DYNAMICAL SYSTEMS

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Contents

1 Dynamical systems on the line $\mathbb{R}$

1.1 Introduction

These notes form an introduction to the qualitative theory of ordinary differential equations (ODEs). The simplest ODEs are equations of variables x, t, and the various derivatives of x with respect to t, e.g.

$$\left(\frac{dx}{dt}\right)^2 + x = \sin(t). \tag{1.1}$$

where $x(t)$ is a function of time t.

The *order* of an ODE is the order of the highest derivative appearing in the equation.

Using $\dot{x} = \frac{dx}{dt}$ and $\ddot{x} = \frac{d^2x}{dt^2}$, consider

(i) $a\ddot{x} + b\dot{x} + cx = 0, \quad a, b, c$ constants.

(ii) $\dot{x} = \sinh(x)$

(iii) $\dot{x}_1 = x_2$ and $\dot{x}_2 = -\frac{b}{a}x_2 - \frac{c}{a}x_1$.

Such equations are said to describe a *dynamical system*, as they determine how a variable x, or several variables $x, y\ldots$ change with time t. A system or equation that does not include any terms explicitly in the variable t other than as derivatives of x is called *autonomous* otherwise the system is said to be *non-autonomous*.

Note that (i) and (ii) are ODEs in a single *dependent* variable $x = x(t)$ which depends on the *independent* variable t, whereas (iii) is a system of two dependent variables $x_1 = x_1(t), x_2 = x_2(t)$ with independent variable t. Note that (i) and (iii) are different formulations of the same ODE. Why? Make the substitution $\dot{x}_1 = x_2$ which implies $\ddot{x}_1 = \dot{x}_2$ and substitute. This operation is reversible - each ODE can be derived from the other.

What are the orders of the ODEs in the examples? (i) is second order in one variable, (ii) is a first order equation in one (dependent) variable, and (iii) is a first order system of equations in two variables. First order equations in n-variables , $(x_1, \ldots, x_n) \in \mathbb{R}^n$, arise when each x_i, $1 \le i \le n$ is dependent on the independent variable *time* t,

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_n), \\ &\ldots, \\ \dot{x}_n &= f_n(x_1, \ldots, x_n). \end{aligned} \tag{1.2}$$

Introducing $\mathbf{x} = (x_1, \ldots, x_n)$, and the vector function

$$\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})) : \mathbb{R}^n \to \mathbb{R}^n$$

enables us to write the system of equations 1.2 as a single vector equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{1.3}$$

where the *vector field* $\mathbf{f}$ on $\mathbb{R}^n$ associates a vector $\mathbf{f}(\mathbf{x})$ to the point $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

These equations are describing how the variables $x_1, \ldots, x_n$ are evolving or changing with time, hence the idea of a **dynamical system**. Each function f_i, $i \in \{1, \ldots, n\}$ is assumed continuously differentiable with respect to each of the component variables x_j of $\mathbf{x}$.

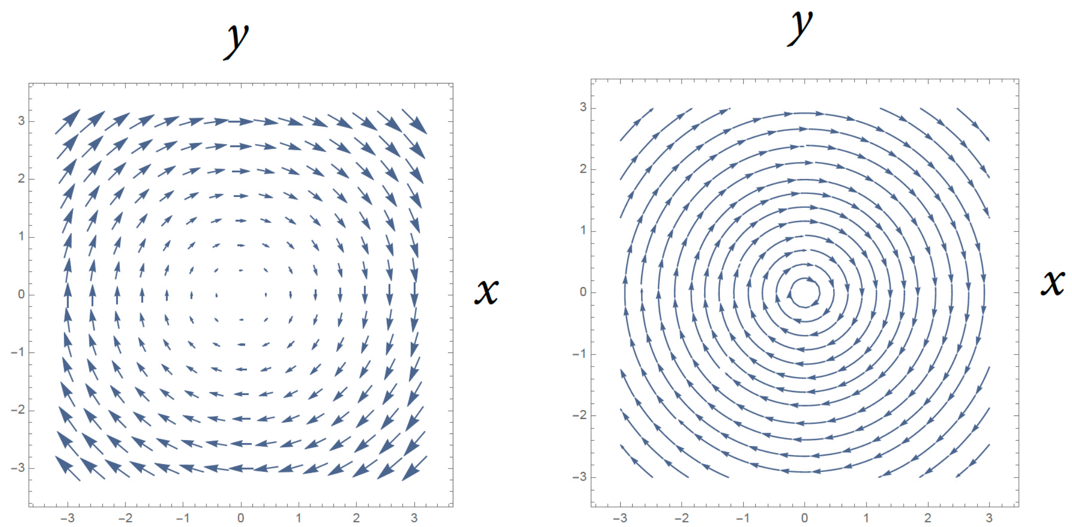


Figure 1: *An illustration of the vector field for equation 1.4 and the resulting solution curves of concentric circles centred on the fixed point $(x, y) = (0, 0)$ which form a circular flow in an clockwise direction.*

In the case of $n = 1$ for equation 1.3, the system reduces to a single *scalar* equation, $\dot{x}_1 = f_1(x_1)$, or simplifying notation, $\dot{x} = f(x)$, with $x \in \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$. The solutions are $x = x(t) \in \mathbb{R}$ such $\frac{d}{dt}(x(t)) \equiv f(x(t))$.

The case $n = 2$ for equation 1.3 has $\dot{x} = f(x, y), \quad \dot{y} = g(x, y)$ where $\mathbf{x} = (x_1, x_2) = (x, y)$ and $(f_1, f_2) = (f, g)$ (for ease of notation). This, similar to the case $n = 1$, gives rise to solutions $(x(t), y(t)) \in \mathbb{R}^2$, which are parametrised curves in the plane. The domain $\mathbb{R}^2$ is called the *phase space*. The collection of parameterised curves of the system of ODEs form a *phase portrait* when seen as a dynamical system. The *solution curves* are also often referred to as *trajectories*, or *orbits*.

Example 1.1. *Consider the system on* $\mathbb{R}^2$

$$\dot{x} = y \quad (= f(x, y)), \quad \dot{y} = -x \quad (= g(x, y)) \tag{1.4}$$

The vector field for this system is $\mathbf{f}(x, y) = (y, -x)$*. It can be shown that solutions take the form* $x(t) = x_0 \cos(t - t_0) + y_0 \sin(t - t_0)$ *and* $y(t) = -x_0 \sin(t - t_0) + y_0 \cos(t - t_0)$ *with the initial condition* $x(t_0) = x_0$ *and* $y(t_0) = y_0$ *and the trajectories are circles centred on the origin. See figure 1.*

1.2 Vector fields

In vector notation $\dot{\mathbf{x}} = (x_1, \ldots, x_n)$ provides the velocity at point $\mathbf{x} \in \mathbb{R}^n$ as

$$\dot{\mathbf{x}} = (\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n) = \mathbf{f}(\mathbf{x}). \tag{1.5}$$

This gives a distribution of vectors (a "field of vectors", or a "vector field") as we sample the various points $\mathbf{x}$ of the *phase space* $\mathbb{R}^n$. A representative sample selection of the trajectories which are tangent to the vectors form a *phase portrait*. The phase portrait should be sufficiently well populated with solution curves to get a clear idea of the likely behaviour for any initial condition given that there is an assumption of non-crossing of trajectories for differentiable choices of the vector field $\mathbf{f}$.

In the above example the solution curves(trajectories) which follow the vector field are circles centred on the origin. We now concentrate on vector fields on the line, before returning to systems on the plane later.

1.3 Quantitative and qualitative approaches to ODEs

We begin with the case $n = 1$. In studying differential equations we can essentially look for insight in two very different ways. One approach is to be quantitative - that is to use the calculus to find a solution for x as a function of t and then find the *particular* solution that satisfies what we call an **initial condition** where for a given value of $t = t_0$, we have $x = x_0$. This is somewhat limited as it is easy to give simple differential equations which either have no analytical solution, or solutions have very different long term behaviours, or a formulaic solution arises from the analysis which is no easier to understand than the original equation!

The alternative approach to investigating the precise functional form of the solutions is to consider the *qualitative* behaviour of the systems where one captures the *nature* of the solutions and not their precise formulaic dependence on the time t. For example, it may be useful to know whether $x = x(t)$, a trajectory of a system, is increasing or decreasing with time. That might be useful if we are considering a population of some kind. Moreover, is it increasing/decreasing in an uncontrolled way or is it moving towards an equilibrium of some kind or a periodic behaviour. These are the sort of issues that we consider in *qualitative dynamical systems*.

Definition 1.1. *A point $x^* \in \mathbb{R}$ is a **fixed point** of $\dot{x} = f(x)$, $f : \mathbb{R} \to \mathbb{R}$ if $f(x^*) = 0$. Note this means that $x(t) \equiv x^*$ for all $t \in \mathbb{R}$ is a solution curve (or trajectory, orbit).*

Fixed points are also called *equilibrium* points from the analogy of when a dynamical system represents the various motions of a mechanical system such as an oscillating pendulum. The pendulum is in stable equilibrium when hanging down or unstable equilibrium when pointing precariously upwards in an exact vertical position. The smallest perturbation would then cause the pendulum to swing back towards its stable equilibrium position and hang vertically downwards.

Example 1.2. *Consider the system on $\mathbb{R}$ given by*

$$\dot{x} = x(x-1)(x-2)^2, \tag{1.6}$$

where $x \in \mathbb{R}$. Deduce the phase portrait of the system on $\mathbb{R}$. Note the form of graph(f) where $f(x) = x(x-1)(x-2)^2$ in figure 2 (a). We see that the system has three fixed points at $x = 0, 1, 2$, the zeroes of f. Also $\dot{x}$ is positive on the intervals $x < 0$, $1 < x < 2$, and $2 < x$, and $\dot{x} < 0$ for $0 < x < 1$ and $1 < x < 2$. We deduce the phase portrait in figure 2(b).

Example 1.3. *Consider the quantitative and qualitative approach to understanding*

$$\dot{x} = \sin(x). \tag{1.7}$$

Quantitatvely

$$\frac{dx}{dt} = \sin(x) \implies \int \frac{dx}{\sin(x)} = \int \csc(x) dx = \int dt \tag{1.8}$$

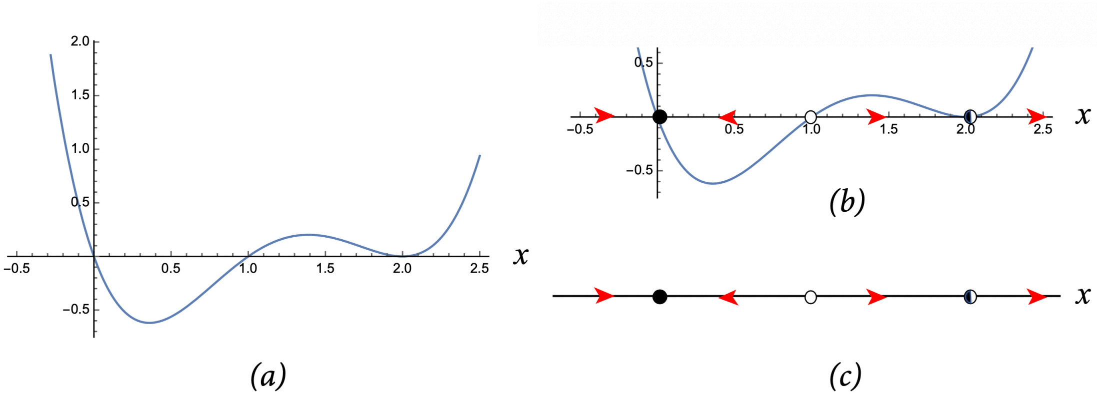


Figure 2: *(a) The graphical form of the function f in equation 1.6. (b) The graph of f with arrows depicting the direction of solution curves with increasing time t of $\dot{x} = f(x)$ on the x-axis. (c) the phase portrait of $\dot{x} = f(x)$.*

which implies

$$t = -\ln(\csc(x) + \cot(x)) + C, \tag{1.9}$$

where C is a constant .

If $x = x_0$ when $t = 0$

$$t = \ln\left|\frac{\csc(x_0) + \cot(x_0)}{\csc(x) + \cot(x)}\right| \tag{1.10}$$

So what does x look like as a function of t, and, even you managed to find it, what insight would it give you?

Note directly from the equation 1.7 we see that we have fixed points for $x^* = n\pi$, $n \in \mathbb{Z}$. Note that checking these fixed point solutions is not easy to do in 1.10! until some simplifying is carried out! For example,

$$\ln\left|\frac{\csc(x_0) + \cot(x_0)}{\csc(x) + \cot(x)}\right| = \ln\left|\frac{\cot(x_0/2)}{\cot(x/2)}\right| \tag{1.11}$$

which then enables the slightly better form of the solution as

$$\cot(\frac{x}{2}) = \cot(\frac{x_0}{2})e^t,$$

with initial condition $x = x_0$ for $t = 0$.

Note further for the system 1.7 that $\dot{x}$ is positive in the intervals $(2n\pi, (2n+1)\pi)$ and negative in the intervals $((2n+1)\pi, (2n+2)\pi)$, $n \in \mathbb{Z}$. With no further information this enables us in figure 3 to give rough sketches of typical solutions for a variety of starting conditions which conveys to the reader a confidence of the behaviour of the trajectories for any initial conditions $x = x_0$ when $t = t_0$.

Example 1.4. *Why is the qualitative behaviour of the system $\dot{x} = g(x)$ on the interval $[a, b] \subset \mathbb{R}$ where $g(x) = (x-a)(b-x)$ similar to that of $\dot{x} = \sin(x)$ on $[0, \pi]$?*

Example 1.5. *Consider $\dot{x} = ax$. This equation can be solved directly to give $x(t) = x_0 \exp a(t - t_0)$ where $x = x_0$ when $t = t_0$. We observe that $|x(t)|$ increases/decreases as t increases for $a > 0$, $a < 0$ respectively.*

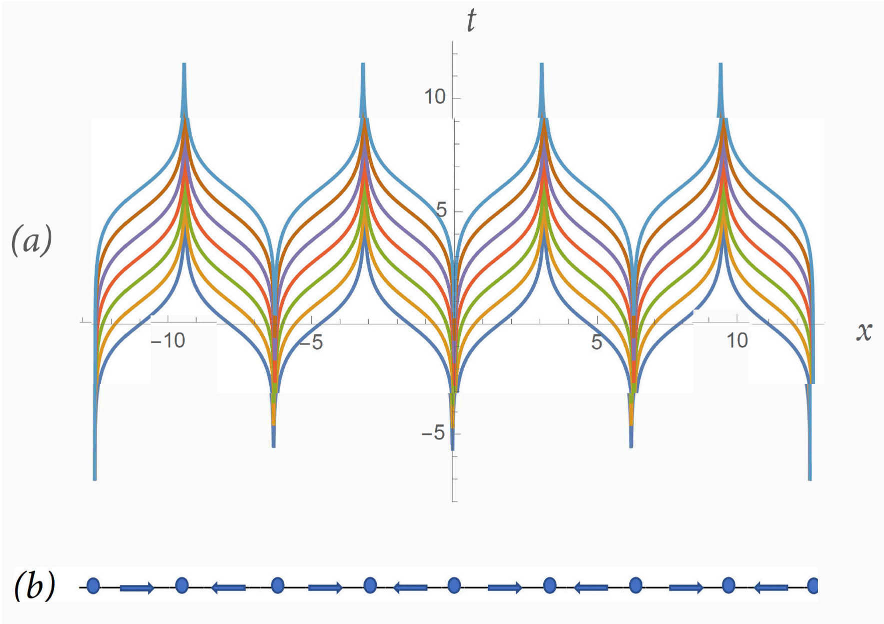


Figure 3: *(a) Qualitative behaviour of solution curves of $\dot{x} = \sin(x)$ in the $xt-$plane. Between the fixed point lines $x = m\pi$, $m \in \mathbb{Z}$, we have monotonic increasing solutions for $x \in (2n\pi, (2n+1)\pi)$ with t, and monotonic decreasing solutions for $x \in ((2n+1)\pi, (2n+2)\pi)$; (b) the corresponding phase portrait with stable fixed points at $x = 2n\pi$, alternating with unstable points at $x = (2n+1)\pi$.*

Thus the dynamical system $\dot{x} = ax$ has a single fixed point $x^* = 0$ which is unstable for $a > 0$ in the sense that solutions with initial condition close to $x^* = 0$ move away with increasing time, and it is stable for $a < 0$ as solutions move towards $x^* = 0$ with increasing time.

Example 1.6. *This is an example of modelling population growth. The logistic equation for population growth has two competing inputs. One is the exponential growth which gives the following dynamics which can occur with uninhibited food supply and is modelled by*

$$\dot{N} = rN \tag{1.12}$$

for $N \geq 0$ with growth rate r. If $N_0 > 0$ at $t = 0$ then N grows (exponentially) to infinity. This is unrealistic as growth requires food supply, and usually runs out of momentum as the supply diminishes. So a second contribution is introduced which inhibits growth, the logistic correction to give

$$\dot{N} = rN(1 - \frac{N}{k}) \tag{1.13}$$

with $r, k > 0$, cf. $\dot{x} = f(x)$.

1.4 Linear stability at a fixed point

Here we consider the *linear stability* at a fixed point of systems $\dot{x} = f(x)$ on the real line $\mathbb{R}$. This process can determine the behaviour of the dynamical system close to a fixed point. Note for 1.13 we have an unstable fixed point at $N = 0$, i.e. as t increases, the population close to $N = 0$ increases. Also note that for populations close to the fixed point $N = k$, we

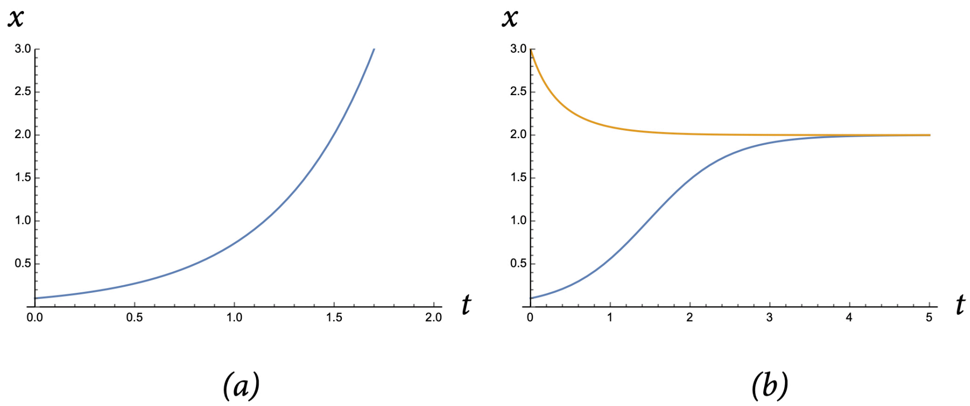


Figure 4: *The phase portraits of the dynamics of: (a) unrestricted population growth re. equation $\dot{x} = 2x$, cf. 1.12; (b) population growth with a logistic correction: for the system $\dot{x} = x(2 - x)$, all non-zero initial populations evolve with increasing time to the equilibrium population $x = 2$, (cf. equation 1.13 with $k = 2$).*

have the population moving towards that value, not away from it. For $N < k$, $\dot{N} > 0$, and for $N > k$, $\dot{N} < 0$.

Let $x = x^*$ be a fixed point of $\dot{x} = f(x)$ and therefore $f(x^*) = 0$. Expand the function $f(x)$ in a Taylor series at $x = x^*$ using $x(t) = x^* + \eta(t)$ to obtain

$$\dot{\eta}(t) = \frac{d}{dt}(x(t) - x^*) = \frac{dx(t)}{dt} = f(x(t)) = f(x^* + \eta(t)) \tag{1.14}$$

and

$$f(x^* + \eta(t)) = f(x^*) + \eta f'(x^*) + O(|\eta|^2) \tag{1.15}$$

We conclude that the linear approximation of the system $\dot{x} = f(x)$ at the fixed point $x = x^*$ is

$$\dot{\eta} = f'(x^*)\eta \tag{1.16}$$

or $\dot{\eta} = a\eta$ where $a = f'(x^*)$ which gives stability for $a < 0$ and instability for $a > 0$, cf. section 1.4.

Returning to the equation 1.13, we can now consider the equilibrium(fixed) points $N = 0$ and $N = k$. If we define $f(N) = rN(1 - \frac{N}{k})$, then $f'(N) = r(1 - 2N/k)$, and so $f'(0) = r > 0$, $f'(k) = -r$ thus illustrating linear instability at $N = 0$ and linear stability at $N = k$ as in the phase portrait in figure 4.

Example 1.7. *Consider the dynamical system $\dot{x} = f(x)$ where $f(x) = x^2$ on $\mathbb{R}$. The linear stability calculation at the fixed point $x = x^*$ gives $a = f'(x)|_0 = 0$. So we do not conclude stability or instability for the fixed point. Noting $\dot{x} > 0$ for $x \neq 0$ we see that we have a "stable/unstable" fixed point at $x^* = 0$. "stable" from the left (x negative) and "unstable" to the right (x positive).*

Definition 1.2. ***Stability at a fixed point***
*A fixed point $x = x^*$ of $\dot{x} = f(x)$ is **stable** if the orbit for any point sufficiently close to x^* remains close to x^*.*
*A stable point is said to be **asymptotically** stable if, in addition, the orbits of all points in a sufficiently small neighbourhood of $x = x^*$ converge to the the fixed point, within that neighbourhood.*

*A fixed point $x = x^*$ of $\dot{x} = f(x)$ is **unstable** if there exist points arbitrarily close to x^* whose orbits move far away in finite time.*
*A fixed point $x = x^*$ is **linearly stable** if $f'(x^*) < 0$.*
*A fixed point $x = x^*$ is **linearly unstable** if $f'(x^*) > 0$.*

The condition $f'(x^*) = 0$ says "further investigation is needed" of the stability behaviour of the fixed point. Graphing the function f locally at $x = x^*$ is usually helpful here.

