[L31]

Quiz 5 due Thursday 11.59pm

Last proper lecture today

Wed = revision lectures (2 hours)

We've been studying $U \subseteq V$ inner product space

$\Rightarrow U^\perp = \{ w \in V \mid u \cdot w = 0 \quad \forall u \in U \}$

Quiz What is $V^\perp$? ☑ $\{0\}$, ☐ undefined ☐ ϕ ☐ other

if w is s.t.

$u \cdot w = 0 \quad \forall u \in V \Rightarrow w = 0$ so $V^\perp = \{0\}$

eg $w \cdot w = 0$ $\Rightarrow$ inner product space

We proved If $\Pi_1 \ldots \Pi_r$ are orthogonal projections

s.t. $\sum_i \Pi_i = I$, $\quad \Pi_i \Pi_j = 0 \quad \forall i \neq j$

then $V = \bigoplus U_i \qquad U_i = \text{Image}(\Pi_i)$

is orthogonal decomposition ie $U_i \subset U_j^\perp \quad \forall i \neq j$

Conversely:

<u>Prop 8.7</u> If $V = U_1 \oplus \cdots \oplus U_r$ is an orthogonal decomposition of an inner product space $(V, \cdot)$ then $\exists$ orthogonal projections $\Pi_1 \ldots \Pi_r$ s.t. (a) $\sum_i \Pi_i = I$, (b) $\Pi_i \Pi_j = 0 \;\; \forall i \neq j$

(c) image $(\Pi_i) = U_i$.

<u>proof</u> As in prop 5.5 we construct Π_i obeying (a) – (c). Here $v = u_1 + \cdots + u_r$, $\quad u_i \in U_i$

and $\Pi_i(v) = u_i$. The new part is to show that $\Pi_i^* = \Pi_i$, ie $\forall i$, $w \cdot \Pi_i(v) := \Pi_i^*(w) \cdot v$

$$\underset{\substack{\uparrow \\ \text{to show}}}{=} \Pi_i(w) \cdot v \quad \forall v, w$$

But, $\forall w_j \in U_j$

$$w_j \cdot \Pi_i(v) = w_j \cdot u_i = \begin{cases} 0 & i \neq j \quad \text{as } U_i \subseteq U_j^\perp \\ w_j \cdot u_i & i = j \end{cases}$$

$$= \Pi_i(w_j) \cdot u_i$$

$$= \Pi_i(w_j) \cdot v \quad \text{as } U_k \subseteq U_i^\perp \text{ if } k \neq i$$

QED.

8.2 Spectral theorem

Theorem 8.8 let $\alpha : V \longrightarrow V$ be a self-adjoint linear map on an inner product space $(V, \cdot)$. Then $\exists$ an orthonormal basis of V consisting of eigenvectors of α. The eigenspaces of α form an orthogonal decomposition of V.

Corollary 8.9 In the context of the theorem, if $\lambda_1, \dots \lambda_r$ are the distinct eigenvalues, then $\exists$ orthogonal projections $\Pi_1, \dots \Pi_r$ s.t. (a) $\sum \Pi_i = I$ (b) $\Pi_i \Pi_j = 0 \ \forall i \neq j$ (c) $\alpha = \lambda_1 \Pi_1 + \dots + \lambda_r \Pi_r$

proof By Thm 8.8, $V = \hat{\bigoplus_{i=1}} E(\lambda_i, \alpha)$ is an orthogonal decomposition $\Rightarrow$ Π_i are orthogonal projections obeying (a)–(b). For (c)

$$\begin{aligned}\alpha(v) &= \alpha\left(\Pi_1(v) + \cdots + \Pi_r(v)\right)\\ &= \lambda_1 \Pi_1(v) + \cdots + \lambda_r \Pi_r(v)\\ &= (\lambda_1 \Pi_1 + \cdots + \lambda_r \Pi_r)(v) \quad \forall v\end{aligned}$$

Q.E.D.

