) = (x_2, r_2)$ is very similar to that at $(x, r) = (x_1, r_1)$.

3.2.3 $\quad \dot{x} = x - rx(1-x)$

$\dot{x} = x(1-r) + rx^2$ Note $x=0$ is a fixed pt for all $r \in \mathbb{R}$ and $\frac{d}{dx}\left(x(1-r)+rx^2\right) = (1-r)+2rx$ so there is a change of stability at $r=1$.

Introduce a local parameter at $r=1$, i.e. $\mu = (r-1)$, then

$$\dot{x} = x + (\mu-1)x(1-x) = \mu x - (\mu-1)x^2$$

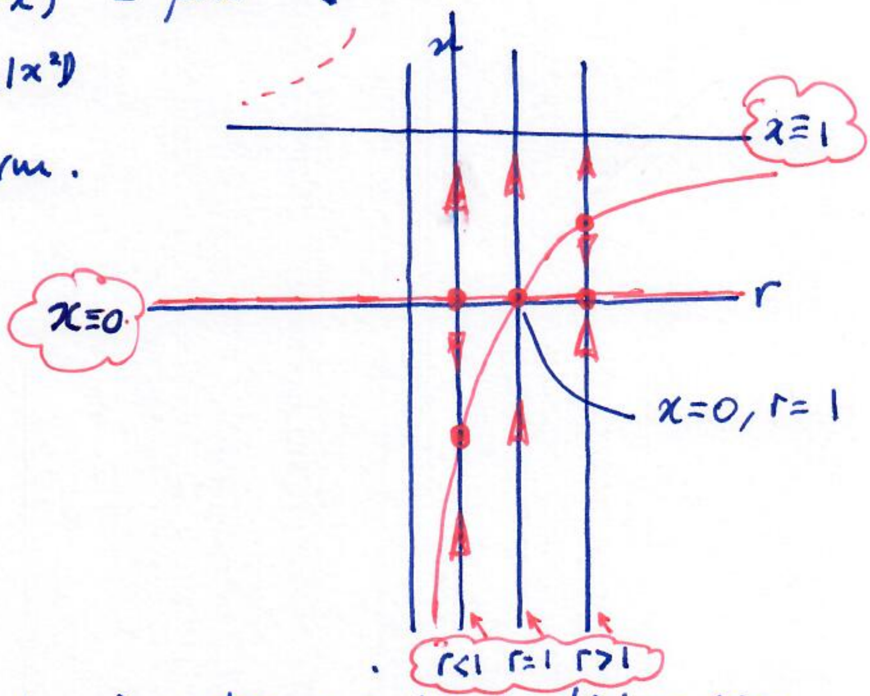
$$= \mu x + x^2 + O(|\mu|, |x^2|)$$

- transcritical normal form.

There are two families of fixed points

$x \equiv 0 \quad \forall r$

and $x = 1 - \frac{1}{r}$.