1.5 Existence and uniqueness of solutions

Key questions are a) Does a solution to an ODE exist for any initial condition $x = x_0$, and $t = t_0$? - and, b), if so, is the solution unique?

Example 1.8. *This is an example of non-uniqueness. Consider $\dot{x} = f(x)$ with $f(x) = x^{\frac{1}{3}}$. The function $f : \mathbb{R} \to \mathbb{R}$, is well-defined and continuous in $\mathbb{R}$, but it is not differentiable at $x = 0$, a fixed point of the system. Note $x_1(t) \equiv 0$ for all $t \in \mathbb{R}$ is a solution by a direct check. Also, the function $x_2(t) = (\frac{2}{3}t)^{\frac{3}{2}}$ satisfies $\dot{x}_2 = (\frac{2}{3}t)^{\frac{1}{2}} = x_2(t)^{\frac{1}{3}}$ with the initial condition $x_2(0) = 0$. So we have two solutions $x_1(t)$ and $x_2(t)$, for all $t \in \mathbb{R}$, both of which have the same initial condition for $t = 0$. To avoid this problem, both existence and uniqueness of solutions is ensured by the following theorem.*

Theorem 1.1. ***(Existence and Uniqueness)*** *Let $f : X(\subset \mathbb{R}) \to \mathbb{R}$ be a differentiable function (vector field) on the open interval $X(\subset \mathbb{R})$. Then for every $x_0 \in X$, the equation $\dot{x} = f(x)$ has a solution $x(t)$, with $x(0) = x_0$, for $t \in I$, where the open interval I contains the point $t = 0$. Moreover, the solution is unique, i.e. any two solutions with the same initial condition coincide on a neighbourhood of $t = 0$.*

1.5.1 Blow-up of solutions

The theorem above guarantees existence and uniqueness of a solution for sufficiently small t. Do solutions exist for all t? Not necessarily, see below!

Example 1.9. *Let $f(x) = 1 + x^2$ and suppose $x = x_0 = 0$ for $t = 0$.*

$$\frac{dx}{dt} = 1 + x^2 \implies \int_{x_0}^{x} \frac{dx}{1 + x^2} = \int_0^t dt, \tag{1.17}$$

and so,

$$\arctan(x) - \arctan(x_0) = t - 0 \implies t = \arctan(x) \implies x = \tan(t). \tag{1.18}$$

Note $x(t) \to \infty$ as $t \to \frac{\pi}{2}$, and so the orbit reaches infinity in finite time!

2 Bifurcations of dynamical systems on $\mathbb{R}$

2.1 Introduction

Often, differential equations contain *parameters*, e.g. $\dot{x} = ax^2 - 1$, where a is a constant that can be chosen. As one might expect, choosing different values of a may give rise to different types of phase portrait. Finding the critical parameter values for where and how the phase portrait changes is the the study of *bifurcation* of systems. We investigate this in the following example.

Example 2.1. *Consider $\dot{x} = ax^2 - 1$ for different values of $a \in \mathbb{R}$.*

(i) let $a = -1$; then $\dot{x} = ax^2 - 1 = -x^2 - 1$. Now $\dot{x} < 0$ for all x, and so there are no fixed points.

(ii) let $a = 1$; then $\dot{x} = x^2 - 1$. This system has two fixed points at $x = \pm 1$, $x = +1$ is unstable, and $x = -1$ is stable, cf. figure 5. Note that the behaviour for all $a < 0$, ($a > 0$) is the

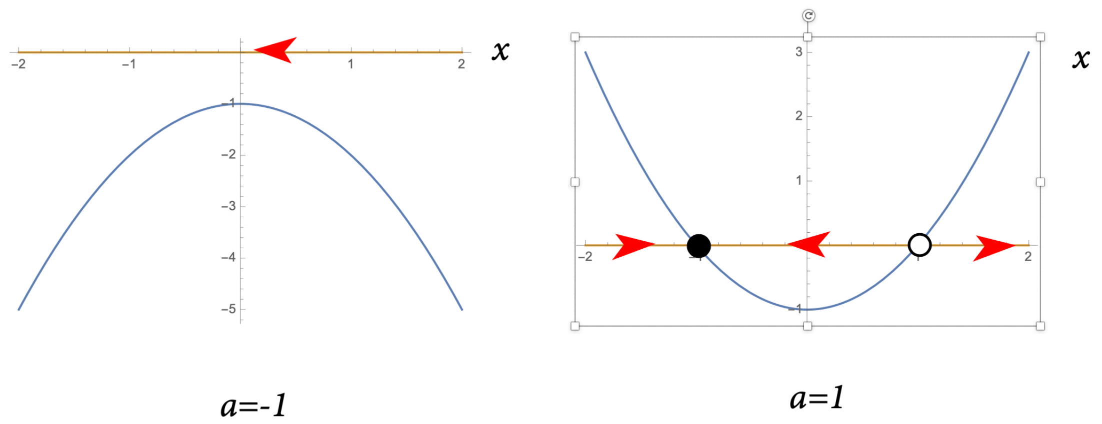


Figure 5: *Phase portraits of the system $\dot{x} = ax^2 - 1$ for $a = -1, a = 1$. We see that by changing the parameter from $a = -1$ to $a = 1$ provides phase portraits which are are qualitatively different. We say that this family of systems, as $a \in \mathbb{R}$ is varied, exhibits various qualitative behaviours.*

same respectively as for (i) (and (ii)). However, there is a change in the structure (a bifurcation) of the phase portrait at $a = 0$.

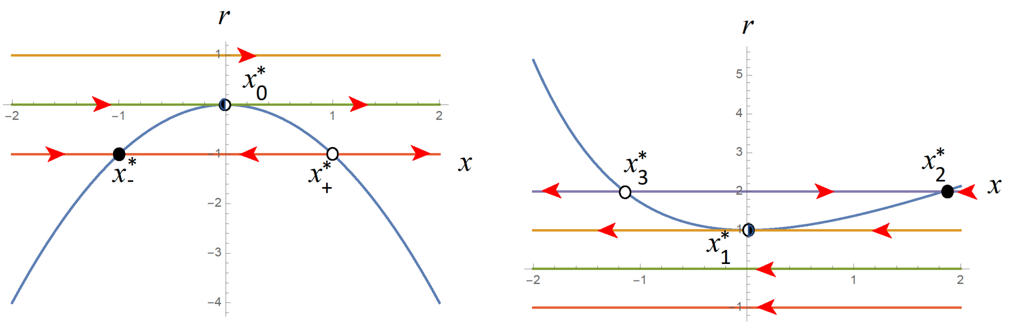
Bifurcations can be classified according to how many parameters are needed for them to occur "stably" as a family of phase portraits. The precise interpretation of this property is not addressed in this module, but we now discuss the bifurcations which occur stably in systems on the line with just a single parameter.

2.2 Saddle-node bifurcation

The saddle-node bifurcation involves the creation or destruction of a pair of fixed points as a parameter is varied. The proto-typical example is given by the 1-parameter family of vector fields on $\mathbb{R}$ given by

$$\dot{x} = r + x^2, \tag{2.1}$$

where $r \in \mathbb{R}$.



(a) (b)

Figure 6: *(a) Phase portraits of the vector field 2.1 for parameter choices* $r < 0$, $r = 0$, $r > 0$. *The diagram shows for equation 2.1 that as r increases from negative values through zero, two fixed points (x_-^* stable; x_+^* unstable) come together to form a single stable-unstable fixed point x_0^* for* $r = 0$*, which then vanishes for all* $r > 0$ *when the phase portraits have no fixed points. (b) Phase portraits of the vector field 2.4. This is a saddle-node bifurcation where a fixed point emerges for* $r = 1$*, and which then splits to form two fixed points x_2^*(stable) and x_3^* (unstable) as r increases beyond* $r = 1$.

In general, the bifurcation points of $\dot{x} = f(x,r) = f_r(x)$ occur when there are changes of stability for the the set of fixed points, i.e. $f(x,r) = 0$, and the potential local change of stability of the fixed point, i.e. $f_r'(x) = 0$. These conditions have to be considered in tandem.

2.2.1 Bifurcation diagrams

A way to exhibit the changing behaviour of a dynamical system with the variation of its parameter is to exhibit a bifurcation diagram.

Example 2.2. *Consider*

$$\dot{x} = r + x^2 = f_r(x). \tag{2.2}$$

A bifurcation diagram for this system is a graph in the rx-plane representing the varying pattern of fixed points of the equation 2.2 and their stabilities. The phase portrait for the evolution of x can be depicted by considering particular, but representative, choices of the parameter r.

First of all we investigate with linear stability analysis. The fixed points occur when $f_r(x) = 0$*, i.e. for* $x = \pm\sqrt{(-r)}$*. Therefore, we have*

(i) no solutions for $r > 0$*;*

(ii) one solution, $x^* = 0$*, for* $r = 0$*;*

(iii) two solutions, $x_+^* = +\sqrt{(-r)}$ *and* $x_-^* = -\sqrt{(-r)}$*, for* $r < 0$.

Linear stability calculations give

$$\begin{aligned} f'(x_+^*) &= 2x|_{x=x_+^*} > 0 - \text{unstable} \\ f'(x_-^*) &= 2x|_{x=x_-^*} < 0 - \text{stable} \end{aligned} \tag{2.3}$$

For the case $r = 0$, compare with example 1.7. The flow of system 2.2 with increasing time can now be completed (see figure 6) reference to the stability properties of the fixed points and the functional form of the vector field f. Note r has no dynamic attached to it, so effectively, the variable can be interpreted as $\dot{r} = 0$.

Example 2.3. *Consider*

$$\dot{x} = r - x - e^{-x}, \ r \in \mathbb{R}. \tag{2.4}$$

Fixed points arise where $r - x - e^{-x} = 0$ as r varies. Drawing the graphs $y = r - x$ and $y = e^{-x}$ and checking where they interest will exhibit the fixed points of the dynamical systems.

The intersection where fixed points appear to vanish will occur at a common tangency of the two graphs, i.e. the curves will touch when $r - x = e^{-x}$ and the common tangency occurs when $-1 = -e^{-x}$ which implies $x = 0$ and, therefore, $r = 1$. Further investigation gives

(i) $r < 1$ *- no intersections - no fixed points*

(ii) $r = 1$ *- 1 intersection - one fixed point $x_1^* = 0$*

(iii) $r > 1$ *- 2 intersections - two fixed points x_2^* and x_3^*.*

To find approximate values for x_2^*, x_3^*, we consider a Taylor expansion in powers of x about the origin to obtain :

$$f_r(x) = r - x - e^{-x} \tag{2.5}$$

$$= r - x - (1 - x + \frac{x^2}{2!} - \dots) \tag{2.6}$$

$$= (r-1) - \frac{x^2}{2!} + O(x^3) \tag{2.7}$$

So $x_2^* \approx +\sqrt{2(r-1)}$ and $x_3^* \approx -\sqrt{2(r-1)}$. Also $f_r'(x) = -x + O(x^2)$. Therefore x_2^* is a stable fixed point and x_3^* is unstable.

2.2.2 Saddle-node normal form

For this section, we regard a one-parameter family of functions as a function of two variables $f : (x,r) \to f(x,r)$. Assume that f is sufficiently differentiable in both variables. Assume $\dot{x} = f(x,r)$ has a fixed point $(x,r) = (x^*, r_c)$, and assume that it is a double root in x, i.e.

$$f(x^*, r_c) = 0 \text{ and } \frac{\partial f}{\partial x}(x^*, r_c) = 0. \tag{2.8}$$

Expanding f in a double power series in the variable x and r gives

$$\begin{aligned} f(x,r) = {} & f(x^*, r_c) + (x - x^*)\frac{\partial f}{\partial x}(x^*, r_c) + (r - r_c)\frac{\partial f}{\partial r}(x^*, r_c) + \\ & + \frac{1}{2!}(x - x^*)^2 \frac{\partial^2 f}{\partial x^2}(x^*, r_c) + (x - x^*)(r - r_c)\frac{\partial^2 f}{\partial x \partial r}(x^*, r_c) \\ & + O(|r - r_c|^2, |x - x^*|^3) \end{aligned} \tag{2.9}$$

If we label key coefficients in the equation 2.9 as

$$A = \frac{\partial f}{\partial r}(x^*, r_c); \quad B = \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(x^*, r_c); \quad C = \frac{\partial^2 f}{\partial x \partial r}(x^*, r_c). \tag{2.10}$$

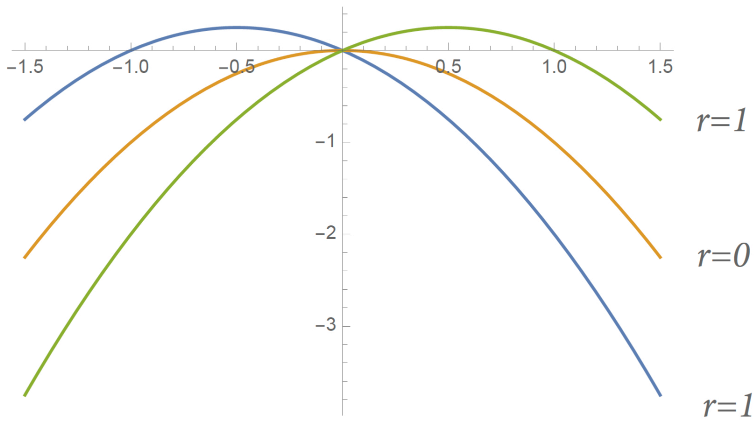


Figure 7: *The graphs of* $f(x) = rx - x^2$ *for* $r < 0 (r = -1)$, $r = 0$, $r > 0 (r = 1)$.

Shifting the origin of the (x, r) plane to (x^*, r_c), (by using local coordinates $u = x - x^*$ and $\mu = r - r_c$) and using 2.8 we obtain

$$\dot{u} = A\mu + Bu^2 + Cu\mu + O(|\mu^2|, |u^3|) \tag{2.11}$$

If $A, B \neq 0$, then the change of coordinates $v = u + \frac{C\mu}{2B}$ and parameter change $\eta = \mu - \frac{C^2\mu^2}{4AB}$ (both invertible) give

$$\dot{v} = A\mu + Bv^2 + O(|\mu^2|, |v^3|) \tag{2.12}$$

the normal form for a *saddle-node* bifurcation. The condition that $A \neq 0$ is called the saddle-node *transversality condition*.

2.3 Transcritical bifurcation

For this bifurcation, as a parameter is varied, two fixed points collide exchanging stabilities.

Example 2.4. *A proto-typical example of a transcritical bifurcation is given by*

$$\dot{x} = rx - x^2 \tag{2.13}$$

where $r \in \mathbb{R}$. *Fixed points are* $x = x_1^* = 0$ *and* $x = x_2^* = r$, *for all* $r \in \mathbb{R}$. *The stability of a fixed point is given by evaluating the derivative of* $f_r(x) = rx - x^2$, *i.e.* $f_r'(x) = r - 2x$.

For $x = x_1^*$, $f_r'(x_1^*) = r$, *and, for* $x = x_2^*$, $f_r'(x_2^*) = r - 2r = -r$.

Therefore for $r < 0$, x_1^* *is stable, and* x_2^* *is unstable, whereas, for* $r > 0$, x_1^* *is unstable, and* x_2^* *is stable; stabilities are exchanged. The fixed point for* $r = 0$ *at* $x = 0$ *is a saddle-node type. Note A=0, B=-1,C=1 in this example.*

Example 2.5. *Consider*

$$\dot{x} = x(1 - x^2) - \alpha(1 - e^{-\beta x}) = f_{(\alpha,\beta)}(x) \tag{2.14}$$

where $\alpha, \beta \in \mathbb{R}$ *are parameters. We represent them as a point* $(\alpha, \beta) \in \mathbb{R}^2$. *Note that* $x = 0$ *implies* $f_{(\alpha,\beta)}(0) = 0$ *and so* $x_1^* = 0$ *is a fixed point for all* $(\alpha, \beta) \in \mathbb{R}^2$.

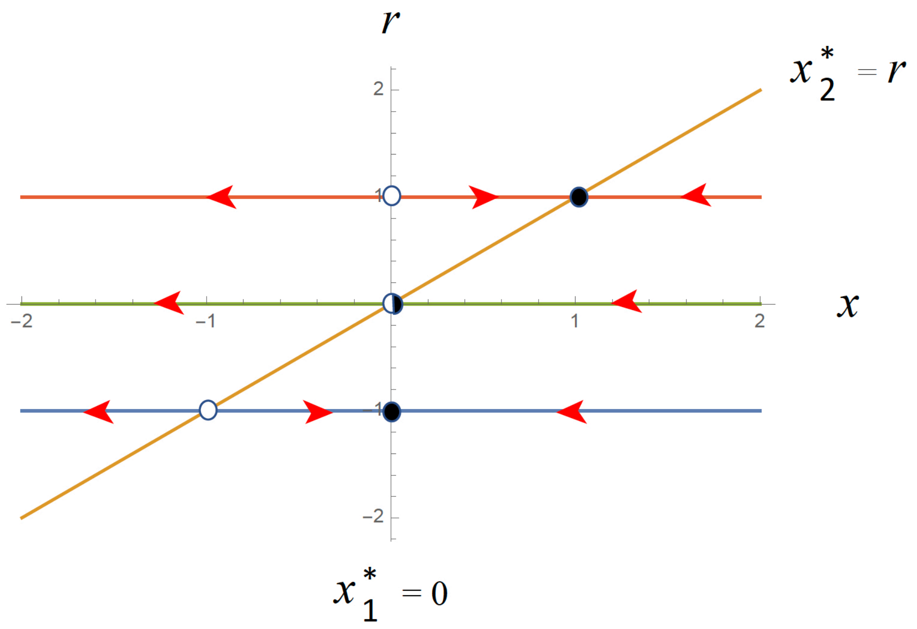


Figure 8: *The two families of fixed points $x_1^* \equiv 0$ and $x_2^* = r$ for the vectorfield 2.13 exchange stability roles at $r = 0$*

We will show that the system undergoes a transcritical bifurcation as the real parameter $\mu = ab$ increases through $\mu = 1$. We require that the vector field have a double root at the bifurcation point. Note

$$\frac{df_{(\alpha,\beta)}(x)}{dx} = 1 - 3x^2 - \alpha\beta e^{-\beta x} \implies \frac{df_{(\alpha,\beta)}(x)}{dx}|x_1^* = 1 - \alpha\beta = 1 - \mu. \tag{2.15}$$

Thus the stability of $x = x_1^*$ switches at $\mu = 1$.

To find approximate values of the fixed points we expand

$$\begin{aligned} f_{(\alpha,\beta)}(x) =& x - \alpha(1 - (1 - \beta x + \frac{1}{2}\beta^2 x^2 + O(x^3))) \\ =& (1 - \alpha\beta)x + \frac{1}{2}\alpha\beta^2 x^2 + O(x^3) \end{aligned} \tag{2.16}$$

We obtain $x_1^* = 0$ (as before) and $x_2^* \approx -2(1 - \alpha\beta)/\alpha\beta^2$, provided $\alpha\beta \neq 0$. Note that at $\mu = \alpha\beta = 1$, $x_1^* = x_2^* = 0$ when the stability of x_1^* changes. We can now check that the stability of x_2^* simultaneously changes at $\mu = 1$ opposite to the stability of x_1^*, as required for a transcritical bifurcation using

$$\frac{df_{(\alpha,\beta)}(x)}{dx} = 1 - \alpha\beta + 2\frac{1}{2}\alpha\beta^2 x + O(x^2) \tag{2.17}$$

gives

x_1^*: $\frac{df_{(\alpha,\beta)}}{dx}(x_1^*) = 1 - \alpha\beta = 1 - \mu$
We obtain: $\mu > 1$, stable; $\mu < 1$, unstable.

x_2^*: $\frac{df_{(\alpha,\beta)}}{dx}(x_2^*) = -(1-\alpha\beta) = -(1-\mu)$
We obtain $\mu > 1$, unstable; $\mu < 1$, stable.

Example 2.6. *Consider the one parameter system*

$$\dot{x} = r\ln(x) + x - 1 = f_r(x). \tag{2.18}$$

where $r \in \mathbb{R}$. We note that $f_r(1) = 0$, and so $x = x_1^* = 1$ is a fixed point for all r. Also

$$\frac{df_r(x)}{dx}|_{x_1^*} = \left(\frac{r}{x} + 1\right)|_{x_1^*} = r + 1, \tag{2.19}$$

.

Therefore we have a double root of f_r for $x = x_1^* = 1$ and $r_c = -1$. Note that the transversality condition for a saddle-node bifurcation fails here, i.e. $a = \frac{df_r(x)}{dr}|_{(x_1^*,r_c)} = 0$, (recall for a saddle-node bifurcation we required $a \neq 0$). However,

$$\frac{\partial^2 f_r(x)}{\partial x^2}|_{(x_1^*,r_c)} = \frac{-r}{x^2}|_{(x_1^*,r_c)=(1,-1)} = 1 \neq 0 \quad (i.e.\ b = 1). \tag{2.20}$$

With $a = 0$ in equation 2.10, we now require $c \neq 0$ - it is the next lowest degree term in the expansion involving the parameter. Checking the value of c, we see that

$$\frac{\partial^2 f_r(x)}{\partial x \partial r}|_{(x_1^*,r_c)} = \frac{1}{x}|_{(x_1^*,r_c)=(1,-1)} = 1 \neq 0 \quad (i.e.\ c = 1). \tag{2.21}$$

Note the two fixed points are $x_1^* \equiv 1$ and the other comes from further consideration of

$$0 = r\ln(x) + x - 1 \tag{2.22}$$

Introduce a local coordinate $y = x - 1$ which implies we consider

$$\begin{aligned} 0 =& r\ln(1+y) + y = r(y + \frac{y^2}{2}) + y + O(|y^3|) \\ =& (r+1)y + r\frac{y^2}{2} + O(|y^3|) \end{aligned} \tag{2.23}$$

and we obtain in local coordinates $y_1^* = 0$ and $y_2^* \approx \frac{-2(r+1)}{r}$

To obtain the normal form for a transcritical bifurcation from equation 2.23, we introduce new coordinates $y = kz$, with k a constant to be determined, and $\mu = r + 1$. Then,

$$\begin{aligned} \dot{y} =& r\ln(1+y) + y \\ \implies \quad k\dot{z} =& (\mu - 1)\ln(1+kz) + kz \\ =& \mu kz + (\mu - 1)\frac{k^2 z^2}{2} + O(|z|^3) \end{aligned} \tag{2.24}$$

We obtain the normal form

$$\dot{z} = \mu z - z^2 + O(|z|^3) \tag{2.25}$$

if we choose $k = -2/(\mu - 1)$.