<u>Corollary 8.10</u> Let A be a real symmetric matrix. Then $\exists$ an orthogonal matrix P s.t. $P^{-1}AP = P^{T}AP$ is diagonal.
(i.e. every real symmetric matrix is orthogonally similar to a diagonal one).

If we arrange the diagonal elements to be in increasing order then we see that every real symmetric matrix has a unique (canonical) diagonal form.

<u>Example 8.12</u> $A = \begin{bmatrix} 10 & 2 & 2 \\ 2 & 13 & 4 \\ 2 & 4 & 13 \end{bmatrix} \Rightarrow P_A(x) = |xI_3 - A|$
$= \cdots = (x-9)^2(x-18)$

so $\lambda_1 = 18$, $\lambda_2 = 9$ are the two distinct eigenvalues $(r=2)$

$\lambda_1 = 18$: $\begin{bmatrix} 10 & 2 & 2 \\ 2 & 13 & 4 \\ 2 & 4 & 13 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18x \\ 18y \\ 18z \end{bmatrix} \Rightarrow$ linear system $\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} \propto \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

normalize this to $\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$ unit vector w.r.t the standard inner product of $\mathbb{R}^3$

$E(\lambda_1, A) = \left\langle \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\rangle$

$\lambda_2 = 9$, $\begin{bmatrix} 10 & 2 & 2 \\ 2 & 13 & 4 \\ 2 & 4 & 13 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9x \\ 9y \\ 9z \end{bmatrix} \Rightarrow x + 2y + 2z = 0$ linear system

$E(\lambda_2, A)$ is 2-dimensional, characterized by

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 0$ ie $E(\lambda_2, A) = \left\langle \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\rangle^{\perp}$

fits with the spectral theorem

$$V = \mathbb{R}^3 = \underset{E(\lambda_1, A)}{U_1} \oplus \underset{E(\lambda_2, A)}{U_2} \qquad U_i \subseteq U_j^{\perp}$$

Choose an orthonormal basis of $E(\lambda_2, A)$ - just choose any two vectors with zero dot product and normalize them to unit vectors. e.g.

$$\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -4/(3\sqrt{2}) \\ 1/(3\sqrt{2}) \\ 1/(3\sqrt{2}) \end{bmatrix}$$

$\Rightarrow$ orthonormal basis of V

$$\Rightarrow P = \begin{bmatrix} 1/3 & 0 & \frac{-4}{3\sqrt{2}} \\ 2/3 & 1/\sqrt{2} & 1/3\sqrt{2} \\ 2/3 & -1/\sqrt{2} & 1/3\sqrt{2} \end{bmatrix}$$

$E(\lambda_1, A)$ $\quad$ $E(\lambda_2, A)$

must obey $P^t A P = P^{-1} A P = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

where $P^t P = I_3 = P P^t$.

Proof of Thm 8.8 We'll do this by induction.

First note that we can extend V to complex inner product space $V_{\mathbb{C}}$ (with $\cdot$ sesquilinear) and extend α to a self-adjoint map $\alpha: V_{\mathbb{C}} \to V_{\mathbb{C}}$. Its minimal polynomial $p_\alpha(x)$ has a root in $\mathbb{C}$.

So $\exists$ an eigenvector λ of α, call it $\tilde{v} \in V_{\mathbb{C}}$.

So then $\lambda(\tilde{v}\cdot\tilde{v}) = (\lambda\tilde{v})\cdot\tilde{v} = \alpha(\tilde{v})\cdot\tilde{v} = \tilde{v}\cdot\alpha(\tilde{v})$ α self-adjoint

$= \tilde{v}\cdot(\lambda\tilde{v}) = \bar{\lambda}\underbrace{(\tilde{v}\cdot\tilde{v})}_{\neq 0}$ (as $\cdot$ sesquilinear)