They cross at $x=0,\ r=1$ in a transcritical bifurcation

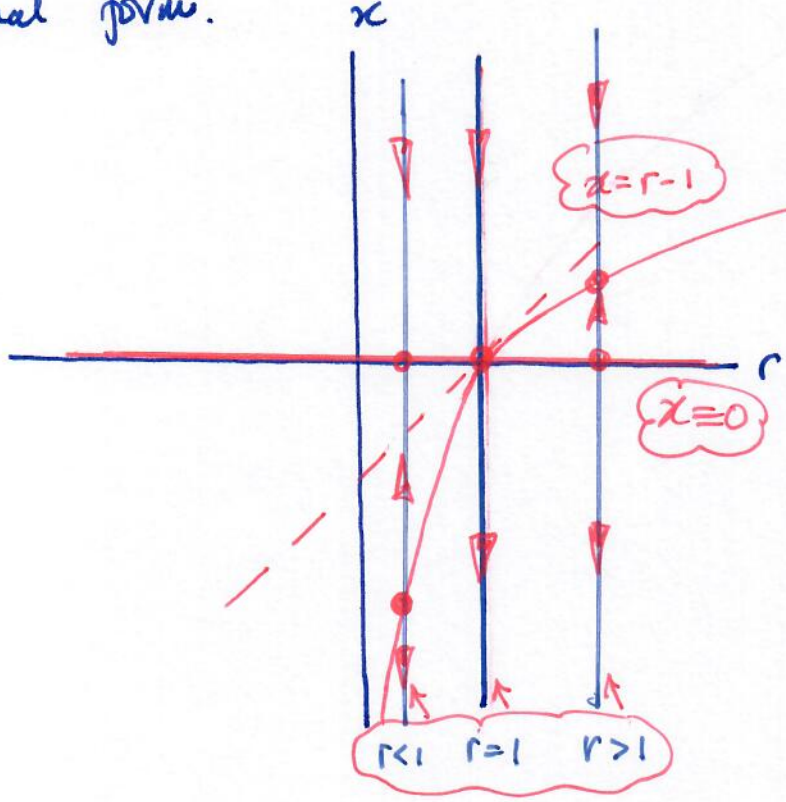
3.2.4 $\dot{x} = x(r - e^{x})$

$$\dot{x} = x\left(r - 1 - x - \frac{x^2}{2!} - \dots\right)$$

$$= x\left((r-1) - x\right)$$

Let $\mu = r-1$ & we obtain

$\dot{x} = x\mu - x^2 + O(|x|^3)$, which is the transcritical normal form.



The two families of fixed points are

$x \equiv 0$ for all $r \in \mathbb{R}$

$e^{x} = r \iff x = \ln(r)$ for $r \in \mathbb{R}^+$

and they cross at $x=0$, $r=1$ with changes of stability

3.4.4 $\quad \dot{x} = x + \frac{rx}{1+x^2}$

$$\dot{x} = x + rx(1+x^2)^{-1} = x + rx(1 - x^2 + x^4) + O(x^6)$$

$$= (r+1)x - rx^3 + O(x^4)$$

Let $\mu = r+1$

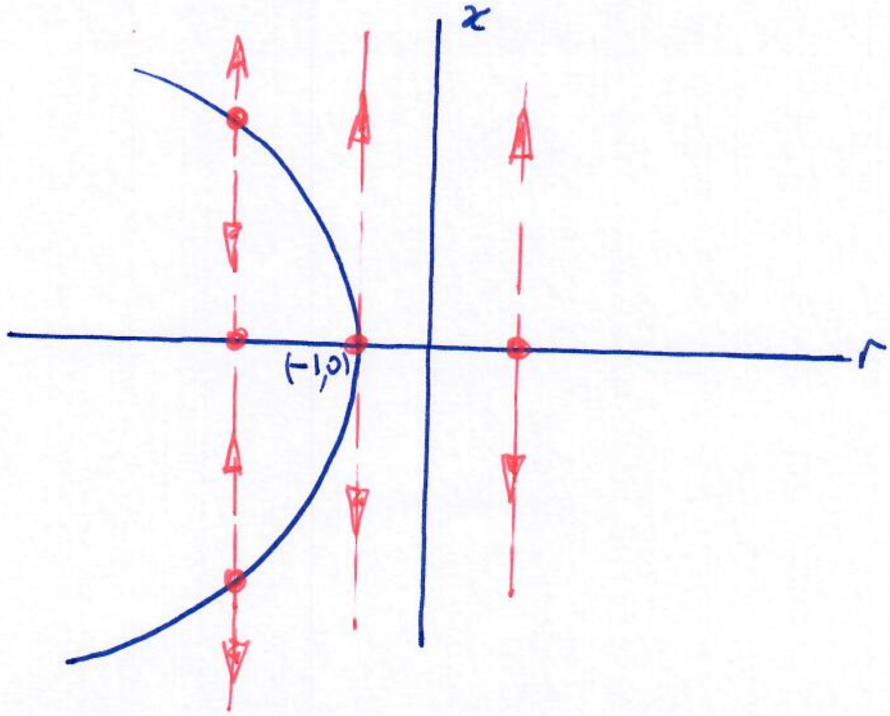
$$\dot{x} = \mu x - (\mu-1)x^3 + O(x^4).$$

$$= \mu x + x^3 + O(|\mu|, |x|^3)$$

So we have a subcritical normal form with fixed points $x \equiv 0 \quad , \forall \mu$

$x = \pm\sqrt{-\mu}$ for $\mu < 0$

<u>NOTE</u>. This is subcritical in the (x,μ) plane but always need to check that parameter has not been reversed i.e. $\frac{d\mu}{dr} \begin{matrix} >0 \\ <0 \end{matrix}$? . Here $\frac{d\mu}{dr} = 1$ (O.K here)



3.4.8 $\dot{x} = rx - \dfrac{x}{1+x^2}$

$$\dot{x} = rx - x(1+x^2)^{-1} = rx - x + x^3 + O(x^5).$$
$$= (r-1)x + x^3 + O(x^5)$$

Subcritical pitchfork at $x=0$, $r=1$.