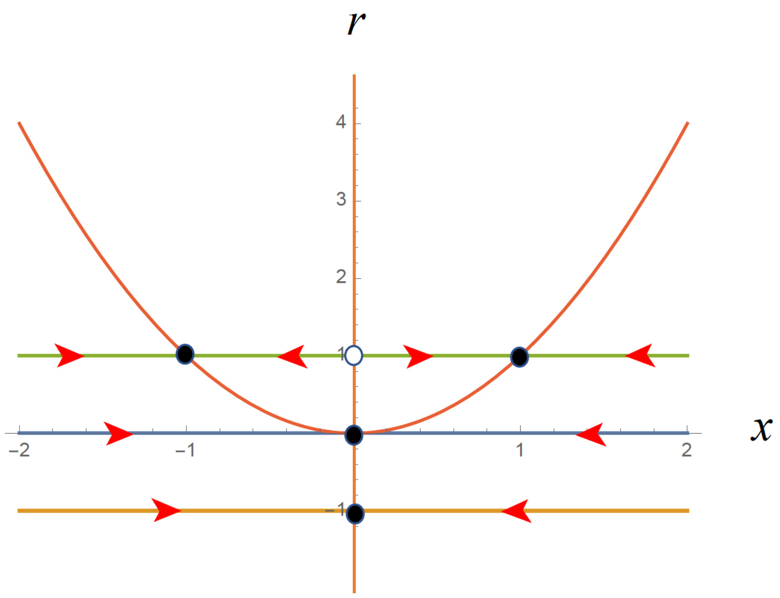


Figure 9: *The bifurcation diagram exhibits a one-parameter family given by equation 2.27, and samples of the phase portraits for the qualitatively different vector fields for $r < 0$, $r = 0$, and $r > 0$ including the fixed points and their stabilities.*

2.4 Pitchfork bifurcation

This is a bifurcation which often occurs in *symmetric* or *equivariant* vector fields where the system $\dot{x} = f_r(x)$ satisfies

$$f_r(x) = -f_r(-x). \tag{2.26}$$

Then, necessarily $x^* = 0$ is a fixed point for all r as $f_r(0) = 0 \ \forall r \in \mathbb{R}$. A saddle-node bifurcation cannot occur with this type of fixed point, as a necessary consequence of the symmetry gives $\frac{\partial^2 f_r}{\partial x^2}(x)|_{x=0} \equiv 0$. Note also that $\frac{\partial f_r}{\partial r}|_{x=0}0 \equiv 0$. So we have no linear term in r and no quadratic terms in x. So we impose the next degeneracy condition $\frac{\partial^3 f_r}{\partial x^3}(x)|_0 \neq 0$.

The normal forms we obtain are

$$\dot{x} = rx - x^3 - -\textit{supercritical pitchfork} \text{ with } \frac{\partial^3 f_r}{\partial x^3}(x)|_0 < 0. \tag{2.27}$$

$$\dot{x} = rx + x^3 - -\textit{subcritical pitchfork} \text{ with } \frac{\partial^3 f_r}{\partial x^3}(x)|_0 > 0. \tag{2.28}$$

The bifurcation diagram resembles a "pitchfork" shape with a fixed point $x = 0$ for all r, and two further fixed points growing from $x = 0$ as $x = \pm\sqrt{(r)}$ as r increases for $r > 0$ (i.e. after the bifurcation) in the *supercritical* case; $x = \pm\sqrt{(-r)}$ in the *subcritical case* (i.e. before the bifurcation for $r < 0$.).

Example 2.7. *Consider the 1-parameter system $\dot{x} = -x + \beta\tanh(x) = f_\beta(x)$. Note $f_\beta(0) = 0$, for all β. Also*

$$\frac{\partial f_\beta}{\partial x}(x)|_{x=0} = (-1 + \beta/\cosh^2(x))|_{x=0} = -1 + \beta. \tag{2.29}$$

So $\beta = 1$ is a bifurcation point at which the stability of the fixed point $x_0^* = 0$ changes.

We have

$$\frac{\partial^3 f_\beta}{\partial x^3}(x)|_{(x,\beta)=(0,1)} = 2\beta \frac{2\cosh^2(x) - 3}{\cosh^4(x)}|_{(x,\beta)=(0,1)} = -2.$$

2.5 Normal forms for bifurcations

In this section, we will interpret the vector field of a one-parameter family of vector fields

$$\dot{x} = f_r(x) \tag{2.30}$$

as a function of two variables, i.e. $f_r(x) = f(x,r)$, as in equation 2.9. This is for ease of notation. Let us assume that f is as differentiable in both variables as many times as we wish (note a differentiable function is continuous).

2.5.1 Non-bifurcation condition

Let $x = x^*$ for $r = r_c$ be a fixed point of $\dot{x} = f_r(x)$, i.e. $f(x^*, r_c) = 0$. Suppose that

$$\frac{\partial f}{\partial x}(x^*, r_c) \neq 0. \tag{2.31}$$

This condition ensures by the *Implicit Function Theorem* that there exists a differentiable function $x = x(r)$ for r in an open neighbourhood of $r = r_c$ with $x(r_c) = x^*$ which satisfies

$$f(x(r), r) \equiv 0. \tag{2.32}$$

In other words, there is a line of fixed points $x = x(r)$, for an open interval of $\mathbb{R}$ containing r_c. Note the equation 2.31 implies that $\frac{\partial f}{\partial x}(x(r), r) \neq 0$ by continuity of f for sufficiently small $r - r_c$. Thus there is no change in the stability of the fixed point $x = x(r)$ for $r - r_c$ small, and it follows **that there is no bifurcation of the fixed point $x = x^*$ at $r = r_c$** if condition 2.31 holds.

Therefore, for bifurcations to occur, it is necessary that

$$f(x,r) = 0, \ \frac{\partial f}{\partial x}(x,r) = 0. \tag{2.33}$$

With these conditions the Taylor expansion of $f(x,r)$ at a fixed point (x^*, r_c) of $\dot{x} = f(x)$ becomes

$$\begin{aligned} f(x,r) =& A(r - r_c) + B(x - x^*)^2 + C(x - x^*)(r - r_c) \\ &+ D(r - r_c)^2 + E(x - x^*)^3 + \ldots \end{aligned} \tag{2.34}$$

with

$$\begin{aligned} A =& \frac{\partial f}{\partial r}(x^*, r_c);\ B = \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(x^*, r_c);\ C = \frac{\partial^2 f}{\partial x \partial r}(x^*, r_c); \\ D =& \frac{1}{2!}\frac{\partial^2 f}{\partial r^2}(x^*, r_c);\ E = \frac{1}{3!}\frac{\partial^3 f}{\partial x^3}(x^*, r_c). \end{aligned} \tag{2.35}$$

Introduce local coordinates and parameters $y = x - x^*$ and $\mu = r - r_c$ so that equation 2.34 becomes

$$f(x,r) = A\mu + By^2 + Cy\mu + D\mu^2 + Ey^3 + \ldots \tag{2.36}$$

with added terms (coefficients D and E).

2.5.2 Normal form reduction

We now refer to equation 2.36 for this section.

(a) **S**addle-node - $A \neq 0; B \neq 0$.

The constraint $A \neq 0$ is the *transversality condition*, and the constraint $B \neq 0$ is the *non-degeneracy condition*.

We have $\dot{y} = A\mu + By^2 + Cy\mu + D\mu^2 + \ldots$. If $C \neq 0$, then the transformation $z = y + \frac{C\mu}{2B}$ - completing the square - gives

$$\dot{z} = A\nu + Bz^2 + O(|\nu|^2, |z|^3)$$

if $\nu = \mu + \mu^2(D - \frac{C^2}{4B})/A$. This gives a saddle-node bifurcation.

(b) **T**ranscritical - $A = 0; B \neq 0, C \neq 0$.

The constraint $C \neq 0$ is the *transversality condition*, and the constraint $B \neq 0$ is the *non-degeneracy condition*.

We have $\dot{y} = By^2 + Cy\mu + O(|\mu^2|, |y^3|)$ and this gives a transcritical bifurcation..

(c) **P**itchfork - $A = 0; B = 0, C \neq 0, E \neq 0$.

Here $E = \frac{\partial^3 f}{\partial x^3}(x^*, r_c)$ and we have $\dot{y} = Cy\mu + Ey^3 + O(|\mu^2|, |y^4|)$.

For an appropriate constant k there are two other solutions $y \approx \pm\sqrt{\frac{-C\mu}{E}}(1 + k\sqrt{\mu})$, depending on whether the real $\sqrt{(\frac{-C\mu}{E})}$ exists for $\mu > 0$ or $\mu < 0$, which gives, respectively, a supercritical or subcritical pitchfork bifurcation.

The constraint $C \neq 0$ is the *transversality condition*, and the constraint $E \neq 0$ is the *non-degeneracy condition*.

Example 2.8. *Investigate the 1-parameter system*

$$\dot{x} = f_r(x) = x^5 - x^3 - rx. \tag{2.37}$$

It can be verified that the system has simultaneous saddle-node bifurcations at $r = -\frac{1}{4}$ and a subcritical pitchfork bifurcation at $r = 0$. The system has 1 fixed point for $r < -\frac{1}{4}$; 3 fixed points for $r = -\frac{1}{4}$, 5 fixed points for $-\frac{1}{4} < r < 0$; and 3 fixed points for $r \geq 0$. See figure 10 for some graphical information.

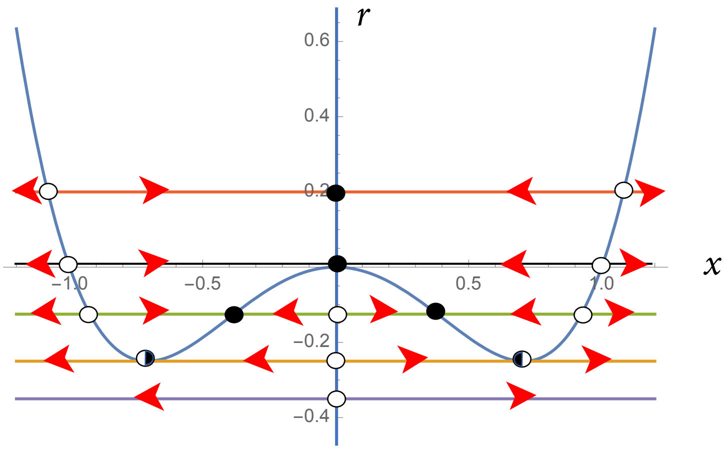


Figure 10: *The set of fixed points in the xr−plane of the vector field* $\dot{x} = f_r(x)$ *in equation 2.37, for* $r \in \mathbb{R}$*. It can be shown that saddle-node bifurcations occur at* $r = -\frac{1}{4}$ *and a sub-critical pitchfork bifurcation occurs at* $r = 0$.

3 Dynamical systems on the circle $\mathbb{S}^1$

3.1 Vector fields and ODEs on the circle $\mathbb{S}^1$

We denote the unit circle by $\mathbb{S}^1$ or $\mathbb{S}$ (cf. $\mathbb{R}^1$ or $\mathbb{R}$). To define an ODE on $\mathbb{S}^1$ requires an angular coordinate to ensure that the vector field has built-in periodicity of 2π radians. We define the angular coordinate $\theta \in \mathbb{S}^1$ by the projection map $p : \mathbb{R} \to \mathbb{S}$ where $p(x) = x(\,mod\, 2\pi)$ and we use the notation $p(x) = \theta$. We now consider vector fields on $\mathbb{R}$ with the following properties:

(i) $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable;

(ii) $f(x + 2k\pi) = f(x)$, for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$.

Note that (i) implies $\frac{df}{dx}(x) = \frac{df}{dx}(x + 2k\pi)$ for all $x \in \mathbb{R}$, and f induces a continuously differentiable map $g : \mathbb{S}^1 \to \mathbb{R}^1$ on the circle to itself by defining

$$g(\theta) = g(p(x)) \stackrel{\text{def}}{=} (f(x)).$$

The function g is well-defined on $\mathbb{S}^1$ given the properties of f. For any $k \in \mathbb{Z}$, the trigonometric functions $\cos(kx)$ and $\sin(kx)$ are automatically periodic, of period 2π, and so the more general periodic construct, the *Fourier* series,

$$f(x) = \sum_{k \in I} a_k \cos(kx) + b_k \sin(kx),$$

where $I \subset \mathbb{Z}$, $a_k, b_k \in \mathbb{R}$, also has periodicity 2π.

Example 3.1. *Consider a constant circular flow (or uniform rotation) on the circle given by the ODE*

$$\dot{\theta} = \omega, \quad \text{which has solution} \quad \theta(t) = \omega t + \theta_0 \;(\,mod\, 2\pi) \tag{3.1}$$

given $\theta = \theta_0$ when $t = 0$. We see that

$$\theta(t) = wt + \theta_0 \equiv w(t + \frac{2\pi}{\omega}) + \theta_0 = \theta(t + \frac{2\pi}{\omega}), \tag{3.2}$$

for all $t \in \mathbb{R}$. The orbit is periodic on $\mathbb{S}^1$ with period $\frac{2\pi}{\omega}$.

Example 3.2. *Now consider the non-uniform rotation given by*

$$\dot{\theta} = \omega - a\sin(\theta), \tag{3.3}$$

with $\omega, a \in \mathbb{R}$ and $\omega > 0$. The fixed points of the flow occur for $\dot{\theta} = 0$ and are given by $\theta = \theta^*$ where $\sin(\theta^*) = \frac{\omega}{a}$, and hence fixed points only occur when $\omega \leq |a|$. Also $\frac{df}{d\theta}(\theta) = -a\cos(\theta)$ and $\cos(\theta^*) = \pm\sqrt{(1 - \frac{\omega^2}{a^2})}$. Hence $\theta = \theta_1 =$ is a stable fixed point, and $\theta = \theta_2$ is an unstable fixed point. The oscillation period for $\omega > |a|$ is given by the following calculation. Let the oscillation have period T, and recall that θ is a periodic coordinate of period 2π. Therefore

$$\int_0^T dt = \int_{-\pi}^{\pi} \frac{d\theta}{\omega - a\sin(\theta)}$$

Let $u = \tan(\frac{\theta}{2})$, then

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1}{\sec^2\left(\frac{\theta}{2}\right)} = \frac{1}{1 + \tan^2\left(\frac{\theta}{2}\right)} = \frac{1}{1 + u^2}.$$

Also

$$\sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = 2\tan\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right) = \frac{2u}{1 + u^2}.$$

Furthermore,

$$du = \frac{1}{2\cos^2\left(\frac{\theta}{2}\right)} d\theta = \frac{1 + u^2}{2} d\theta,$$

and we conclude

$$T = 2\int_{-\infty}^{\infty} \frac{du}{\omega u^2 - 2au + w} = \frac{2}{\sqrt{(\omega^2 - a^2)}} \arctan\left(\frac{\omega u - a}{\sqrt{(\omega^2 - a^2)}}\right)\Big|_{-\infty}^{\infty} = \frac{2\pi}{\sqrt{(\omega^2 - a^2)}} \tag{3.4}$$

We then see that as $a \to \omega^-$, we have $T \to \infty$, and as $a \to 0$, we have $T \to \frac{2\pi}{\omega}$ which agrees with the uniform rotation period obtained in the previous example.

3.2 Stability definitions refined

The possible dynamical behaviour on the circle means that we need to be more precise about stability definitions.

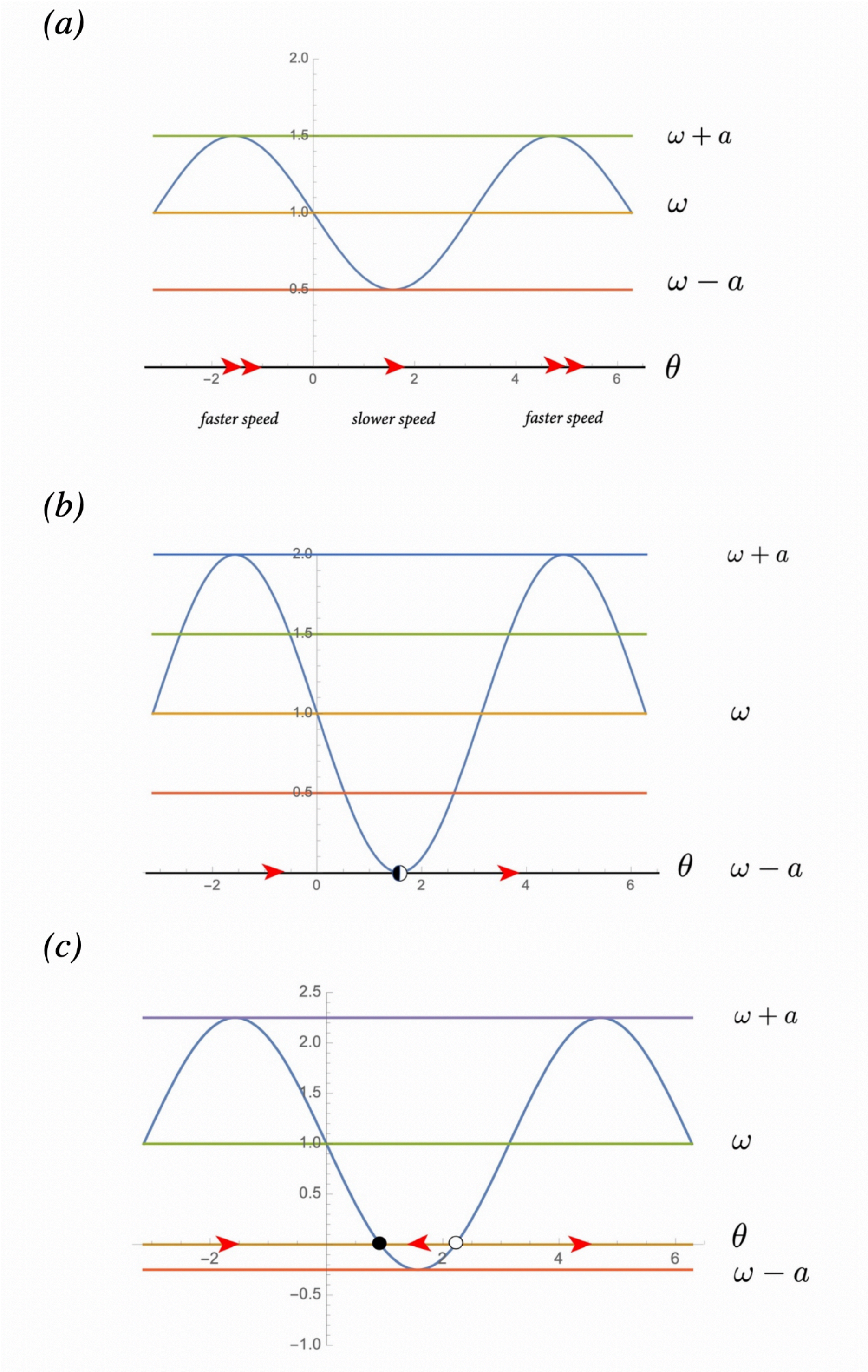


Figure 11: *Examples of the phase portraits the vector field 3.3 of the circle. In (a), representing $|\omega/a| > 1$, there are no fixed points but the rotation rate $\dot{\theta}$ is at its slowest(a bottle-neck) when $\theta = \frac{\pi}{2}$ and is at its fastest for $\theta = \frac{3\pi}{2}$; in (b), for $\frac{\omega}{a} = 1$, there is a single stable/unstable fixed point at $\theta = \frac{\pi}{2}$; in (c), two fixed points occur(one stable and one unstable) close to $\theta = \frac{\pi}{2}$ when $\omega/a \approx 1^-$.*

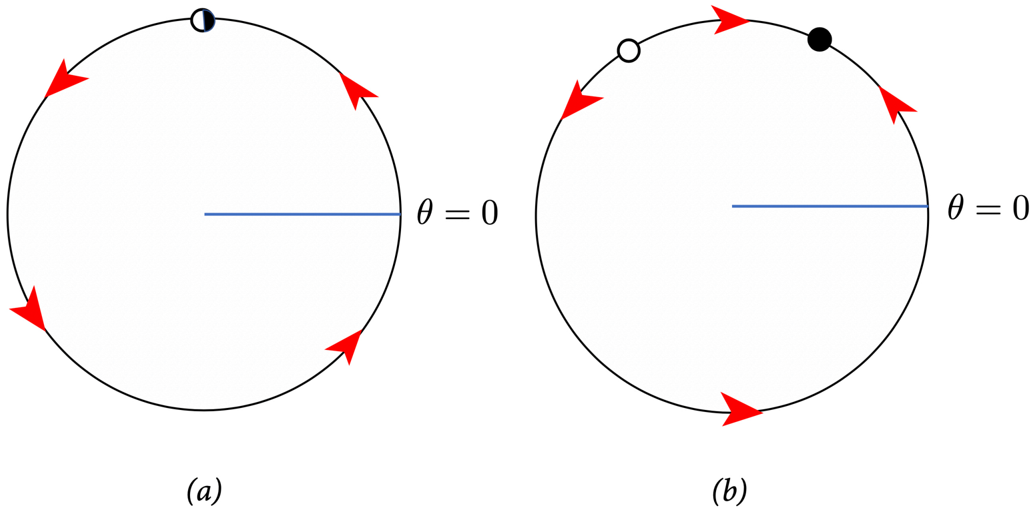


Figure 12: *Fixed points for systems on the circle are stable/unstable by reference to the stability criteria of fixed points on the real line. However, for the vectorfield 3.3 on* $\mathbb{S}^1$*, observe that every orbit through* $\theta = \theta_0$*, for all* $\theta_0 \in \mathbb{S}^1$*, at* $t = 0$*, tends asymptotically to the single fixed point* $\theta = \theta^* = \frac{\pi}{2}$ *(for* $\frac{\omega}{a} = 1$*) as* $t \to \infty$ *even though the fixed point itself is unstable!*