$\therefore$ $\lambda = \bar{\lambda}$

$\therefore$ the real part of $\alpha(\tilde{v}) = \lambda\tilde{v}$

$\Rightarrow$ a real eigenvector v with $\alpha(v) = \lambda v$

Now return to the real setting knowing that α has an eigenvector $v \in V$. Let

$U = \langle v\rangle^\perp$ and check that α restricts to a map $\alpha: U \longrightarrow U$:

$$\alpha(u)\cdot v \underset{\text{def of adjoint}}{=} u\cdot\alpha^*(v) \underset{\alpha\text{ self-adjoint}}{=} u\cdot\alpha(v) = u\cdot\lambda v = \lambda u\cdot v = 0 \quad \forall u \in \langle v\rangle^\perp$$

$\therefore$ $\alpha(u) \in U$

But U has smaller dimension than V so we can suppose as induction hypothesis that the theorem holds on U. The rest follows. QED

L32 Revision Lecture (I)

Exam – 3 hrs online ($+\frac{1}{2}$ hour to upload)
(dont leave till last min)

Revision Tips ① Look at all exam dates and draw up a revision schedule

② Review written works and quizzes (follow the solns)

③ Make a "mind map" of the contents of the course (also print out the typed lecture notes)

④ Do as many past exams as possible (at least one timed)

– on module web page, with solns.

help: Email me at any time – try to be self-contained.

Outline 8 chapters

1 – 5.2 basic elementary theory
5.3 – 8 more advanced theory
(doesn't nec. mean harder)

Format similar to previous years: 5 questions,
3 relate to elementary part
2 relate to more advanced
(2020 is not representative)

Ch 1 Vector spaces Spanning, L.i. lists of vectors excision lemma etc.

Subspaces $U \subseteq V$, $U+W$, $U \cap W$, $U \oplus W$

Ch2 Matrices row/col ops, rank etc

Ch3 Determinants Leibniz formula (laplace expansion)

Ch4 Linear maps $\alpha: V \longrightarrow W \xleftrightarrow[\text{basis}]{}$ $m \times n$ matrices A, $\exists P, Q$ s.t. PAQ = canonical form for equivalence

Ch5 Linear maps $\alpha: V \to V \xleftrightarrow[\text{basis}]{}$ $n \times n$ matrix A

Change basis $\leftrightarrow$ Similarity $A \sim P^{-1}AP$

$\pi: V \to V$ projection $\pi^2 = \pi$

ker α, im (α), rank(α) + nullity(α)

$P_A(x)$, $P_A(A) = 0$, $m_A(x)$ min poly $\Rightarrow$ test for diagonalizability.

Ch6 quadratic forms $q: V \longrightarrow \mathbb{K} \longleftrightarrow$ bilinear form $b: V \times V \to \mathbb{K}$

wrt basis $\Updownarrow$ $\qquad$ $\Updownarrow$

$q(x_1 \dots x_n)$ polynomial. $= q_A(x_1 \dots x_n)$ $\longleftrightarrow$ A symmetric

Change basis: congruence-equivalence. $A \sim P^t AP$

Ch7/8 Inner product spaces and spectral theorem

build on quadratic forms with b denoted "dot product", positive-definite.

$\left.\begin{array}{l} \alpha: V \to V \\ \pi: V \to V \end{array}\right\}$ add dot product and related concepts

$A \longrightarrow P^{-1}AP = P^t AP$ ($P^{-1} = P^t$ P "orthogonal")

related to changing orthonormal basis.

$\Rightarrow$ Every self-adjoint map α can be diagonalized with an orthonormal basis of eigenvectors.

More details by chapter

- Axioms of a field $\mathbb{K}$ e.g. $\mathbb{R}, \mathbb{C}, \mathbb{Q}$,
  $\mathbb{F}_2 = \{0,1\}$, $\mathbb{F}_3 = \{0,1,2\}$ etc ($\mathbb{F}_p$ p prime $= \mathbb{Z}/p\mathbb{Z}$)
- defn of a v.s. over $\mathbb{K}$ (V, +, scale by $t \in \mathbb{K}$)
- basic examples $\mathbb{K}^n = \{$Column vectors $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, $c_i \in \mathbb{K}\}$

e.g. V of dim n wrt a basis $\{v_i\}$

$$V \longleftrightarrow \mathbb{K}^n, \quad v = \sum_{i=1}^{n} c_i v_i$$

e.g. $\mathbb{K}[x]$ polys in x over $\mathbb{K}$

$$\longleftrightarrow \quad f = f_0 1 + f_1 x + f_2 x^2 + \cdots f_n x^n$$

e.g. $\mathbb{K}[x]_n$ poly's of degree $\leq n$ (or zero)

e.g. $M_n(\mathbb{K})$ as vector space.

