Queen Mary
University of London

Main Examination period 2022 – January – Semester A

MTH786: Machine learning with Python

Duration: 4 hours

The exam is available for a period of **4 hours**, within which you must complete the assessment and submit your work. **Only one attempt is allowed – once you have submitted your work, it is final**.

All work should be **handwritten** and should **include your student number**.

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Examiners: N. Perra

Question 1 [45 marks].

Consider the following matrix $\mathbf{A} = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}$ where $a \in \mathbb{R}$.

(a) Compute the eigenvalues and correspondent eigenvectors of $\mathbf{A}$. **[10]**

(b) Compute the inner product of the eigenvectors of $\mathbf{A}$. What do we learn from it? **[5]**

(c) Consider the following matrix $\mathbf{B} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$ compute its singular values. **[10]**

(d) Compute the left and right singular vectors of the matrix $\mathbf{B}$. **[10]**

(e) Consider a matrix $\mathbf{D} \in \mathbb{R}^{m,n}$, a vector $\mathbf{x} \in \mathbb{R}^{n,1}$, and the function $E(\mathbf{x}) = \frac{1}{2}\|\mathbf{D}\mathbf{x}\|^2$. Show that $\nabla E(\mathbf{x}) = \mathbf{D}^\top \mathbf{D}\mathbf{x}$. **[10]**

Question 2 [25 marks].

Consider the following data samples $\left(x^{(1)}, y^{(1)}\right) = (1, 0)$, $\left(x^{(2)}, y^{(2)}\right) = (2, 1)$, $\left(x^{(3)}, y^{(3)}\right) = (3, 2)$

(a) Write down, in explicit matricial form, the normal equation assuming a simple linear model. **[5]**

(b) Determine the solution of the normal equation. **[5]**

(c) Let us now assume that you made some errors measuring the output variables $y^{(i)}$ with $i \in \{1, 2, 3\}$. The perturbed measurements $\mathbf{y}_\delta$ read $y_\delta^{(1)} = \epsilon$, $y_\delta^{(2)} = 1 + \epsilon$ and $y_\delta^{(3)} = 2 - \epsilon$. Determine the solution of the normal equation considering these perturbed samples and considering the same initial data matrix. **[5]**

(d) Compute the error between $\hat{\mathbf{w}}$ and $\hat{\mathbf{w}}_\delta$ in the Euclidean norm. **[5]**

(e) Compare the error computed in the previous question (i.e., question d) with the data error $\delta := \|\mathbf{y} - \mathbf{y}_\delta\|$. **[5]**

Question 3 [30 marks].

(a) Consider the following probability density function of a (continuous) random variable x: $p(x|\alpha) = Ax^{-\alpha}$ for $x \geqslant 1$ where $A \in \mathbb{R}$ and $\alpha \in \mathbb{R}$. Compute the value A as function of α and discuss where it is defined. **[10]**

(b) Compute the expectation value $\mathbb{E}[x]$ and the second moment of the distribution $\mathbb{E}[x^2]$ as function of α and discuss where they are defined. **[10]**

(c) Discuss the convex properties of the function $f(x) = ax^2$ where $a \in \mathbb{R}$. **[5]**

(d) Discuss the convex properties the function $f(x) = -a\log(x)$ where $a \in \mathbb{R}$. **[5]**

End of Paper.