Example 3.3. *Consider the dynamics of* $\dot{\theta} = \omega - a\sin(\theta)$*. We already have that for* $|a| < |\omega|$*, there are no fixed points and so the flow is either clockwise and anti-clockwise. The period lengthens to infinity as* $a \to \omega^-$*. Increasing* $|a|$ *produces a 'bottleneck' of a slower flow in the neighbourhood of* $\theta = \theta^* = \frac{\pi}{2}$ *where* $\sin(\theta^*) \approx (\frac{\omega}{a})$ *for* $\frac{\omega}{a} \approx 1$*. For* $\frac{\omega}{a} = 1$*, the vector field has a single fixed point at* $\theta = \frac{\pi}{2}$*, and the "bottlemeck" becomes a "stopper", see figure 12.*

This situation, as a result of the nature of the circle on which the dynamics occurs requires further investigation, since the fixed point is not stable, but it has some asymptotic stability properties! Note in (b) of figure 11, observe that every orbit through $\theta = \theta_0$ *at* $t = 0$ *tends asymptotically to the single fixed point* $\theta = \theta^* = \frac{\pi}{2}$ *(for* $\frac{a}{\omega} = 1$*) as* $t \to \infty$ *even though the fixed point has "stable/unstable" characteristics!*

Definition 3.1. *A fixed point* $\theta = \theta^* \in \mathbb{S}^1$ *of* $\dot{\theta} = f(\theta)$ *- [*$x = x^* \in \mathbb{R}$ *of* $\dot{x} = f(x)$*] - is* ***stable*** *if for every neighbourhood* V *of* θ^* *[*x^**], there is a neighbourhood* U *of* θ^* *[*x^**] such that all solutions* $\theta(t)$ *with* $\theta(0) \in U$ *- [*$x(t)$ *with* $x(0) \in U$*]- remain in the set* V *for all* $t > 0$*.*
A fixed point $\theta = \theta^* \in \mathbb{S}^1$ *of* $\dot{\theta} = f(\theta)$ *- [*$x = x^* \in \mathbb{R}$ *of* $\dot{x} = f(x)$*] - is* ***unstable*** *if it is not stable.*

Definition 3.2. *A stable fixed point* $\theta = \theta^* \in \mathbb{S}^1$ *- [*$x = x^* \in \mathbb{R}$*] - is* ***asymptotically*** *stable if the neighbourhood* U *can also be chosen so that* $\lim_{t\to\infty}\theta(t) = \theta^*$ *for* $\theta(0) \in U$ *- [*$\lim_{t\to\infty} x(t) = x^*$ *for* $x(0) \in U$*].*

3.3 Attractors and basins of attraction

There is a more general definition *attractor* which is applicable to more complex invariant sets than fixed points - we will be considering limit cycles in the last chapter.

Definition 3.3. *An* ***attractor*** *is a subset* A *of the phase space (*$\mathbb{R}$ *or* $\mathbb{S}$ *so far) which is invariant as time increases, i.e no orbits escape from* A *as time increases, and there exists a neighbourhood* N

of A such that for every orbit $x(t)$ with $x(0) \in N$ remains in N and $x(t) \to A$ as $t \to \infty$. Also the set A is minimal.

Comment: The only attractors on the line and circle are fixed points.

Definition 3.4. *The largest such neighbourhood, $\mathcal{B}(A)$, which consists of all points b whose orbits asymptotically approach the attractor A as $t \to \infty$, is called the* ***basin of attraction*** *for A. More formally, $\mathcal{B}(A)$ is the set of all points b in the phase space with the following property: for any open neighborhood N of A, there is a positive constant T such that the solution curve $x(t) \in N$, where $x(0) = b$, for all $t > T$.*

Comment. We often us the notation $\phi_t(b)$ to to indicate the point $x(t)$ on the solution curve where $x(0)$. So $\{\phi_t(b), t \in \mathbb{R}\}$ is the **solution curve** or **orbit** through the point b at time zero. Think of the colloquial use of the word *orbit* for the trajectory as time evolves of a planet's path around the sun parametrised by time

Comment The basin of attraction $B(\theta_0)$, - $[B(x_0)]$ - for a fixed point $\theta = \theta^* \in \mathbb{S}^1$ - $[x = x_0 \in \mathbb{R}]$ - is the set of all points $\theta_0 \in \mathbb{S}^1$ - $[x_0 \in \mathbb{R}]$,- the solution curve $\theta(t)$ - $[x(t)]$ - with $\theta(0) = \theta_0$ - $[x(0) = x_0]$ - is such that $\lim_{t\to\infty} \theta(t) = \theta^*$, $[\lim_{t\to\infty} x(t) \to x^*]$.

These definitions address the issue of the unique fixed point $\theta = \theta^*$ in example 3.3, in the case $\omega/a = 1$ The fixed point is not *asymptotically stable*, in fact it is unstable, even though it is globally attracting in that all orbits converge to the fixed point $\theta = \theta^*$ as time $t \to \infty$! The conundrum is addressed by observing that the apparent "asymptotic" behaviour is obtained from the global properties of the orbits, together with the non-Euclidean structure of $\mathbb{S}$, rather than local dynamical behaviour being contained wholly arbitrarily small neighbourhoods of the fixed point.

3.4 Comment on stability

The above discussion highlights how some descriptions seem almost self-contradictory without closer scrutiny and an accommodation of topologies other that $\mathbb{R}^n$ as the state space. For example we have just highlighted a fixed point which while it is attracting is not stable cf. figure 12 (a).

We also, more generally, can have fixed points which are stable but not attracting, see, for example, figure 1. We can also have points which are stable and attracting, namely asymptotically stable fixed points. Recall the saddle-node fixed point given by the basic example $\dot{x} = x^2$ on $\mathbb{R}$. This type of fixed point at $x = 0$ is often described, loosely, as *stable/unstable* because it has orbits which move away on one side of the fixed point, and move towards the fixed point on the other side. The fixed point for $\dot{x} = x^2$ is actually an *unstable* fixed point, but the *loose* description *stable/unstable* has a clear visual interpretation on the real line. However, it is better to describe it as a *saddle-node* for reasons which will become more obvious in later sections.

Two-dimensional Dynamical Systems

4 Linear dynamical systems on the plane $\mathbb{R}^2$

4.1 Introduction

We consider vector fields $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$, where $\mathbf{f}$ is linear in the variables $\mathbf{x} = (x, y) \in \mathbb{R}^2$, i.e. $\mathbf{f}(\alpha\mathbf{x} + \beta\mathbf{x}') = \alpha\mathbf{f}(\mathbf{x}) + \beta\mathbf{f}(\mathbf{x}')$ for $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$. So the linear function, and the corresponding dynamical system, can be represented in matrix form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{4.1}$$

or

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}, \tag{4.2}$$

where

$$\mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{4.3}$$

Note that a linear system always has a fixed point at $(x, y) = (0, 0)$ and we are able to determine the nature of the fixed point from the eigenvalues of the coefficient matrix $\mathbf{A}$. Recall that the eigenvalues of the matrix $\mathbf{A}$ are given by the solutions of the quadratic equation in λ given by $Det(\lambda\mathbf{I} - A) = 0$. The equation takes the form

$$\lambda^2 - \tau\lambda + \delta = 0, \tag{4.4}$$

where $\tau = Tr(\mathbf{A}) = a + d$ and $\delta = Det(\mathbf{A}) = ad - bc$, cf. equation 4.3.

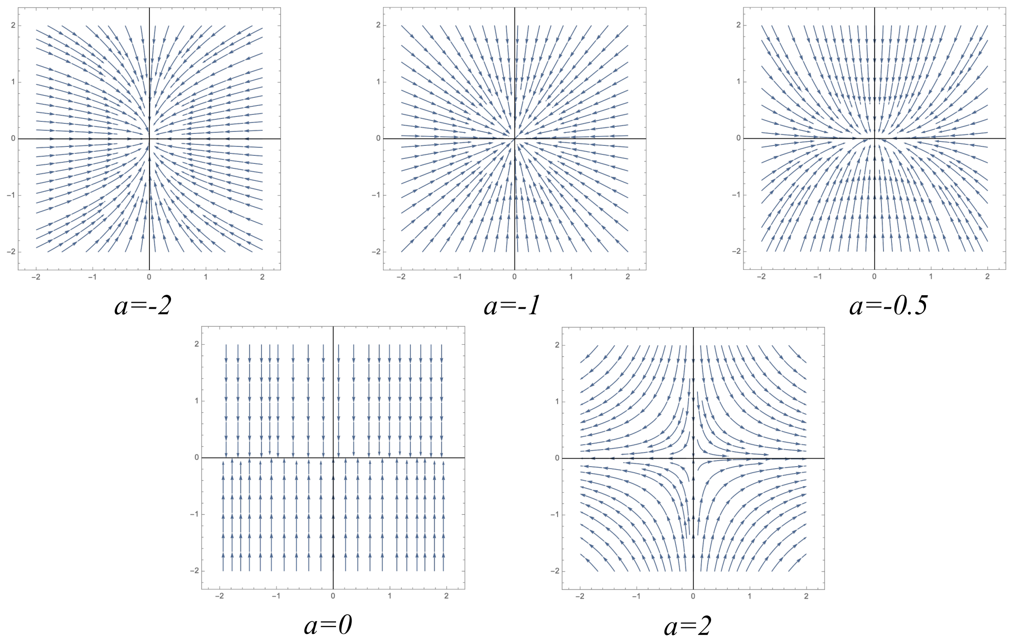


Figure 13: *The system 4.6 exhibits a variety of phase portraits in the $xy-$plane as the parameter a is varied. The five distinct phase portraits for the fixed point $x^* = (0, 0)$ are for the parameter intervals $a < -1$, $a = -1$, $-1 < a < 0$, $a = 0$, $a > 0$.*

Example 4.1. *Let the coefficient matrix be*

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \tag{4.5}$$

which gives the dynamical system. Phase portraits for this system can be seen in figure 13.

$$\dot{x} = ax, \ \dot{y} = -y. \tag{4.6}$$

Note that the nature of the solution curves of the systems are easy to address, because of the decoupled(diagonal) form of matrix A, which has solution curves in parametrised form as $x(t) = x(0)e^{at}$ and $y(t) = y(0)e^{-t}$.

Example 4.2. *Consider the linear system with $\mathbf{f}(x,y) = (y, -\omega^2 x)$. That is*

$$\dot{x} = y, \quad \dot{y} = -\omega^2 x \tag{4.7}$$

Note the curves in $\mathbb{R}^2$ are invariant curves for the system.

$$H(x,y) = \omega^2 x^2 + y^2. \tag{4.8}$$

Hence, if a solution of the system 4.7, $\mathbf{x}(t) = (x(t), y(t))$, has an initial point $\mathbf{x}(t) = (x_0, y_0)$ for

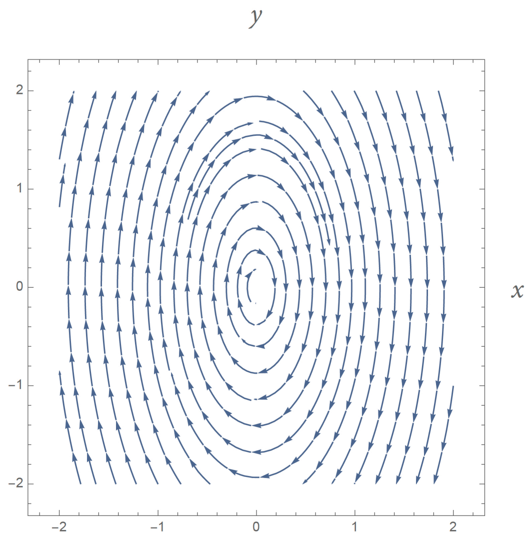


Figure 14: *The phase portrait of the system 4.7 with a centre fixed point surrounded by elliptic closed orbits ($\omega = 2$ in this illustration).*

$t = 0$, and $H(\mathbf{x}(0)) = H_0$ then $H(\mathbf{x}(t)) \equiv H_0$, for $t \in \mathbb{R}$.

To check this, we note that

$$\frac{d}{dt}(H(x,y)) = \frac{d}{dt}(\omega^2 x^2 + y^2) = 2\omega^2 x \frac{dx}{dt} + 2y\frac{dy}{dt} \equiv 0. \tag{4.9}$$

Hence, the orbits(solution curves) of the system 4.7 remain on the constant (elliptical) curves of the functions H - we say the orbits follow the level contours of the function H. Note the fixed point at $\mathbf{x} = (0,0)$ of the system in equation 4.7 is said to be stable, but not asymptotically stable. Note the earth's elliptical orbit is stable in that it not only remains close to the sun, but it is neither spiralling into or away from the sun - fortunately, it is not asymptotically stable!

4.1.1 Equivalence of linear systems

Let $\mathbf{z} = \mathbf{P}\mathbf{w}$ be a linear change of coordinates, with $\mathbf{P}$ ia non-singular matrix, where

$$\mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} u \\ v \end{bmatrix}. \tag{4.10}$$

Since $\mathbf{P}^{-1}$ exists, it follows that

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} \implies \mathbf{P}\dot{\mathbf{w}} = \mathbf{A}\mathbf{P}\mathbf{w},$$

and therefore, if

$$\dot{\mathbf{w}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{w}.$$

So we have changed the system $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}$ to $\dot{\mathbf{w}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{w}$, and a natural question to ask is whether we can choose a matrix $\mathbf{P}$ so that the matrix $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is easier to work with? It should also be noted that this change of coordinates is reversible, i.e. $\mathbf{w} = \mathbf{P}^{-1}\mathbf{z}$, which exists as $\mathbf{P}$ is non-singular, converts the system $\dot{\mathbf{w}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{w}$ back to $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}$, and hence, the two systems are interchangeable!

4.2 Classification of linear systems

Theorem 4.1. *Let $\mathbf{A}$ be a 2×2 real matrix, then there is a 2×2 non-singular real matrix $\mathbf{P}$ such that*

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{J},$$

where $\mathbf{J}$ is a Jordan canonical matrix, which take the following forms:

$$\mathbf{J}_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \mathbf{J}_2 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \mathbf{J}_3 = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}. \tag{4.11}$$

The matrices $\mathbf{A}$ and $\mathbf{J}$ have the same eigenvalues as they are similar, as shown in the following lemma.

Lemma 4.1. *The matrices $\mathbf{A}$ and $\mathbf{J}$ have the same eigenvalues as they are similar by the conjugating matrix $\mathbf{P}$.*

Proof Recall that the eigenvalues of $\mathbf{A}$ are given by the roots of the polynomial in λ defined by $Det(\lambda\mathbf{I} - \mathbf{A}) = 0$, but

$$Det(\mathbf{P}^{-1}) \cdot Det(\mathbf{P}) = Det(\mathbf{P}^{-1}\mathbf{P}) = Det(\mathbf{I}) = 1,$$

and so,

$$\begin{aligned} Det(\lambda\mathbf{I} - \mathbf{A}) =& Det(\mathbf{P}^{-1}) \cdot Det(\lambda\mathbf{I} - \mathbf{A}) \cdot Det(\mathbf{P}) \\ =& Det(\mathbf{P}^{-1}(\lambda\mathbf{I} - \mathbf{A})\mathbf{P}) = Det(\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P}) \\ =& Det(\lambda\mathbf{I} - \mathbf{J}) \end{aligned} \tag{4.12}$$

Note $\mathbf{J}$ can have several eigenvalue types:

1. $\mathbf{J}_1$ eigenvalues - $\{\lambda_1, \lambda_2\} \subset \mathbb{R}$
2. $\mathbf{J}_2$ eigenvalues - $\{\lambda, \lambda\} \subset \mathbb{R}$,
3. $\mathbf{J}_3$ eigenvalues - $\{\alpha - i\beta, \alpha + i\beta\}, \ \{\alpha, \beta\} \subset \mathbb{R}$

4.2.1 Linear systems with Jordan coefficient matrices

(i)

$$\dot{\mathbf{w}} = \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{4.13}$$

with $\mathbf{w}(t) = (u(t), v(t)) = (u(0)e^{\lambda_1 t}, v(0)e^{\lambda_2 t})$

(ii)

$$\dot{\mathbf{w}} = \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{4.14}$$

with $\mathbf{w}(t) = (u(t), v(t)) = (u(0)e^{\lambda t} + v(0)te^{\lambda t}, v(0)e^{\lambda t})$

(iii)

$$\dot{\mathbf{w}} = \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{4.15}$$

with $\mathbf{w}(t) = (u(t), v(t))$, where

$$u(t) = e^{\alpha t}(u(0)\cos(\beta t) - v(0)\sin(\beta t)$$

and

$$v(t) = e^{\alpha t}(u(0)\sin(\beta t) + v(0)\cos(\beta t)).$$

4.2.2 Polar coordinates: a nonlinear change of coordinates

Sometimes, it is advantageous to either directly model a system in polar coordinates, or transform a model described in cartesian coordinates into the equivalent model in polar coordinates for a clearer understanding of the system behaviour. This can be particularly useful when considering circular or spiral phase portrait behaviour as polar descriptions lend themselves to simple descriptions of circular or spiral shapes.

We have the connecting equations $r^2 = x^2 + y^2$ and $\tan(\theta) = \frac{y}{x}$. It follows that we can use for $\dot{r}$

$$r\dot{r} = x\dot{x} + y\dot{y},$$

and for $\dot{\theta}$,

$$\frac{d}{dt}(\tan(\theta)) = \frac{d}{dt}\left(\frac{y}{x}\right) \implies \sec^2(\theta)\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{x^2}.$$

Using $\sec^2(\theta) = 1 + \frac{y^2}{x^2} = \frac{r^2}{x^2}$, we obtain an equation for $\dot{\theta}$ as:

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}.$$

Note that polar coordinates (r, θ) are more revealing of the behaviour of the system in equation 4.15. The change of coordinates $r = \sqrt{(}u^2 + v^2)$ and $\theta = \arctan(\frac{v}{u})$ reduces the system to

$$\dot{r} = \alpha r, \quad \dot{\theta} = \beta.$$

4.2.3 Construction of the matrix P

To identify for a given coefficient matrix $\mathbf{A}$, the Jordan form $\mathbf{J} = \mathbf{J}_i$, where $\mathbf{J}_i$, $i = 1, 2, 3$ (as defined in equation 4.11), recall that $\mathbf{A}$ and $\mathbf{J}$ have the same eigenvalues as they are similar matrices, and we have:

(i) $\mathbf{A}$ reduces to $\mathbf{J}_1$ if it has real distinct eigenvalues λ_1, λ_2.

(ii) $\mathbf{A}$ reduces to $\mathbf{J}_3$ if it has complex eigenvalues $\alpha \pm i\beta$, with $\beta \neq 0$.

(iii) $\mathbf{A}$ reduces to $\mathbf{J}_2$ if it has real repeated eigenvalues λ, λ and is not diagonal, else it reduces to $\mathbf{J}_1$ if diagonal (in this latter case $\mathbf{A}$ is already in the form of $\mathbf{J}_1 = \mathbf{A} = \lambda\mathbf{I}$).

Theorem 4.2. *Given matrix $\mathbf{A}$, the conjugating matrix $\mathbf{P}$ which reduces to Jordan form is for:*

(i) $\mathbf{J} = \mathbf{J}_1, \quad \mathbf{P} = [\mathbf{v}_1, \mathbf{v}_2]$*, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of the distinct eigenvalues λ_1 and λ_2 respectively.*

(ii) $\mathbf{J} = \mathbf{J}_2, \quad \mathbf{P} = [\mathbf{v}_1, \mathbf{v}_2]$*, where $\mathbf{v}_1$ is an eigenvector of $\mathbf{A}$ corresponding to the repeated eigenvalue λ and $\mathbf{v}_2$ satisfies the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$.*

(iii) $\mathbf{J} = \mathbf{J}_3, \quad \mathbf{P} = [\mathbf{v}_1, \mathbf{v}_2]$*, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the real and imaginary parts of the complex eigenvector of $\mathbf{A}$ corresponding to the eigenvalue $\lambda_1 = \alpha - i\beta$.*

Check:

(i) $\mathbf{J}_1$:

$$\mathbf{AP} = \mathbf{A}\,[\mathbf{v}_1, \mathbf{v}_2] = [\mathbf{Av}_1, \mathbf{Av}_2] = [\lambda_1\mathbf{v}_1, \lambda_2\mathbf{v}_2] = \mathbf{PJ}_1$$

(ii) $\mathbf{J}_2$:

$$\mathbf{AP} = \mathbf{A}\,[\mathbf{v}_1, \mathbf{v}_2] = [\mathbf{Av}_1, \mathbf{Av}_2] = [\lambda\mathbf{v}_1, \mathbf{v}_1 + \lambda\mathbf{v}_2] = \mathbf{PJ}_2.$$

(iii) $\mathbf{J}_3$:

Let $\mathbf{v}_1 + i\mathbf{v}_2$ be a complex eigenvalue of $\mathbf{A}$ for the eigenvalue $\lambda_1 = \alpha - i\beta$. Then

$$\mathbf{A}(\mathbf{v}_1 + i\mathbf{v}_2) = (\alpha - i\beta)\,[\mathbf{v}_1 + i\mathbf{v}_2] = (\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) + i(-\beta\mathbf{v}_1 + \alpha\mathbf{v}_2)$$

Taking real and imaginary parts we have

$$\mathbf{Av}_1 = (\alpha\mathbf{v}_1 + \beta\mathbf{v}_2); \qquad \mathbf{Av}_2 = (-\beta\mathbf{v}_1 + \alpha\mathbf{v}_2). \tag{4.16}$$

Now let $\mathbf{P} = [\mathbf{v}_1, \mathbf{v}_2]$ and we obtain

$$\mathbf{AP} = [\mathbf{Av}_1, \mathbf{Av}_2] = [\alpha\mathbf{v}_1 + \beta\mathbf{v}_2, -\beta\mathbf{v}_1 + \alpha\mathbf{v}_2] = \mathbf{PJ}_3. \tag{4.17}$$

Thus we can show for $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{J}_k$, for some $k = 1, 2, 3$.