$A + B = (c_{ij})$, $A = (a_{ij})$, $B = (b_{ij})$, $c_{ij} = a_{ij} + b_{ij}$

Symmetric matrices $\{A \in M_n(\mathbb{K}) \mid A^t = A\}$

antisymmetric " $\{A \in M_n(\mathbb{K}) \mid A^t = -A\}$

(same over $\mathbb{F}_2$)

Def 1.7 $v_1 \ldots v_n$ are l.i. (linearly independent)

if $c_1 v_1 + \cdots + c_n v_n = 0 \Rightarrow c_1 = c_2 \cdots = c_n = 0$

(ie do not admit a nontrivial linear relation)

$v_1 \ldots v_n$ spanning V ($V = \langle v_1 \ldots v_n \rangle$)

for every $v \in V$ $\exists$ c_i s.t. $v = c_1 v_1 + \cdots c_n v_n$

$v_1 \ldots v_n$ basis if both.

Basic facts (1) $|$any l.i. list$| \leq |$any spanning list$|$

$\Rightarrow$ all bases have same number $n = \dim(V)$ of vectors.

(2) Any spanning list can be cut down to a basis (if V is finite dimensional)

(3) Any l.i. list can be extended to a basis (if V is f.d.)

Subspaces $U \subseteq V$ (test: U not empty and closed under $+$ and scaling)

If $U, W \subseteq V$ subspaces then $U \cap W$, $U+W = \{u+w \mid u \in U, w \in W\}$ are subspaces

$V = U \oplus W$ if $V = U + W$ and $U \cap W = \{0\}$.

"direct sum"

$V = U_1 \oplus \cdots \oplus U_r$ if U_i subspaces s.t. $\forall v \in V$, $v = u_1 + \cdots + u_r$, $u_i \in U_i$ uniquely $\leftrightarrow \oplus$

$$\dim(U \cap W) + \dim(U+W) = \dim(U) + \dim(W)$$

Ch 2 Row (and col) ops on $m \times n$ matrices

(1) add a multiple of jth to ith (eg $r_1 - 2r_3$ etc. means add -2 times 3rd row to 1st row)

(2) Scale ith

(3) Swap ith and jth $(i \neq j)$

$r = \text{rank} = \dim$ col space (spanned by column vectors) $= \dim$ row space (" " row ")

unchanged by row and col ops.

$\Rightarrow$ Thm 2.10 (etc) Every $m \times n$ matrix A can be put in canonical form for equivalence by row / col ops,

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

$r = \text{rank}(A)$, P $m \times m$ invertible $\leftrightarrow$ row ops

Q $n \times n$ invertible $\leftrightarrow$ col ops.

Ch 3 $\quad \det(A) = \sum_{\pi \in S_n} \text{sign}(\pi) \, a_{1\pi(1)} \cdots a_{n\pi(n)}$

$A \in M_n(\mathbb{k})$

group of permutations $\quad \pm 1 \quad$ eg. $= (-1)^{\# \text{transpositions}}$ (can write π as product of transpositions)

eg (123) is 3-cycle sending

$$\begin{array}{ccc} 1 & \longrightarrow & 2 \\ & \nwarrow \; \swarrow & \\ & 3 & \end{array}$$

$\Rightarrow$ $\det(A)$ characterized uniquely as a function D on $n \times n$ matrices ~~to~~ obeying

1. $D(A)$ is linear in every row (rest held fixed)
2. $D(A) = 0$ if A has a repeated row
3. $D(I_n) = 1$

(any function with these props, eg laplace formula, $= \det(A)$.)