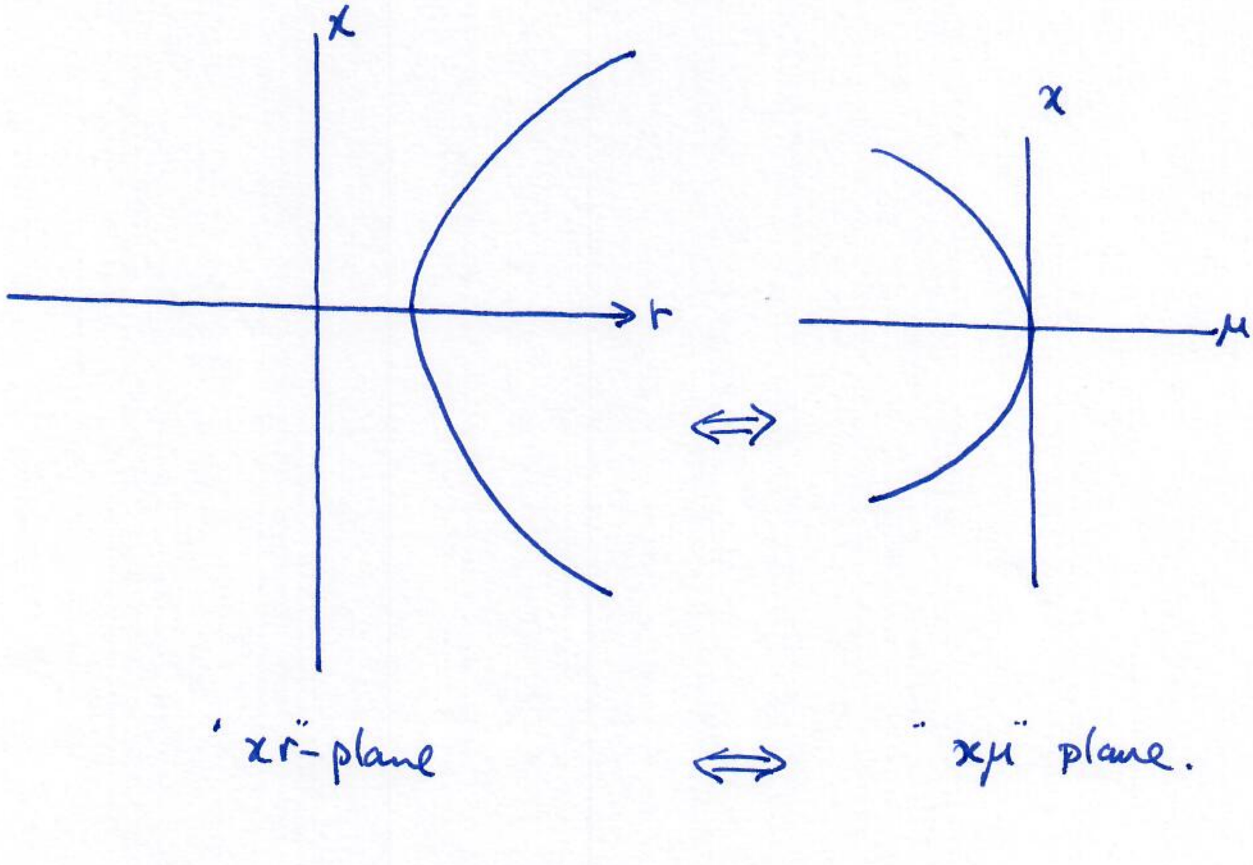
Compare with $\dot{x} = x - \dfrac{rx}{1+x^2}$

$\dot{x} = x(1-r) + rx^3 + O(|x|^5, |r|)$. Let $\mu = 1-r$, then

$$\dot{x} = x\mu + x^3 + O(|\mu|, |x|^3)$$

So a subcritical "pitchfork" in (x,μ) but a

<u>supercritical</u> in (x,r) at $x=0$, $\mu=0$ $(r=1)$.



"xr"-plane $\Longleftrightarrow$ "xμ" plane.