4.3 Jordan matrix calculations

Consider the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with coefficient matrix

$$\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}. \tag{4.18}$$

The eigenvalues are solutions of $Det(\lambda\mathbf{I} - \mathbf{A}) = 0$ which gives $\lambda^2 + \lambda - 6 = 0$ and so $\lambda_1 = 2, \lambda_2 = -3$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ using $\mathbf{A}\mathbf{v}_1 = 2\mathbf{v}_1$, and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ using $\mathbf{A}\mathbf{v}_2 = -3\mathbf{v}_2$. We can now construct the conjugating matrix $\mathbf{P}$ for $\mathbf{A}$ by using the eigenvectors as its columns to obtain

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix},$$

i.e.

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{J} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}.$$

Thus $\mathbf{A}$ gives a saddle fixed point with unstable manifolds on the eigen-direction $\mathbf{v}_1$ and stable manifolds along the eigen-direction $\mathbf{v}_2$.

4.3.1 Change of coordinates and eigendirections

We introduced the linear transformation $\mathbf{z} = \mathbf{P}\mathbf{w}$ with coordinates $\mathbf{z} = (x, y)$ $(= \begin{bmatrix} x \\ y \end{bmatrix})$ and $\mathbf{w} = (u, v)$ $(= \begin{bmatrix} u \\ v \end{bmatrix})$ which showed the equivalence of the system $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}$ to

$$\dot{\mathbf{w}} = (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})\mathbf{w} = \mathbf{J}\mathbf{w}. \tag{4.19}$$

Note that

$$\mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

and

$$\mathbf{z} = \mathbf{P}\mathbf{w} = [\mathbf{v}_1|\mathbf{v}_2]\begin{bmatrix} u \\ v \end{bmatrix} = u\mathbf{v}_1 + v\mathbf{v}_2. \tag{4.20}$$

Hence:

$$\begin{bmatrix} x \\ y \end{bmatrix} = u\mathbf{v}_1 + v\mathbf{v}_2.$$

This equation shows that the point $(x, y) \in \mathbb{R}^2$ has coordinates (u, v) relative to the ordered base vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$.

4.3.2 Eigenvalues and node/saddles

Consider $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

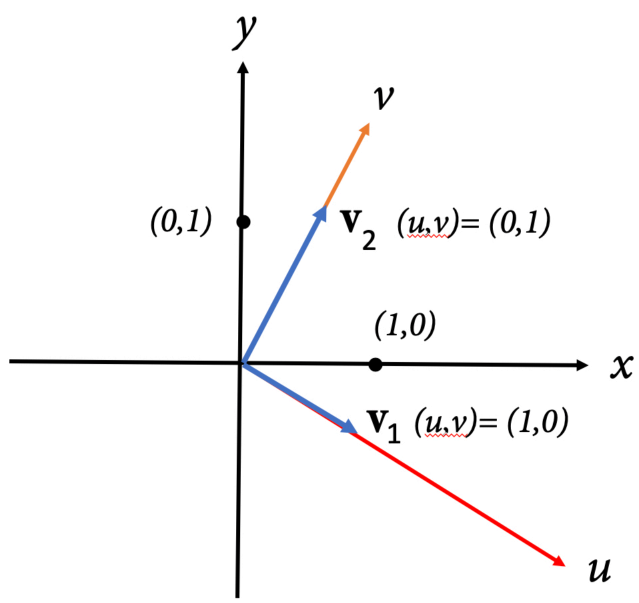


Figure 15: *The $u, v-$ coordinate axes of the transformed system in equation 4.19 with axes given by the eigenvector $\mathbf{v}_1$ which has coordinates $(1, 0)$ in the $uv-$ plane, and the eigenvector $\mathbf{v}_2$ which has coordinates $(0, 1)$.*

Then $\dot{x} = \lambda_1 x$ and $\dot{y} = \lambda_2 y$ which gives, by eliminating dt, $\frac{dy}{dx} = (\lambda_2 y)/(\lambda_1 x)$ and thus the integral curves are of the form $y = Cx^{\lambda_2/\lambda_1}$, C constant. Thus we have when λ_1, λ_2 are of the same sign a node with orbits being tangent to the x-axis for $\lambda_2/\lambda_1 > 1$, and tangent to the y-axis for $\lambda_2/\lambda_1 < 1$. For $\lambda_2/\lambda_1 = 1$, we have a star node where every trajectory is radial.

For λ_1, λ_2 opposite sign, we have a saddle, with unstable/stable manifolds along the axes. Stability is determined by the signs of the eigenvalues $\lambda_1 < 0$ - stable; $\lambda_2 > 0$ - unstable;

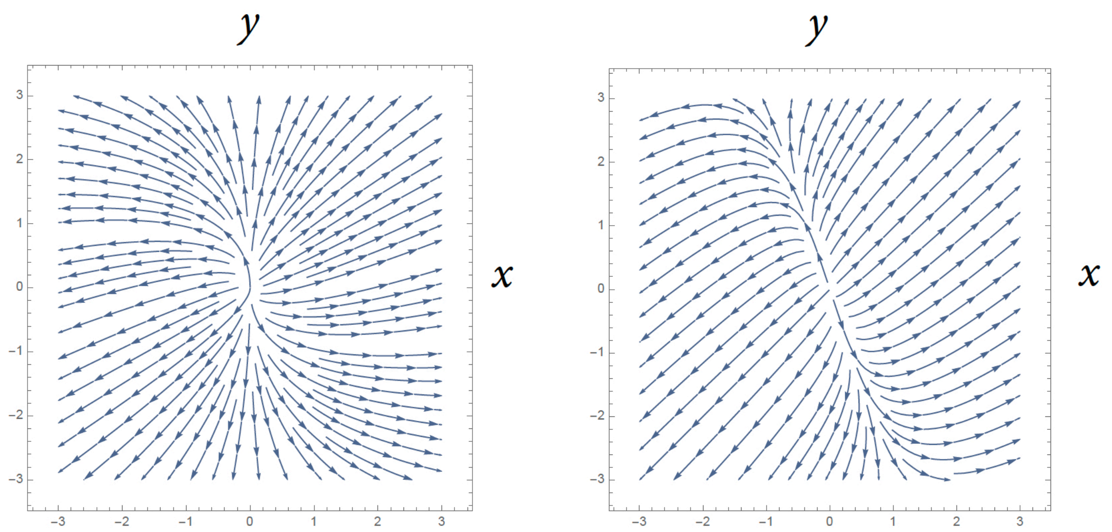
Example 4.3. *This example addresses node-tangencies for fixed points. Consider the coefficient matrices*

1.
$$\mathbf{A} = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}. \tag{4.21}$$

We have $\delta = 8$ and $\tau = 6$, so we have an unstable node. The fixed point at the origin has eigenvalues $\lambda_1 = 4$ with eigenvector $\mathbf{v}_1 = [2, 1]^T$ and $\lambda_2 = 2$ with eigenvector $\mathbf{v}_2 = [0, 1]^T$, see figure 16.

2.
$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \tag{4.22}$$

We have $\delta = 5$ and $\tau = 6$, so we have an unstable node with $\lambda_1 = 1$ and $\lambda_2 = 5$. The eigenvectors are $\mathbf{v}_1 = [1, -3]^T$ and $\mathbf{v}_2 = [1, 1]^T$, see figure 16.



(a) (b)

Figure 16: *(a) The phase portrait for system 4.21 is an unstable node, with tangency of orbits along the* $\mathbf{v}_2$ *eigendirection as* $\lambda_2/\lambda_1 < 1$ *and* $\mathbf{v}_2 = [0,1]^T$. *For (b) The phase portrait for system 4.22 is an unstable node, with tangency of orbits along the* $\mathbf{v}_1$ *eigendirection as* $\lambda_2/\lambda_1 > 1$ *and* $\mathbf{v}_1 = [1,-3]^T$.

4.3.3 Eigenvalues and spirals/centres

Consider $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}.$$

Converting to polar coordinates gives us $\dot{r} = \alpha r$, $\dot{\theta} = \beta$. For $\beta \neq 0$, we have stable spirals to the origin for $\alpha < 0$, and unstable spirals for $\alpha > 0$. The direction of rotation (clockwise-anti clockwise) is determined by the sign β. $\beta = 0$ provides a star node with radial trajectories.

4.3.4 Eigenvalues and improper nodes

Consider $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

We have a stable improper node for $\lambda < 1$ and an unstable improper node for $\lambda > 1$.

4.4 The $Trace(\mathbf{A}) - Det(\mathbf{A})$ classification of fixed points

Recall that the set of eigenvalues of the matrix $\mathbf{A}$ are given by the characteristic polynomial $Det(\lambda\mathbf{I} - \mathbf{A}) = 0$ in the variable λ. For a 2×2 matrix $\mathbf{A}$ the polynomial takes the form, cf. equation 4.4,

$$\lambda^2 - \tau\lambda + \delta = 0,$$

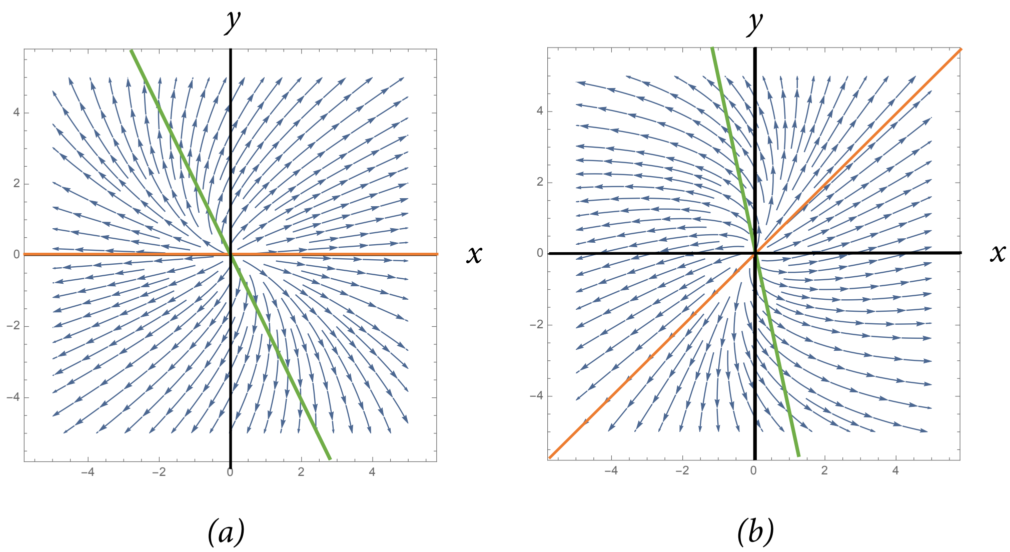


Figure 17: *The phase portraits for unstable improper nodes systems given in 4.23 , with tangency of orbits along the* $[1,0]^T$ *eigendirection in (a), along the* $[1,1]$ *eigendirection in (b). Note also, as a help in seeing the form of the phase portraits, null clines are illustrated in the figure (green), i.e. for (a): the* $\dot{x} = 0$ *nullcline is the line* $y + 2x = 0$*, whereas for (b): the corresponding nullcline for* $\dot{u} = 0$*, is given by* $\frac{dy}{dx} = -1$*, (perpendicular to* $\mathbf{v}_1 = [1,1]^T$*) which is given by the line* $5x + y = 0$.

where $\tau = Tr(\mathbf{A})$ and $\delta = Det(\mathbf{A})$, and the roots are then given by

$$\lambda = \frac{\tau \pm \sqrt{(\tau^2 - 4\delta)}}{2}.$$

Example 4.4. *Sketch the phase portraits of* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$*, for*

$$(a)\ \mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad (b)\ \mathbf{A} = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}. \tag{4.23}$$

(a) The single eigendirection for the repeated eigenvalue $\lambda = 2$ *is* $[1,0]^T$*, i.e. the* $x - axis$*. The nullclines is* $y = 0$ *for* $\dot{y} = 0$*, and* $2x + y = 0$ *for* $\dot{x} = 0$.

(b) The matrix $\mathbf{A} = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$ *has a repeated eigenvalue* $\lambda = 3$*.* $\mathbf{A}$ *is not diagonal, therefore the phase portrait for* $\mathbf{A}$ *is not a star-node (note that if* $\mathbf{A}$ *is conjugate to the diagonal matrix* $\lambda\mathbf{I}$*, then* $A = \lambda\mathbf{I}$ *!), and is therefore an unstable improper node.*

The eigenvector equation gives $\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} x \\ y \end{bmatrix}$*, and the eigendirection is given by* $x = y$*, i.e. we can choose* $\mathbf{v}_1 = [1,1]^T$*. The second requirement is a vector* $\mathbf{v}_2$ *which satisfies*

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1.$$

This has a solution $\mathbf{v}_2 = [2,1]^T$*. It can be checked that* $\mathbf{P} = [\mathbf{v}_1|\mathbf{v}_2] = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ *satisfies*

$$\mathbf{AP} = \mathbf{P}\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}\left(= \begin{bmatrix} 3 & 7 \\ 3 & 4 \end{bmatrix}\right) \text{ (much easier to check!)}$$

and so we conclude that $\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$*, a Jordan matrix of type* $\mathbf{J}_2$*.*

We can capture the regions of the Trace-Det plane which have the different types of associated fixed points. See figure 18.

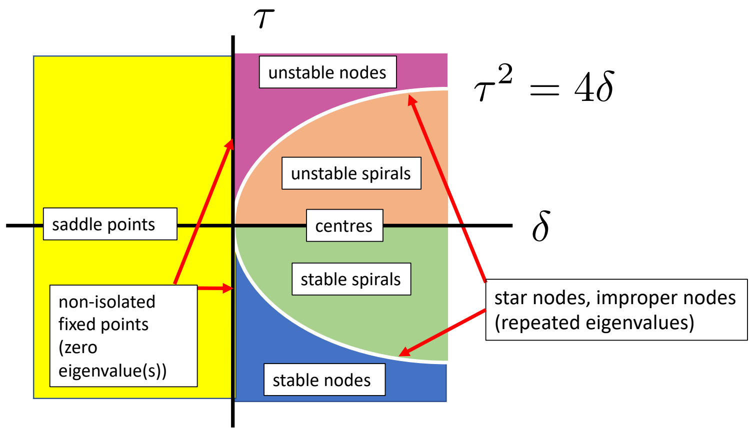


Figure 18: *The types of fixed points which occur in different regions of the Trace-Det($\tau - \delta$) plane. The axes and the discriminant of $\lambda^2 - \tau\lambda + \det = 0$, i.e. $\tau^2 = 4\delta$ form boundaries for the regions.*

4.5 Stability definitions revisited

(i) The *stable manifold* of the fixed point x^* is the set of points x_0 such that the solution curve $x(t)$ with $x(0) = x_0$ satisfies $x(t) \to x^*$ as $t \to \infty$.

(ii) The *unstable manifold* of the fixed point x^* is the set of points x_0 such that the solution curve $x(t)$ with $x(0) = x_0$ satisfies $x(t) \to x^*$ as $t \to -\infty$.

(iii) The unstable manifold of a *linear saddle* fixed point x^* is the bi-infinite line through the fixed point x^* given by the unstable eigendirection.

(iv) The stable manifold of a *linear saddle* is the bi-infinite line through the fixed point x^* given by the stable eigendirection.

(v) Stable and unstable manifolds of a saddle fixed point are also referred to as *separatrices*.

(vi) A fixed point x^* is *attracting* if there exists an open neighbourhood U of x^* such that all orbits with $x(0) \in U$ satisfy $x(t) \to x^*$ as $t \to \infty$. It can be *locally* or *globally* attracting. For locally attracting, the orbits would need to remain in arbitrarily small neighbourhoods as $t \to \infty$, i.e be asymptotically stable.

(vii) The *basin of attraction* of a fixed point x^* is the set of initial conditions x_0 of trajectories $x(t)$ (i.e. $x(0) = x_0$) such that $\lim_{t\to\infty} x(t) = x^*$.

(viii) A fixed point x^* is *Liapounov stable* if all trajectories starting sufficiently close to x^* remain close to it for all time(and not just as $t \to \infty$). More precisely, given any neighbourhood V of x^*, there exists a neighbourhood U of x^* such that all trajectories passing through U remain in V for all $t \in \mathbb{R}^+$. (Note same as *stable* but we consider *Liapunov functions* to identify stability of fixed points in chapter 5)

(ix) Liapounov stable fixed points which are not asymptotically stable are said to be *neutrally stable*.

(x) A fixed point x^* is *asymptotically stable* if is both Liapounov stable and *locally* attracting, i.e. all orbits asymptotically approach x^* as t increases and remain in any local neighbourhood.

Example 4.5. *Some examples of the use of the terms stable and unstable manifolds for linear systems.*

(i) The stable manifold of a stable linear node is $\mathbb{R}^2$.

(ii) The unstable manifold of an stable linear spiral is $\{\mathbf{0}\}$*, the origin of* $\mathbb{R}^2$.

(iii) The stable manifold of an stable linear spiral is $\mathbb{R}^2$.

(iv) The stable manifold of an unstable linear spiral is $\{\mathbf{0}\}$*, the origin of* $\mathbb{R}^2$.

(v) The unstable manifold of a stable linear node is $\{\mathbf{0}\}$*, the origin of* $\mathbb{R}^2$.

(vi) A linear centre fixed point is neutrally stable, stable but not asymptotically stable.

5 Nonlinear dynamical systems on $\mathbb{R}^2$

5.1 Phase plane and phase portraits

Recall the concept of a first-order ODE in n-variables, $x_1, \ldots, x_n \in \mathbb{R}$, where

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_n), \\ &\ldots, \\ \dot{x}_n &= f_n(x_1, \ldots, x_n). \end{aligned} \tag{5.1}$$

and $x_1, \ldots, x_n \in \mathbb{R}$, depend (or are functions of) the independent variable *time* t. Also, by introducing the vector notation $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and the vector function

$$\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})) : \mathbb{R}^n \to \mathbb{R}^n$$

allows the system of equations 5.1 to be written as a single vector equation,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

The *vector field* $\mathbf{f}$ on $\mathbb{R}^n$ associates a vector, $\mathbf{f}(\mathbf{x})$, to each point $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. These equations are describing how the variables $x_1, \ldots, x_n$ are evolving or changing with time, hence the concept of a *dynamical system*. Each function f_i, $i = \{1, \ldots, n\}$ is assumed continuously differentiable with respect to $\mathbf{x}$. The case $n = 2$, which is the focus of this section, is therefore of the form $\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2)$. Given the low number of variables, it is useful to simplify the notation of this chapter to

$$\dot{x} = f(x, y), \; \dot{y} = g(x, y) \tag{5.2}$$

where $\mathbf{x} = (x_1, x_2) \stackrel{\text{def}}{=} (x, y)$ and $\mathbf{f} = (f_1, f_2) \stackrel{\text{def}}{=} (f, g)$.

Our main objects of interest will be, again, *fixed points* or *equilibrium points*. However, the new phenomenon in 2-dimensional phase portraits to be considered is the *closed* or *periodic* orbit.

Definition 5.1. *The point $\mathbf{x}^* = (x^*, y^*) \in \mathbb{R}^2$ is said to be a **fixed point** of the system 5.2 if the solution $(x(t), y(t)) \equiv \mathbf{x}^*$ for $t \in \mathbb{R}$ is satisfied by the system. This requires $(f(\mathbf{x}^*), g(\mathbf{x}^*)) = (0, 0)$.*

Definition 5.2. *The solution curve $(x(t), y(t)) \in \mathbb{R}^2$ of the system 5.2 is said to be a **periodic** or **closed** orbit of **period** T if $\mathbf{x}(t) \equiv \mathbf{x}(t+T)$ for all $t \in \mathbb{R}$, and $T > 0$ is the minimum such value, i.e. $\mathbf{x}(t) \neq \mathbf{x}(0)$, for any $t \in (0, T)$.*

A fixed point could be interpreted as a *trivial* periodic orbit if $T = 0$ were to be allowed, but it is *non-trivial* periodic orbits, with positive period $T > 0$, that we are interested in this section.

5.1.1 Existence and uniqueness of solutions for systems in $\mathbb{R}^n$

Theorem 5.1. *Let $f : U(\subseteq \mathbb{R}^n) \to \mathbb{R}^n$, where U is an open set, and consider the initial value problem of*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$, and $\mathbf{x}(0) = \mathbf{x}_0 \in U$.

- If $\mathbf{f}$ is continuous, then there is a solution $\mathbf{x}(t)$ such that $\mathbf{x}(0) = \mathbf{x}_0$ defined for some open interval $(a, b) \in \mathbb{R}$ which contains $0 \in \mathbb{R}$.

- if $\mathbf{f}$ is continuously differentiable, then the solution $\mathbf{x}(t)$ is unique.

5.2 Linearisation

This is an important 2-dimensional extension of the linear stability we saw earlier in dynamical systems on the real line $\mathbb{R}$. Suppose that $\mathbf{x}^* = (x^*, y^*)$ is a fixed point of 5.2, i.e $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$. Introduce *local coordinates* $(u, v) \in \mathbb{R}^2$ at $\mathbf{x}^*$, with $\mathbf{x} = \mathbf{x}^* + (u, v)$.

The Taylor expansion of $\mathbf{f}$ at the fixed point $\mathbf{x}^*$ gives

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \mathbf{f}(\mathbf{x}^*) + D\mathbf{f}(\mathbf{x}^*) \begin{bmatrix} u \\ v \end{bmatrix} + O(|(u,v)|^2). \tag{5.3}$$

We are using column vector notation here to allow for simpler matrix expressions..

The first order coefficient matrix $D\mathbf{f}(\mathbf{x})$ is called the *Jacobian* matrix and has the form

$$D\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(\mathbf{x}^*) & \frac{\partial f_1}{\partial y}(\mathbf{x}^*) \\ \frac{\partial f_2}{\partial x}(\mathbf{x}^*) & \frac{\partial f_2}{\partial y}(\mathbf{x}^*) \end{bmatrix}, \tag{5.4}$$

where $\mathbf{f} = (f_1, f_2)$.

Ignoring terms of higher order than degree 1 in the local coordinates u, v, at the fixed point $\mathbf{x} = \mathbf{x}^*$, where $\mathbf{f}(\mathbf{x}^*) = 0$, we obtain a linear system

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = D\mathbf{f}(\mathbf{x}^*) \begin{bmatrix} u \\ v \end{bmatrix}. \tag{5.5}$$

called *the linearised system* of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ at $\mathbf{x}^*$.

5.2.1 Hartman-Grobman Linearisation Theorem (HGLT)

A subset of the linear systems that we have considered remain qualitatively the same under sufficiently small perturbations of the entries of the coefficient matrix. Such linear systems are said to be *hyperbolic*.

For example, a sufficiently small change in the coefficient matrix $\mathbf{A}$ for a (un)stable node or spiral, does not change the fixed point type. Why? - because a small change in the entries of the matrix $\mathbf{A}$ results in a small change in the values of $Tr(\mathbf{A})$ and $Det(\mathbf{A})$ and therefore the roots of the eigenvalue equation λ_1, λ_2.

Similarly, a sufficiently small change in a saddle also preserves the opposite sign eigenvalues and therefore the saddle itself. However, an *arbitrarily small change* in a coefficient matrix with purely imaginary eigenvalues can perturb it into a matrix with non-zero trace, and hence a centre fixed point can become a spiral.

Definition 5.3. *A linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is said to be **hyperbolic** if none of the eigenvalues of the matrix $\mathbf{A}$ have zero real part.*

These are precisely the linear systems which do not change their qualitative type under sufficently small perturbations. For example, a linear system of centre-type is *non-hyperbolic*, whereas a linear system of stable node-type is *hyperbolic*. Linearisations at fixed points which are hyperbolic are key to the following Hartman-Grobman Linearisation Theorem

Theorem 5.2. ***(HGLT)*** *The qualitative behaviour of a dynamical system and its linearised system on a sufficiently small neighbourhood of a fixed point are the **same** provided the linearisation is hyperbolic.*

The following examples exhibit systems where HGLT fails to give clarification on the nature of a fixed point of a nonlinear system.

Example 5.1. *Consider the system*

$$\dot{x} = y - x^3, \quad \dot{y} = -x - y^3 \tag{5.6}$$

at the fixed point $(x, y) = (0, 0)$*. The Jacobian matrix at a point* $\mathbf{x} = \mathbf{x}^* = (x, y)$ *for* $\mathbf{f}(\mathbf{x}) = (y - x^3, -x - y^3)$ *is*

$$D\mathbf{f}(\mathbf{x}^*) = \begin{bmatrix} -3x^2 & 1 \\ -1 & -3y^2 \end{bmatrix} \tag{5.7}$$

which gives the linearised system $\dot{x} = y$*,* $\dot{y} = -x$ *at the fixed point* $\mathbf{x} = \mathbf{0}$ *which is a linear centre. However the non-linear system it approximates, the dynamic 5.6 gives rise to a stable spiral (note, in polar coordinates,* $r\dot{r} = -(x^4 + y^4) \implies \dot{r} = -r^3(\cos^4(\theta) + \sin^4(\theta))$*, i.e. which implies that* $r(t) \to 0$ *as* $t \to \infty$*).*

Example 5.2. *Consider the system*

$$\dot{x} = -x^2, \quad \dot{y} = -y \tag{5.8}$$

at the fixed point $(x, y) = (0, 0)$*. The Jacobian matrix at a point* $\mathbf{x} = \mathbf{x}^*$ *is*

$$D\mathbf{f}(\mathbf{x}^*) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{5.9}$$

and the corresponding linear system has a line of fixed points on the x-axis. However the non-linear system it approximates, is a 2-dimensional saddle-node ("half saddle and half node") which has just one fixed point, cf. figure 19 (c).

Example 5.3. *Consider the system*

$$\dot{x} = y - xy^2, \quad \dot{y} = -x + yx^2, \tag{5.10}$$

at the fixed point $\mathbf{x}^* = (0, 0)$*. The Jacobian matrix at the point* $\mathbf{x}^*$ *is*

$$D\mathbf{f}(\mathbf{x}^*) = \begin{bmatrix} -y^2 & 1 - 2xy \\ -1 + 2xy & x^2 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{5.11}$$

Again, this linearisation is not hyperbolic, eigenvalues are $\lambda = \pm i$*, and so the fixed point does not satisfy the requirements of HGLT - it's linearisation is a centre. However the non-linear system it approximates, the dynamic system 5.10 , also has a* ***centre*** *at the origin. Check the polar form of the equation which gives* $\dot{r} = 0$*. Does the local centre extend out to be a global centre on the whole plane* $\mathbb{R}^2$*? [Hint: Consider the complete set of fixed points for the system.]*

The other two non-linear examples given above have phase portraits different from their linearisations at the fixed points.

Example 5.4. *Show that the system*

$$\dot{x} = -y + ax(x^2 + y^2), \ \dot{y} = x + ay(x^2 + y^2)$$

has the polar form $\dot{r} = ar^3$*,* $\dot{\theta} = 1$*. Explain why the HGLT does not offer a clear conclusion on the nature of the fixed point at the origin. What can you deduce from the polar form for the cases* $a < 0$*,* $a = 0$*, and* $a > 0$*?*

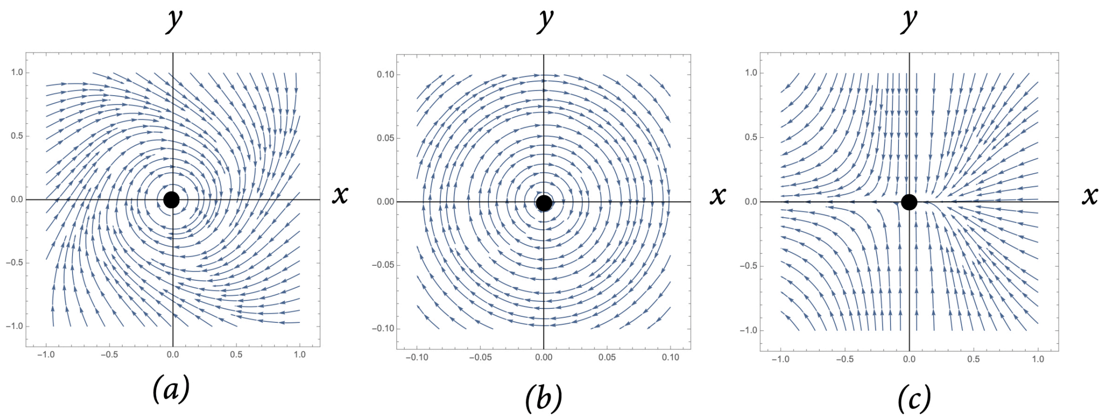


Figure 19: *(a) is the phase portrait of the non-linear system 5.6 which is linearly approximated by a centre at the origin. It can be shown that 5.6 has a spriral orbit structure at the origin by changing to polar coordinates and examining $\dot{r}$ and its sign. (b) is the phase portrait (a) magnified, and the spiral structure of the fixed point becomes less clear, as the pitch of the cycle becomes relatively smaller close to the fixed point; (c) is the phase portrait of 5.8 with a non-hyperbolic linearisation so it does not satisfy the requirements of HGLT. The linearised system is $\dot{x} = 0,\ \dot{y} = -1$. The phase portrait is easy to discern from one dimensional considerations as the system is decoupled and is a product of a 1-dimensional saddle-node on the $x-$ coordinate ($\dot{x} = -x^2$), and a stable fixed point on the $y-$ coordinate ($\dot{y} = -y$).*

5.3 Examples of non-linear systems

We consider some more examples of systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$

Example 5.5. *Consider the system with* $\mathbf{f}(\mathbf{x}) = (-x + x^3, -2y)$.(5.12)

The equations are decoupled with $\dot{x} = -x + x^3$ and $\dot{y} = -2y$. Fixed points are given by $\mathbf{x}_1^* = (0,0)$, $\mathbf{x}_2^* = (1,0)$, $\mathbf{x}_3^* = (-1,0)$. The Jacobian matrix is

$$D\mathbf{f}(\mathbf{x}^*) = \begin{bmatrix} -1+3x^2 & 0 \\ 0 & -2 \end{bmatrix} \tag{5.13}$$

$\mathbf{x}_1^*$: $D\mathbf{f}(\mathbf{x}^*)|_{\mathbf{x}_1^*} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$; Eigenvalues: $\lambda_1 = -1, \lambda_2 = -2; \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix};$

$\mathbf{x}_2^*$: $D\mathbf{f}(\mathbf{x}^*)|_{\mathbf{x}_2^*} = \begin{bmatrix} 2 & 0 \\ -0 & -2 \end{bmatrix}$; Eigenvalues: $\lambda_1 = 2, \lambda_2 = -2;\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix};\ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix};$

$\mathbf{x}_3^*$: $D\mathbf{f}(\mathbf{x}^*)|_{\mathbf{x}_3^*} = \begin{bmatrix} 2 & 0 \\ -0 & -2 \end{bmatrix}$; Eigenvalues: $\lambda_1 = 2,\ \lambda_2 = -2; \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix};\ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix};$

Example 5.6. *Consider the system with*

$$\mathbf{f}(\mathbf{x}) = (y, x - x^3). \tag{5.14}$$

Fixed points are given by $\mathbf{x}_1^* = (0,0)$, $\mathbf{x}_2^* = (1,0)$, $\mathbf{x}_3^* = (-1,0)$.
The Jacobian matrix is

$$D\mathbf{f}(\mathbf{x}^*) = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{bmatrix} \tag{5.15}$$

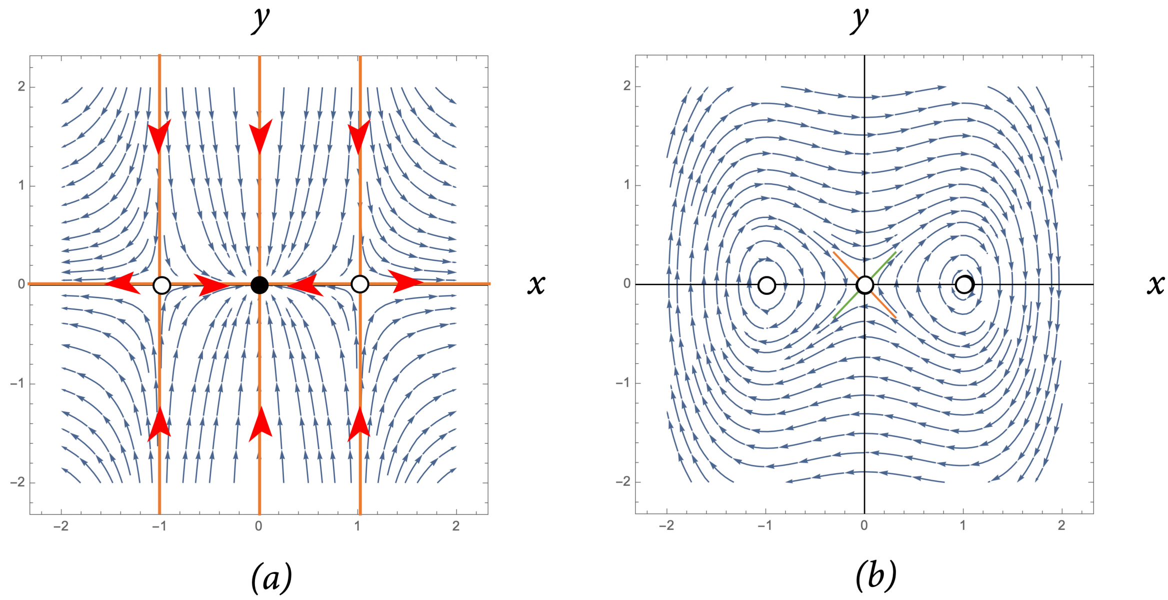


Figure 20: *(a) The phase portrait of the system 5.12. The full phase portrait in the plane shows the phase portraits of two decoupled ODES given in the system 5.12 by looking at the flows on the x and y-axes. It can be seen that the 2-dimensional phase portrait is a "product flow" of the two component 1-dimensional flows. (b) The phase portrait of 5.14, showing the eigendirections of the saddle fixed point at the origin.*

1. $\mathbf{x}_1^*$: $D\mathbf{f}(\mathbf{x}^*)|_{\mathbf{x}_1^*} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; *Eigenvalues:* $\lambda_1 = 1$, $\lambda_2 = -1$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; *saddle, HGLT applies;*
2. $\mathbf{x}_2^*$: $D\mathbf{f}(\mathbf{x}^*)|_{\mathbf{x}_2^*} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$; *Eigenvalues:* $\lambda_1 = i\sqrt{2}$, $\lambda_2 = -i\sqrt{2}$; *non-hyperbolic(centre) - HGLT does not apply ;*
3. $\mathbf{x}_3^*$: $D\mathbf{f}(\mathbf{x}^*)|_{\mathbf{x}_3^*} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$; *Eigenvalues:* $\lambda_1 = i\sqrt{2}$, $\lambda_2 = -i\sqrt{2}$; *non-hyperbolic(centre) - HGLT does not apply.*

So by using the HGLT we cannot be sure of the non-linear status of the fixed points at $(1,0)$ *and* $(-1,0)$. *Fortunately we can find integral curves for this system if we reconfigure the system equations 5.14 as*

$$dt = \frac{dx}{y} = \frac{dy}{x - x^3} \tag{5.16}$$

from which we get, by separating variables,

$$V(x,y) = \frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2} = constant \tag{5.17}$$

i.e. the quantity V *is conserved by the system 5.14 which means integral curves of the system are confined to the level curves of* V. *Of course,* $V = constant$ *curves are equivalent to contour lines or level curves (i.e* $z = constant$ *of the surface* $z = V(x,y)$ *in 3-dimensional xyz-space.*

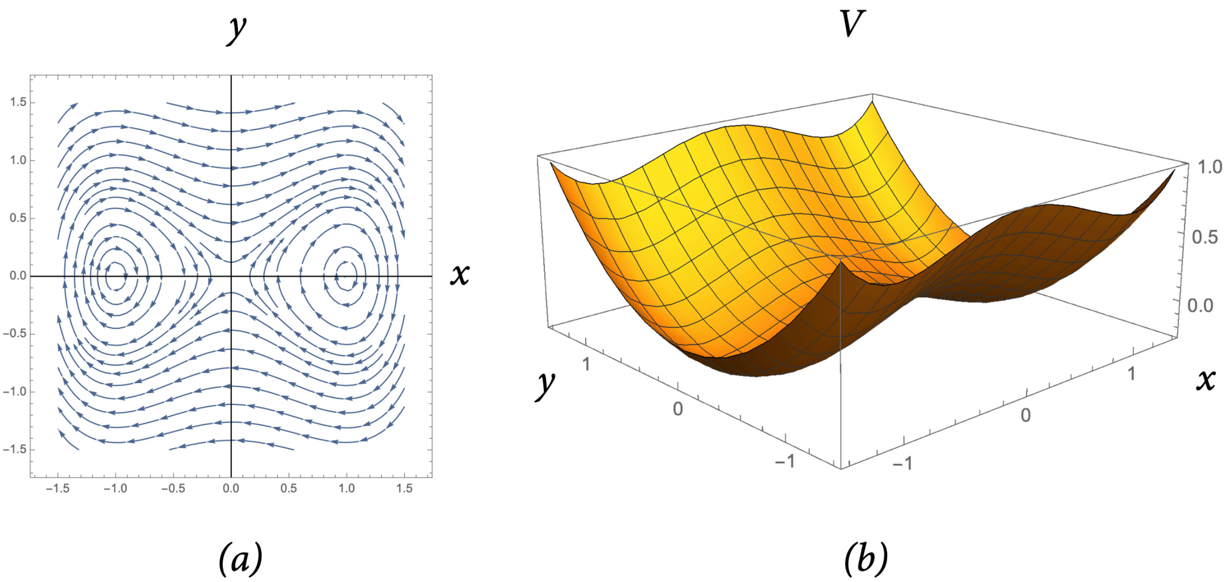


Figure 21: *(a) The phase portrait for the system 5.23 follows the level curves of the first integral V. (b) Note the 'Mathematica' picture of the surface $z = V(x,y)$ does not capture the saddle point at the origin and its unstable/stable manifolds, but it does show up the non-linear centres at $(\pm 1, 0)$ well.*

The critical(stationary) points of the surface $z = V(x,y)$ occur when the condition $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$. These are the fixed points of the underlying system. The type of critical point - maximum, minimum, saddle is determined by the discriminant

$$D = V_{xx}V_{yy} - V_{xy}^2,$$

which gives: maximum for $D > 0$ and $V_{xx} < 0$; minimum for $D > 0$ and $V_{xx} > 0$; saddle for $D < 0$.

So the critical points are $(0,0)$ - saddle and $(1,0)$, $(-1,0)$ - both minimums. Thus this means that the orbits around the fixed points $(1,0)$, $(-1,0)$ are closed (periodic orbits), and the "saddle point" or "col" of the surface confirms the existence of the saddle fixed point by the HGLT. It should be noted that the contours show the global nature of the orbits of the system. We see that the unstable and stable saddle manifolds coincide, i.e. the saddle unstable manifolds leave the fixed point $(0,0)$ and fold around to return as the stable manifolds! The contour through the saddle point has the form of a "figure-8". This is a highly non-linear feature which contrasts with the straight lines of a linear saddle.It is called a pair of saddle-connections as the unstable manifold and the stable manifolds of the saddle are the same in each of the two branches!

Example 5.7. *Consider the system with*

$$\mathbf{f}(\mathbf{x}) = (x(3 - (x + 2y)), y(2 - (x + y))) \tag{5.18}$$

The Lotka-Volterra model of competing species. The simplest model for each species with different reproductive rates and carrying capacity could be modelled by the decoupled logistic equations

$$\mathbf{f}(\mathbf{x}) = (x(3 - x), y(2 - y)). \tag{5.19}$$

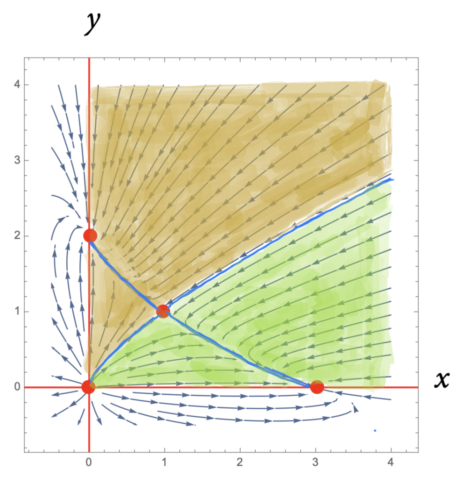


Figure 22: *(a) The phase portrait of the prey-predator system 5.20 showing the four fixed points in the positive quadrant. An approximate sketching of the stable and unstable manifolds of the saddle have also been added; (b), the phase portrait with superimposed (i) nullclines(green), (ii) the direction of flow on the nullclines(small red arrows), and (iii) eigendirections at the fixed points(purple).*

If coupling terms are introduced to model the interaction between the two species, then we obtain

$$\dot{x} = (x(3-(x+2y)), \ \dot{y} = y(2-(x+y)) \tag{5.20}$$

The fixed points are $\mathbf{x}_1^* = (0,0)$, $\mathbf{x}_2^* = (3,0)$, $\mathbf{x}_3^* = (0,2)$, $\mathbf{x}_4^* = (1,1)$. *The Jacobian matrix is*

$$D\mathbf{f}(\mathbf{x}^*) = \begin{bmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{bmatrix} \tag{5.21}$$

$\mathbf{x}_1^*$: $D\mathbf{f}(\mathbf{x}^*)|_{\mathbf{x}_1^*} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$; *Eigenvalues:* $\lambda_1 = 3$, $\lambda_2 = +2$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; *(unstable node)*

$\mathbf{x}_2^*$: $D\mathbf{f}(\mathbf{x}^*)|_{\mathbf{x}_2^*} = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}$; *Eigenvalues:* $\lambda_1 = -3$, $\lambda_2 = -1$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$; *(stable node)*

$\mathbf{x}_3^*$: $D\mathbf{f}(\mathbf{x}^*)|_{\mathbf{x}_3^*} = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}$; *Eigenvalues:* $\lambda_1 = -1$, $\lambda_2 = -2$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$; $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; *(stable node)*

$\mathbf{x}_4^*$: $D\mathbf{f}(\mathbf{x}^*)|_{\mathbf{x}_4^*} = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$; *Eigenvalues:* $\lambda_1 = -1+\sqrt{2}$, $\lambda_2 = -1-\sqrt{2}$; $\mathbf{v}_1 = \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}$; $\mathbf{v}_2 = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$ *(saddle).*

The nullcline for horizontal motion is given by $\dot{y} = 0$ *and is the set of points* (x, y) *such that either* $x + y = 2$ *or* $y = 0$ *and for vertical motion we have* $x = 0$ *and* $x + 2y = 3$. *For initial conditions in the (realistic) positive quadrant almost all orbits have limiting points at either the point* $(0, 2)$ *or* $(3, 0)$ *in which case either the* x *or the* y *population dies out respectively.*

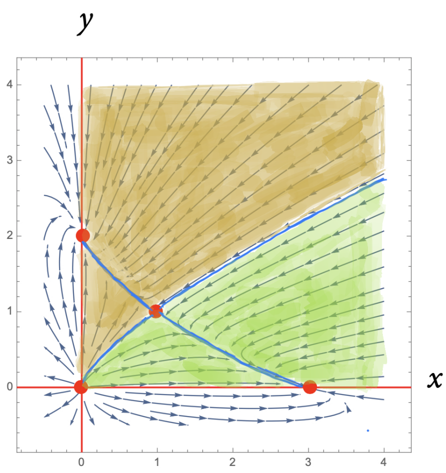


Figure 23: *The phase portrait of the prey-predator system 5.20 showing the four fixed points in the positive quadrant together with an approximate sketching of the stable and unstable manifolds of the saddle point at* $(x, y) = (1, 1)$. *The basins of attraction of the fixed points at* $(0, 2)$ *and* $(3, 0)$ *are shaded and separated by the stable manifold of the saddle fixed point with coordinates (1,1). Note that every orbit in the positive quadrant asymptotically approaches just one or the other of these two fixed points except for the stable manifold of the saddle point and the origin.*

The only initial conditions which avoid that outcome are those on the stable manifold of the saddle point $\mathbf{x}_4^* = (1, 1)$. *So, except for this special set of of initial points on the saddle stable manifold, the quadrant splits into two regions of initial conditions, A and B, which are respectively asymptotic to either the fixed point* $(0, 2)$ *(A) or* $(3, 0)$*(B), see figure 23. The regions A and B are said to be the respective basins of attraction for the fixed points.*

The nullcline for horizontal motion is given by $\dot{y} = 0$ and is the set of points (x, y) such that either $x + y = 2$ or $y = 0$ and for vertical motion we have $x = 0$ and $x + 2y = 3$. For initial conditions in the (realistic) positive quadrant almost all orbits have limiting points at either the point $(0, 2)$ or $(3, 0)$ in which case either the x or the y population dies out respectively.

The only initial conditions which avoid that outcome are those on the stable manifold of the saddle point $\mathbf{x}_4^* = (1, 1)$. So, except for this special set of of initial points on the saddle stable manifold, the quadrant splits into two regions of initial conditions, A and B, which are respectively asymptotic to either the fixed point $(0, 2)$ (A) or $(3, 0)$(B), see figure 23. The regions A and B are said to be the respective *basins of attraction* for the fixed points.

5.4 Conservative and gradient systems

A real-valued function $H : \mathbb{R}^2 \to \mathbb{R}$ is a *constant of the motion* or *first integral* of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$, and $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ if H is constant for any solution curve, i.e. $H(\mathbf{x}(t)) \equiv H(\mathbf{x}(0))$ for all $t \in \mathbb{R}$. The "trivial constant" of the motion that $H(x, y) \equiv C$, a constant, on an *open* set in $\mathbb{R}^2$ is not allowed as it offers no information on the nature of the solution curves.

For $H(\mathbf{x})$, where $\mathbf{x} = (x, y)$ to be a constant of the motion, we require

$$\frac{d}{dt}(H(\mathbf{x}(t)) \equiv 0,$$

which implies

$$\frac{d}{dt}(H(\mathbf{x}(t)) = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} \equiv 0.$$

5.4.1 Conservative systems

Newton's second law of motion, (II), states that the force applied to an object is proportional to its acceleration, with the constant of proportionality being the mass of the object. This law can be written as an ODE in the form

$$m\ddot{x} = F(x)$$

where x is the position coordinate, m is the mass, and $F(x)$ is the force applied at the position x. Converting this second order ODE into a first order equation, we obtain

$$\dot{x} = y; \ \dot{y} = \frac{1}{m}F(x).$$

Define the function $E : \mathbb{R}^2 \to \mathbb{R}$ by

$$E(\mathbf{x}) = E(x, y) = \frac{1}{2}my^2 + V(x), \tag{5.22}$$

where

$$V(x) = -\int F(x)dx.$$

Now

$$\frac{d}{dt}(E(\mathbf{x}(t)) = \frac{\partial E}{\partial x}\dot{x} + \frac{\partial E}{\partial y}\dot{y} = -F(x)y + my\dot{y} = y(-F(x) + m\ddot{x}) \equiv 0,$$

by Newton II.

The function E given in 5.22, the *energy*, is a constant of the motion for Newton II. Energy is conserved in this system - i.e. it is a *conservative system*. The energy E is seen as being comprised of two components: $\frac{1}{2}my^2$ is the *kinetic* energy; $V(x)$ is the *potential* energy

Theorem 5.3. *Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$, be a conservative system, with constant of motion H, then the system has no attracting points.*

Proof. If there were to exist a neighbourhood U of a fixed point $\mathbf{x}^*$ of the system for which every solution $\mathbf{x}(t)$ with $\mathbf{x}_0 \in U$ satisfied $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$, then by continuity of H, $H(\mathbf{x}(t)) \to H(\mathbf{x}^*)$, as $t \to \infty$.

Since $H(\mathbf{x}(t))$ is constant as t varies, it follows that $H(\mathbf{x}(0)) = H(\mathbf{x}^*)$ on the neighbourhood U of x^*, i.e. $H(\mathbf{x}) = H(\mathbf{x}^*)$ for all $x \in U$ and is therefore a *trivial* constant of the motion.

Example 5.8. *Consider a Newton II system with potential energy* $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$. *The system we obtain using the equation 5.22 is*

$$\dot{x} = y, \; \dot{y} = x - x^3 \; \left(= -\frac{\partial V}{\partial x}\right), \tag{5.23}$$

which we considered earlier in equation 5.14. We have chosen $m = 1$ *as its numerical value does not change the qualitative behaviour of the system. The system has fixed points at* $(0,0)$ *- a saddle, and* $(\pm 1, 0)$*, are both centres. See figure 21.*

5.4.2 Gradient systems

Consider a differentiable function $F : \mathbb{R}^2 \to \mathbb{R}$. The *gradient system* with potential function F is

$$\dot{\mathbf{x}} = -\nabla F(\mathbf{x}), \tag{5.24}$$

where $\nabla F(\mathbf{x}) = \left(\frac{\partial F}{\partial x}(\mathbf{x}), \frac{\partial F}{\partial y}(\mathbf{x})\right)$. The fixed points of the gradient system are precisely those points for which $\nabla F(\mathbf{x}) = \mathbf{0}$, the so called *critical points* of F.

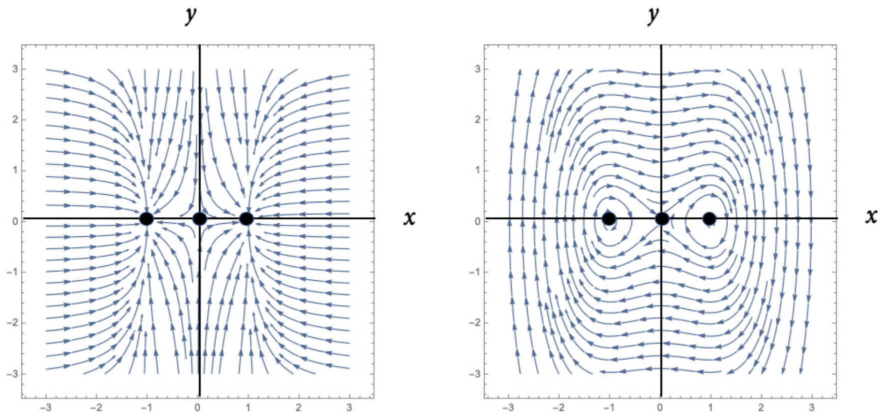


Figure 24: *The phase portraits of (a) the gradient system 5.24 and (b) the corresponding conservative system 5.23 with the same potential function* $E(x,y) = \frac{1}{2}my^2 + \frac{1}{4}x^4 - \frac{1}{2}x^2$.

Theorem 5.4. *A gradient system has no periodic orbits of positive period* $T > 0$.

Proof. If such a periodic orbit $\mathbf{x}(t)$ for $0 \leq t \leq T$ existed for the system 5.24, then the change in the value ΔF of F would be zero since $F(\mathbf{x}(0)) = F(\mathbf{x}(T))$, given $\mathbf{x}(0) = \mathbf{x}(T)$. But

$$\Delta F = \int_0^T \frac{dF}{dt} dt = \int_0^T \nabla F(\mathbf{x}).\frac{d\mathbf{x}(t)}{dt} dt = -\int_0^T ||\dot{\mathbf{x}}||^2 dt < 0.$$

which provides a contradiction.

We should note that this is the very opposite (or, perhaps, *orthogonal*!) to the behaviour of a conservative system.

Example 5.9. *Prove that a conservative system with energy* $E(x,y) = \frac{1}{2}my^2 + V(x)$*, and a gradient system* $\dot{\mathbf{x}} = -\nabla E(\mathbf{x})$*, with the same energy* E *have orthogonal trajectories in their respective phase portraits, i.e. the vectors fields of the two systems are mutually orthogonal (hint: expand* $\frac{dE}{dt}$

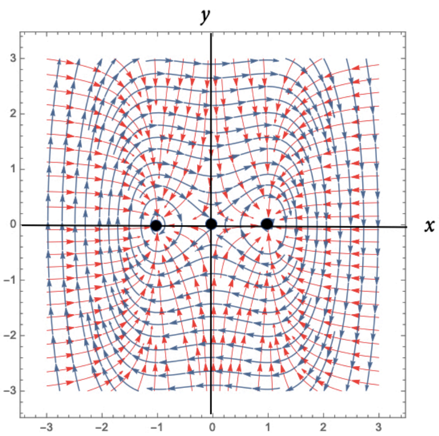


Figure 25: *The conservative (blue) and gradient (red) systems with the same potential, E - illustrated separately in figure 24 - are superimposed to show the mutual orthogonality of the flows (the dot product of the vector fields at every point of $\mathbb{R}^2$ is zero).*

and interpret the expression as a dot product of vectors). Note that the gradient system is now seen in the context of the "total" potential energy being $E(x,y)$, not just the potential energy $V(x,y)$ of the mixed energy $E(x,y)$ in equation 5.22.

Proof. Given the potential energy function $V(x)$, the total energy function for the corresponding conservative system is

$$E(x,y) = \frac{1}{2}my^2 + V(x),$$

cf. 5.22 with system equations

$$\dot{x}_C = y,\ \dot{y}_C = -\frac{V'(x)}{m}$$

The corresponding gradient system for the energy function $E(x,y)$ has the form

$$\dot{x}_G = -\frac{\partial E}{\partial x} = -V'(x);\ \dot{y}_G = -\frac{\partial E}{\partial y} = -my.$$

It follows that the dot product of the two vector fields is

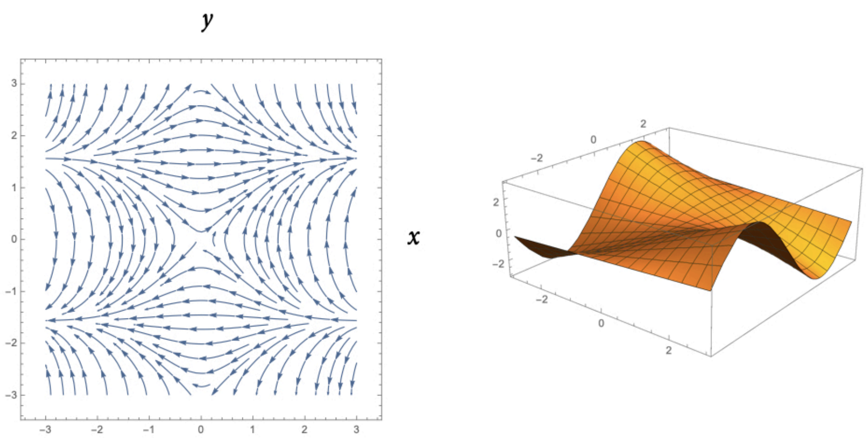
$$(\dot{x}_G,\dot{y}_G).(\dot{x}_C,\dot{y}_C) = \dot{x}_G.\dot{x}_C + \dot{y}_G.\dot{y}_C = (-V'(x)).y + (-my).(-\frac{V'(x)}{m}) \equiv 0. \tag{5.25}$$

Therefore, the conservative and gradient vector fields are orthogonal, see figure 25. This means that gradient systems follow the lines of maximum slope which are always perpendicular to level curves, a good direction to avoid when walking down a mountain!

A more general approach to proving orthogonality is the use the fact that energy $E = E(x,y)$ is conserved for conservative(C) systems, so that

$$\frac{dE}{dt} = \frac{\partial E}{\partial x}\frac{dx_C}{dt} + \frac{\partial E}{\partial y}\frac{dy_C}{dt} = -(\dot{x}_G.\dot{x}_C + \dot{y}_G\dot{y}_C) \equiv 0, \tag{5.26}$$

since $(\dot{x}_G,\dot{y}_G) = (-\frac{\partial E}{\partial x}, -\frac{\partial E}{\partial y})$. Interpreting the dot product $\dot{x}_G.\dot{x}_C + \dot{y}_G\dot{y}_C = 0$ of equation 5.26 geometrically implies that the vector fields $(\dot{x}_G,\dot{y}_G)$ and $(\dot{x}_C,\dot{y}_C)$ are orthogonal for any energy function.



(a) (b)

Figure 26: *(a) The phase portrait of the system 5.27 , and (b) the corresponding surface plot, S, of the potential $F(x, y) = -x\sin(y)$. Note that the vector field is not following the level curves of S, but the perpendicular directions of steepest slope.*

Example 5.10. *Consider the following systems:*

(i) The system

$$\dot{x} = \sin(y); \;\; \dot{y} = x\cos(y) \tag{5.27}$$

is a gradient system with potential function $F(x, y) = -x\sin(y)$. So the system has no periodic orbits, see figure 26.

(ii) The system obtained from the ODE $\ddot{x} + \dot{x}^3 + x = 0$ has no periodic solutions. The corresponding system is $\dot{x} = y$, $\dot{y} = -x - y^3$. If the system is of gradient type, thenit could be concluded that there are no periodic orbits. This is not a gradient system: if a potential $F(x, y)$ existed, then $\frac{\partial F}{\partial x}(\mathbf{x}) = -y$ and $\frac{\partial F}{\partial y}(\mathbf{x}) = x + y^3$, but then we would have

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right), \tag{5.28}$$

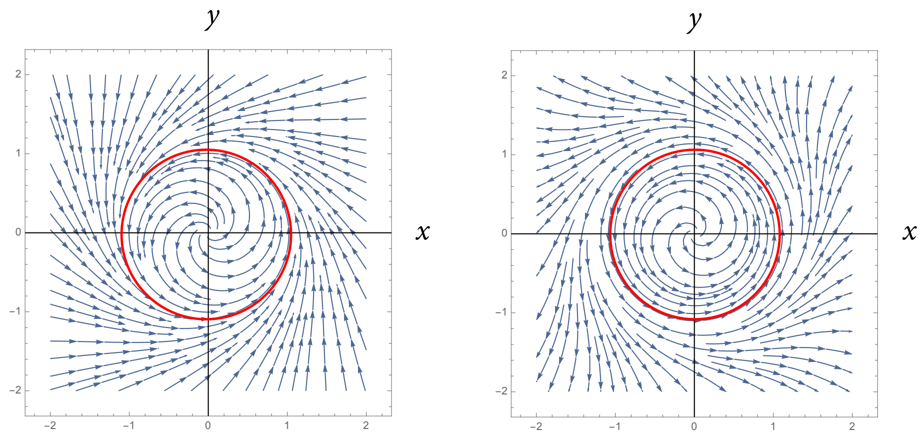
which is not true for this system. The condition 5.28 is, in fact, a necessary and sufficient condition for the existence of the potential function F. Hence, we need to show the existence of non-periodic solutions in a different way. Note that the linearised system at the origin is $\dot{x} = y$; $\dot{y} = -x$, which has energy $E(x, y) = \frac{1}{2}(x^2 + y^2)$ which invites the possibility of an investigation using polar coordinates. Calculating $\frac{dE}{dt}$, we obtain $\frac{dE}{dt} = x\dot{x} + y\dot{y} = -y^4 \leq 0$. It follows that $\Delta E = \int_0^T \dot{E}dt = 0$ only if $y(t) \equiv 0$ along an orbit. But,

$$y(t) \equiv 0 \implies \dot{x}(t) \equiv 0 \text{ and } \dot{y}(t) \equiv 0,$$

which means that the orbit is a fixed point and is not a period orbit.

5.5 Limit cycles

We now consider the phenomenon of an *isolated periodic orbit or closed curve*, called a limit cycle. This is to be compared with many examples already considered of infinite families



(a) *(b)*

Figure 27: *Phase portraits: (a) for the system 5.29 - an unstable spiral fixed point at the origin, and a stable limit cycle on the circle $r = 1$; (b) for the system 5.30 - an unstable spiral fixed point at the origin, and a limit cycle on the circle $r = 1$ which is stable from the region $0 < r < 1$ and unstable into the region $1 < r$.*

of concentric closed curves around a centre-type fixed point such as arise in conservative systems.

Definition 5.4. *A limit cycle L is an isolated periodic orbit, which is* **N***OT a fixed point. Isolated means there exists a neigbourhood set $U \supset L$ which does not contain any other closed orbits.*

Simple examples of phase portraits with limit cycles are most easily described using polar coordinates.

Example 5.11. *Both of the following examples have a limit cycle formed by the circle $r = 1$. Noting the signs of $\dot{r}$ show that (i) has a stable limit cycle, and (ii) has a limit cycle which is stable from the "inside" region, $r < 1$, and unstable on the "outside" region, $r > 1$.*

(i)

$$\dot{r} = r(1-r), \ \dot{\theta} = 1. \tag{5.29}$$

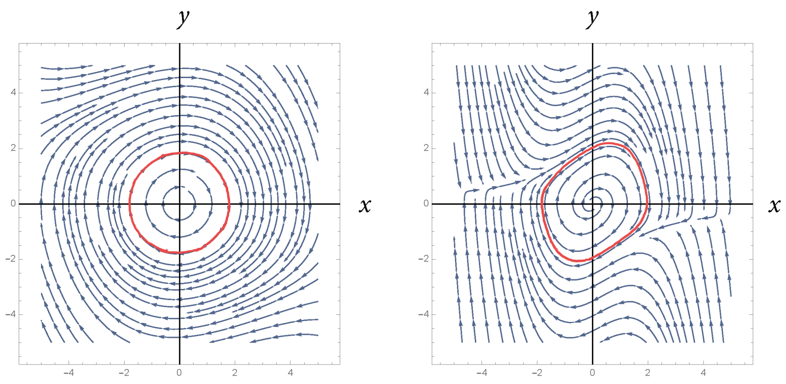
Note $\dot{r} > 0$ for $0 < r < 1$, and $\dot{r} < 0$ for $1 < r$.

(ii)

$$\dot{r} = r(1-r)^2, \ \dot{\theta} = 1. \tag{5.30}$$

Note $\dot{r} > 0$ for $0 < r < 1$, and for $1 < r$.

The existence of a limit cycle which is stable or attracting means that the system over longer time will settle down to a specific oscillation for annular neighbourhood of initial conditions which contains the cycle. This cyclic behaviour can often be a very important feature to observe and understand and maintain, as many biological, physiological, physical, chemical, industrial processes, and electronic devices are usually driven and sustained by a stable periodic process which is exhibited by a stable limit cycle.



(a) *(b)*

Figure 28: *The Van der Pol oscillator for (a)* $\mu = 0.05$*, and (b)* $\mu = 0.5$*. For (a), the phase portrait, although very qualitatively different from a centre with the limit cycle present, is, nevertheless, quantitatively close to the flow on concentric circles which occurs for* $\mu = 0$*. In (b), the phase portrait is more distorted from the circular form it takes for* μ *very small, but the limit cycle is now more apparent.*

Example 5.12. ***Van der Pol Oscillator*** *This is a celebrated system discovered by the Dutch engineer in 1920. It modelled fluctuations and has been widely used in physics, engineering and biological modelling. Called the Van der Pol Oscillator, it has the form of a second order ODE in one real variable* x *as:*

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0,$$

where $\mu \in \mathbb{R}$ *is a parameter. The corresponding first order system in 2 variables,*

$$\dot{x} = y, \; \dot{y} = -x - \mu(x^2 - 1)y,$$

has a fixed point at the origin $(x, y) = (0, 0)$*. The Jacobian matrix is*

$$D\mathbf{f}(\mathbf{x}^*) = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix},$$

where $\mathbf{f}(\mathbf{x}) = (y, -x - \mu(x^2 - 1)y)$*, with eigenvalues* $\lambda = (\mu \pm \sqrt{(\mu^2 - 4)})/2$*.*

Therefore, by HGLT, we have spirals for $|\mu| < 2$ *which are unstable for* $\mu > 0$ *and stable for* $\mu < 0$*. It can also be shown that orbits with initial values at sufficiently large radial distance* r *spiral inwards which are then met by orbits spiralling out from the origin. The resulting collision of competing orbits is resolved by the existence of a stable limit cycle, see figure 28.*

5.6 Liapounov Functions

Liapounov functions focus on a generalisation of the sort of argument in the previous example where polar coordinates were used. Reconsider the system

$$\dot{x} = y - x(x^2 + y^2); \; \dot{y} = -x - y(x^2 + y^2),$$

which has a linearised centre fixed point at the origin $\dot{x} = y$; $\dot{y} = -x$. Consider the function $L(x, y) = x^2 + y^2$. The level curves are concentric circles centred on the origin. It follows $\frac{dL}{dt}(t) = 2x\dot{x} + 2y\dot{y} = -2(x^2 + y^2)$, is negative for all $(x, y) \neq \mathbf{0}$.

Therefore $L(\mathbf{x}(t))$ decreases with time as t increases, and so the distance of $\mathbf{x}(t)$ to the origin decreases with time t. This already shows that the origin is a stable fixed point. In fact $\dot{r} = -r^3$, and $\dot{\theta} = 1$ and so we explicitly know that the origin is a stable spiral (with a non-linear (cubic) radial contraction!).

We now formalise this idea for a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with a fixed point at $\mathbf{x}^* = \mathbf{0}$.

Definition 5.5. *Let U be an open set of $\mathbb{R}^2$ containing the origin. A real-valued C^1 function $L : U \to \mathbb{R}$ is said to be positive definite (PD) on U if*

(i) $L(0, 0) = 0$,

(ii) $L(\mathbf{x}) > 0, \forall\ \mathbf{x} \in U \setminus \{(0, 0)\}$.

Definition 5.6. *The function $L : U \to \mathbb{R}$ is said to be negative definite (ND) if $-L$ is positive definite.*

Lemma 5.1. *The quadratic function $L(x, y) = ax^2 + bxy + cy^2$, $a, b, \in \mathbb{R}$, is positive definite on $U = \mathbb{R}^2$ iff $a > 0$, $b^2 - 4ac < 0$.*

Proof Suppose L is PD, then $L(1, 0) = a$, and so $a > 0$. By completing the square on x, we see

$$L(x, y) = a\left(x + \frac{b}{2a}y\right)^2 + \left(c - \frac{b^2}{4a}\right)y^2. \tag{5.31}$$

Choose $x = -\frac{b}{2a}$ and $y = 1$ to obtain

$$L\left(-\frac{b}{2a}, 1\right) = c - \frac{b^2}{4a} > 0,$$

which implies $b^2 - 4ac < 0$, since $a > 0$.

The reverse argument is trivial; using equation 5.31, $L(x, y) \geq 0$ and if we impose $L(x, y) = 0$, we have $x + \frac{b}{2a}y = 0$ and $y = 0$ which implies $(x, y) = 0$, so the function L is PD.

Definition 5.7. *A positive definite function L defined on an open neighbourhood U of the origin is said to be a **Liapounov function** for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ if $\frac{dL}{dt}(\mathbf{x}) \leq 0$ for $\forall\ \mathbf{x} \in U \setminus \{\mathbf{0}\}$.*

Definition 5.8. *A positive definite function L defined on an open neighbourhood U of the origin is said to be a **strict Liapounov function** for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ if $\frac{dL}{dt}(\mathbf{x}) < 0$ (ND) for $\forall\ \mathbf{x} \in U \setminus \{\mathbf{0}\}$.*

Theorem 5.5. *Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ have a fixed point at $\mathbf{x}^* = \mathbf{0}$, then*

(i) the origin $\mathbf{x}^*$ is stable, (also referred to as Liapounov stable) if a Liapounov function exists,

(ii) the origin $\mathbf{x}^*$ is asymptotically stable if a strict Liapounov function exists.

Example 5.13. *Construct a Liapounov function for the system*

$$\dot{x} = -x + 4y,\ \dot{y} = -x - y^3,$$

to show the asymptotic stability of the fixed point at the origin.

Consider the function $L(x,y) = x^2 + ay^2$. For any $a > 0$, L will be PD. The choice of a needs to be further refined to ensure $\frac{dL}{dt}(t)$ is ND. We obtain

$$\frac{dL}{dt}(x,y) = 2x\dot{x} + 2ay\dot{y} = 2x(-x+4y) + 2ay(-x-y^3) \tag{5.32}$$

$$= -2x^2 + 2(4-a)xy - 2ay^4. \tag{5.33}$$

By choosing $a = 4$, we obtain $\frac{dL}{dt}(x,y) = -2x^2 - 8y^4$ which is ND . It follows that L is a strict Liapounov function and by theorem 5.5, and the fixed point $\mathbf{x}^* = \mathbf{0}$ is asymptotically stable.

5.6.1 Poincaré-Bendixson Theorem

Definition 5.9. *For a planar system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, a point $\mathbf{y}$ is an element of the ω-limit set of the orbit $\mathbf{x} = \mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ if there exists a sequence t_n with $\lim_{n\to\infty} t_n = \infty$ such that $\lim \mathbf{x}(t_n) \to \mathbf{y}$ as $n \to \infty$. For the α-limit set, $\lim_{n\to\infty} t_n = \infty$ is replaced by $\lim_{n\to\infty} t_n = -\infty$.*

We consider the possible ω-limit sets $\omega = \omega(\mathbf{x}_0)$, where x_0 is the initial condition determining the orbit.

Example 5.14. *Consider the ω-limit of the following systems:*

(i) $\dot{r} = -r$, $\dot{\theta} = 1$. *All orbits converge to the origin which is a fixed point, i.e.* $\dot{r} = -r$, $\dot{\theta} = 1$. *All orbits converge to the origin which is a fixed point, i.e.* $\omega(\mathbf{x}_0) = \{(0,0)\}$.

(ii) $\dot{r} = r(1-r)$, $\dot{\theta} = 1$. *All orbits, apart from the origin, approach the unit circle, so there are two possible limit sets i.e.* $\omega(\mathbf{0}) = \{(0,0)\}$ *and* $\omega(\mathbf{x}_0) = \{(x,y) | x^2 + y^2 = 1\}$ *for* $\mathbf{x}_0 \neq \mathbf{0}$.

(iii) $\dot{r} = r(1-r)$, $\dot{\theta} = 1 - \cos(\theta) + (r-1)^2$. *There are just two fixed points* $\mathbf{x}_1^* = (0,0)$ *and* $\mathbf{x}_2^* = (1,0)$ *All orbits on the unit circle* Γ *approach* $\mathbf{x}_2^*$, *so* $\omega(\Gamma) = \{\mathbf{x}_2^*\}$. *A more delicate investigation is needed to see that* $\mathbf{x}_2^*$ *is a saddle node fixed point with a stable eigen-direction on the x-axis, and a saddle node eigen-direction tangent to the circle. We are able to conclude that* $\omega(\mathbf{x}_0) = \mathbf{x}_2^*, \forall \mathbf{x}_0 \in \mathbb{R}^2 \setminus \{\mathbf{x}_1^*)\}$, *see figure 30.*

The following theorem describes the general case.

Theorem 5.6. ***(Poincaré-Bendixson)*** *Suppose that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a planar system with a finite number of fixed points. If the positive orbit $\mathbf{x}_0^+ = \{\mathbf{x}(t,\mathbf{x}_0);\ t \geq 0\}$, where $\mathbf{x}(0,\mathbf{x}_0) = \mathbf{x}_0$, is bounded, then one of the following is true.*

The ω-limit set $\omega(\mathbf{x}_0)$ is a

(a) *single point* $\mathbf{x}^*$, *which is a fixed point, and* $\mathbf{x}(t,\mathbf{x}_0) \to \mathbf{x}^*$ *as* $t \to \infty$.

(b) *periodic orbit* Γ *and either* $\mathbf{x}_0^+ = \Gamma$, *or* $\mathbf{x}_0^+$ *spirals towards* Γ *on one side of* Γ.

(c) *union of fixed points and orbits whose* $\alpha-$ *and* $\omega-$ *limit sets are fixed points. Such orbits are known as heteroclinic and homoclinic orbits, c.f. figure 29.*

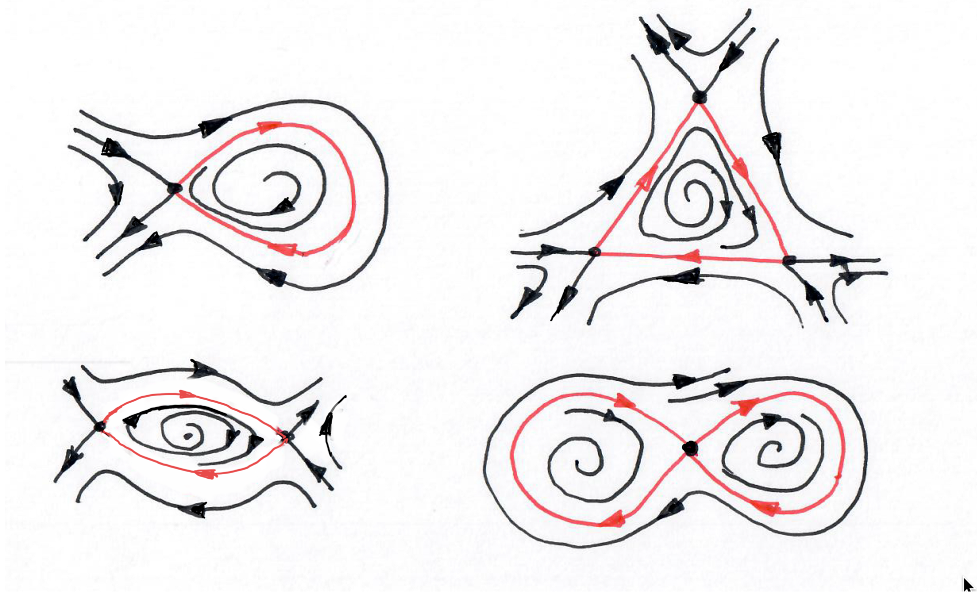


Figure 29: *Some examples of the α and ω-limit sets in phase portraits for case (c) of the Poincaré-Bendixson Theorem. The closed orbits, illustrated by the red curves, consist of a finite set of heteroclinic or homoclinic orbits (these are orbits which connect saddle points).*

To use PBT to exhibit a limit cycle, we may attempt to construct an open bounded set which contains no fixed points and such that all orbits with initial conditions in D remain in D for all time. We also have to show that for $\mathbf{x}_0 \in D$, $\omega(\mathbf{x}_0)$ contains no point on the boundary of D.

Example 5.15. *Consider*

$$\dot{x} = y; \;\; \dot{y} = -x + y(1 - x^2 - 2y^2). \tag{5.34}$$

We note that the origin is the only fixed point. We wish to find a positively time invariant annular region $\mathcal{A}$. Consider the function $L(x, y) = \frac{1}{2}(x^2 + y^2)$. We compute the derivative of L along the orbits of our system:

$$\frac{dL}{dt}(t) = (x, y) \cdot (y, -x + y(1 - x^2 - 2y^2)) = y^2(1 - x^2 - 2y^2).$$

For $\frac{dL}{dt}(t) > 0$, we require $x^2 + 2y^2 < 1$ which is implied by $2x^2 + 2y^2 < 1$ or $x^2 + y^2 < \frac{1}{2}$.

For < 0, we require $x^2 + 2y^2 > 1$ which is implied by $x^2 + y^2 > 1$. Thus the annulus $\mathcal{A}$ given by $\frac{1}{2} - \epsilon < x^2 + y^2 < 1 + \epsilon$ for sufficiently small $\epsilon > 0$ has the desired properties. Furthermore, the the fixed point at the origin is exterior to this region. So PBT applies and the system 5.34 has a ***trapping region****, $\mathcal{A}$, given by the annular region bounded by the circles $r = \sqrt{(}1/2 - \epsilon)$ and $r = \sqrt{(}1 + \epsilon)$ which contains at least one periodic orbit. On the boundary circles all orbits enter the region $\mathcal{A}$ with increasing time.*

Example 5.16. *Consider the polar system*

$$\dot{r} = r(1 - r^2) + \mu r \cos(\theta), \dot{\theta} = 1, \tag{5.35}$$

For $\mu = 0$, there is a stable limit cycle ($r = 1$). We want to show that a limit cycle still exists for $0 < \mu < 1$. We again construct an annular trapping region bounded by two circles $r = r_{min}$

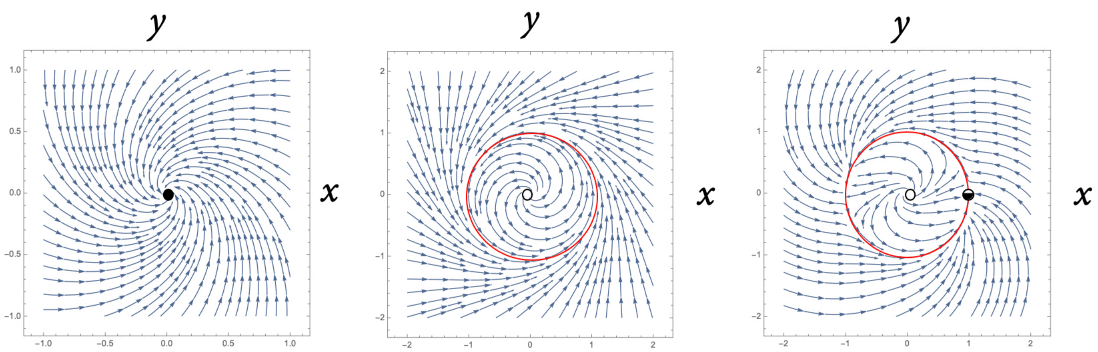
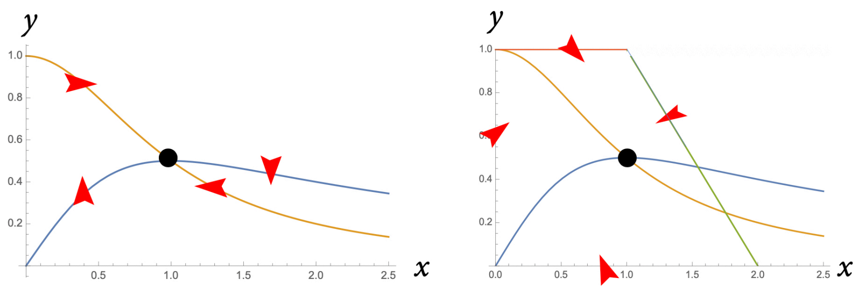


Figure 30: *The ω-limit sets for the three systems of example 5.14. (i) A fixed point at the origin; (ii) a stable limit cycle of radius 1; (iii) the fixed point $(1,0)$, and the circle $r=1$.*



(i) *(ii)*

Figure 31: *(i) the isoclines of the system 5.36 for the choice of $a=b=1$ with the appropriate horizontal and vertical flow directions ; (ii) the indicative direction of the flow on the quadrilateral with all flow directions pointing to the interior of the region.*

and $r=r_{max}$ such that $\dot{r}<0$ on the outer circle, and $\dot{r}>0$ on the inner circle, so that orbits entering the annulus are trapped there, as none can escape. If we can find such circles then, by PBT, there exists at least one limit cycle as $\dot{\theta}\neq 0$ in the annulus and therefore there are no fixed points. To find r_{min}, we require $r(1-r^2)+\mu r\cos(\theta)>0$. Since $\cos(\theta)\geq -1$, we require $r(1-r^2)-\mu r=r(1-\mu-r^2)>0$ and since $r>0$, we find $r_{min}<\sqrt{(1-\mu)}$, which is real for $\mu<1$. Likewise we find $r_{max}>\sqrt{(1+\mu)}$.

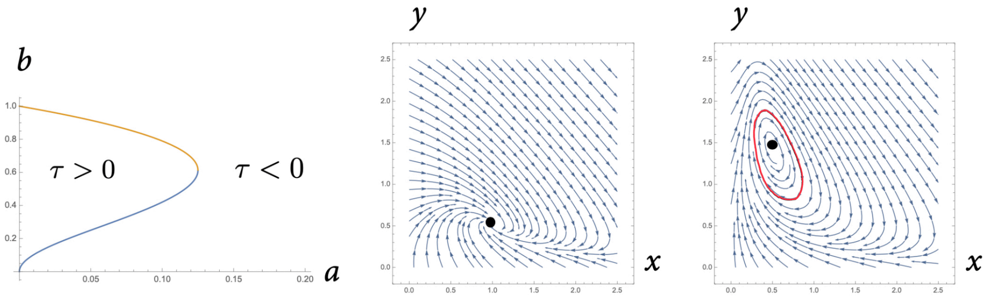
Example 5.17. *A dynamic modelling of a biological system has the form*

$$\dot{x}=-x+ay+x^2y;\ \ \dot{y}=b-ay-x^2y, \tag{5.36}$$

where $x,y\geq 0$, and $a>0$, $b>0$. We need to find a trapping region for this system.

nullclines are : $\dot{x}=0$: $y=\frac{x}{a+x^2}$; $\dot{y}=0$: $y=\frac{b}{a+x^2}$, see figure 31, and the unique fixed point is $\mathbf{x}^*=(b,b/(a+b^2)$. We investigate a quadrilateral trapping region D consisting of the axes, a line $x+y=C$, for an appropriate constant C, and a horizontal line, see figure 31. On the x-axis we have $\dot{y}=b>0$; on the y-axis, we have $\dot{x}=ay>0$ for $y>0$. To check the direction of flow on the line $x+y=C$, note a dot product of the vector field with inward normal vector gives

$$(\dot{x},\dot{y})\cdot(-1,-1)=x-ay-x^2y-b+ay+x^2y=x-b>0$$



(i) *(ii)* *(iii)*

Figure 32: *(i) The curve defined by $\tau = 0$ in the $ab-$ parameter plane given by the equation 5.38, and the regions for which $\tau > 0$ and $\tau < 0$; (ii) the phase portrait for equation 5.36 with $a = b = 1$ where $\tau < 0$ and the phase portrait has a stable spiral fixed point; (iii) the phase portrait for equation 5.36 with $a = 0.1$ and $b = 0.5$ for which $\tau > 0$. A limit cycle arises using PBT as a result of the unstable spiral fixed point at $\mathbf{x}^* = (0.5, \frac{10}{7})$, and the trapping region is shown in figure 31.*

which requires $x > b$ for flow to be inwards. We choose $C_0 = b + b/a$.

The requirement $x > b$ on the line $x + y = C$ necessitates that we contsruct a further segment connecting the point $(0, b/a)$ and the point $(b, b/a)$ on the line $x + y = C_0$. Note $\dot{x} = -x + b + x^2 b/a$; $\dot{y} = -x^2 b/a \leq 0$. To apply PBT we must now exclude the fixed point and a small disc around it, from D to obtain an annular region. This is only possible if the fixed point $\mathbf{x}^* = (b, \frac{b}{a+b^2})$ is unstable to form an annular trapping region. We compute the Jacobian at the general point (x, y) and then at the fixed point $\mathbf{x}^* = (b, b/(a^2 + b))$, to obtain

$$D\mathbf{f}(\mathbf{x}) = \begin{bmatrix} -1 + 2xy & a + x^2 \\ -2xy & -(a + x^2) \end{bmatrix} ; \quad \text{and } D\mathbf{f}(\mathbf{x}^*) = \begin{bmatrix} \frac{-a+b^2}{a+b^2} & a + b^2 \\ -\frac{2b^2}{a+b^2} & -(a + b^2) \end{bmatrix}, \tag{5.37}$$

with $\delta = a + b^2 > 0$ and,

$$\tau = -\frac{b^4 + b^2(2a - 1) + (a + a^2)}{a + b^2}. \tag{5.38}$$

For instability, we need $\tau > 0$, and the boundary line $\tau = 0$ between the regions $\tau > 0$ and $\tau < 0$ is given by equation 5.38, see figure 32.

5.6.2 Dulac's Criterion

Theorem 5.7. ***(Dulac)*** *Let $U \subseteq \mathbb{R}^2$ be a simply connected open set and let $B : U \to \mathbb{R}$ be a C^1 function, and let $\mathbf{f}(\mathbf{x})$ be a vector field on U with components f_1 and f_2. If the function*

$$\nabla(B(\mathbf{f})v(\mathbf{x})) = \frac{\partial(B(\mathbf{x})f_1(\mathbf{x}))}{\partial x} + \frac{\partial(B(\mathbf{x})f_2(\mathbf{x}))}{\partial y}$$

is of constant sign and not identically zero in U, then $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has no periodic orbit lying entirely in U. The function B is called a ***Dulac*** *function.*

Proof of criterion. Suppose there is a periodic orbit C in U. Let A be the bounded region whose boundary is C. *Green's theorem* relates an integral around the perimeter curve C to a double integral on the region A by

$$\int_A \nabla(B\mathbf{f})d\mathbf{x} = \oint_C B\mathbf{f}.\mathbf{n}dl$$

where $\mathbf{n}$ is the outward normal to the curve C which has length parameter l. The integral on the left is non-zero since $\nabla(B\mathbf{f})$ has only one sign in A. The integral on the right is zero because the vector field $\mathbf{f}$ is tangent to the curve C at every point and is always normal to $\mathbf{n}$ giving a zero dot-product. The special case $B(x, y) \equiv 1$ is known as *Bendixson's* criterion.

Example 5.18. *We show that the system* $\dot{x} = x(2 - x - y)$, $\dot{y} = y(4x - x^2 - 3)$ *has no closed orbit in the first quadrant. Choose* $B(x, y) = \frac{1}{xy}$. *We find, for* $x, y > 0$ *that*

$$\nabla(B\mathbf{v}) = \frac{\partial}{\partial x}\left(\frac{2 - x - y}{y}\right) + \frac{\partial}{\partial y}\left(\frac{4x - x^2 - 3}{x}\right) = -\frac{1}{y}.$$

which is of constant sign for $y > 0$.