$\det(AB) = \det(A)\det(B)$, laplace expansion, adjugate

$P_A(x) := \det(xI_n - A)$ characteristic poly.

obeys $P_A(A) = 0$ (cayley hamilton theorem)

$P_A(x)$ is unchanged under similarity $A \to P^{-1}AP$ ($\Rightarrow$ $P_\alpha(x)$ is a prop of $\alpha : V \to V$ independent of basis)

we see later that $P_A(x)$ contains Trace(A) det(A) in coeffs of x^0, x^{n-1}

$\boxed{L33}$ Revision lecture (II)

Chapter 4 $\quad \alpha : V \to W$, $\beta : W \to U$ then $\beta\alpha : V \to U$

If V, W have bases $\{v_i\}, \{w_i\}$ then

$\alpha(v_i) = \sum_j a_{ji} w_j$, $A = (a_{ij})$

$= \begin{bmatrix} \alpha(v_1) \cdots\cdots & \alpha(v_n) \end{bmatrix}$

as column vectors w.r.t basis w_j

① under change of basis

$A \to A' \doteq PAQ$ ← transition matrix for V

P ← transition matrix for W

② $\ker(\alpha) \subseteq V$, $\mathrm{image}(\alpha) \subseteq W$

$\ker(\alpha)$: vectors that are killed by α

$\mathrm{image}(\alpha)$: vectors reached by α.

rank + nullity $= \dim(\mathrm{im}(\alpha)) + \dim(\ker(\alpha)) = \dim(V)$

Chap 5 : $\alpha : V \longrightarrow V$

$\pi : V \longrightarrow V$ projection if $\pi^2 = \pi$

$\Rightarrow \quad V = \mathrm{im}(\pi) \oplus \ker(\pi)$

conversely if $U, W \subseteq V$, $V = U \oplus W \Rightarrow \exists\, \pi : V \to V$

$\mathrm{im}(\pi) = U$, $\ker(\pi) = W$

Here $\pi(v = u + w) = u$

Extend to $V = U_1 \oplus \cdots \oplus U_r$, $\Pi_1 + \cdots + \Pi_r = I$ identity map

$\Leftrightarrow \pi_i$ projections,

$\Pi_i \Pi_j = 0 \quad \forall i \neq j$

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change of basis

$\alpha \leftrightarrow A$ w.r.t basis , $A \to A' = P^{-1}AP$ "similar"

properties of α can be computed in a basis

but indep't of it eg $\det(\alpha)$, $\mathrm{tr}(\alpha)$, $p_\alpha(x)$

Eigenvalue of α , eigenvector is $v \neq 0$ s.t. $\alpha(v) = \lambda v$

λ is the eigenvalue

$E(\lambda, \alpha) \subseteq V$

$= 0$ and all eigenvectors of eigenvalue λ

Thm 5.14 α diagonalizable $\Longleftrightarrow V = \bigoplus_{i=1}^{r} E(\lambda_i, \alpha)$,

(i.e $\exists$ a basis of eigenvectors) λ_i distinct eigenv

$\Leftrightarrow \alpha = \lambda_1 \Pi_1 + \dots \lambda_r \Pi_r$, Π_i as above.

Theorem 5.18 λ an eigenvalue $\Leftrightarrow \lambda$ a root of $P_\alpha(x)$

$\Leftrightarrow \lambda$ a root of $m_\alpha(x)$ defined as the lowest degree monic poly s.t. $m_\alpha(\alpha) = 0$ as map $V \to V$

$\Rightarrow m_\alpha(x)$ divides $P_\alpha(x)$ ie $P_\alpha(x) = m_\alpha(x)\underset{\text{some other poly}}{q(x)}$

Note λ a root of $\underset{\text{poly}}{f(x)} \Leftrightarrow x - \lambda$ divides f ie a factor of f.

methods to find $m_\alpha(x)$: (1) compute $P_\alpha(x)$

(2) look at its factors working over $\mathbb{K}$, including all linear factors

(3) take the lowest degree $m_\alpha(x)$ that contains all the linear factors and divides $P_\alpha(x)$ and obeys $m_\alpha(\alpha) = 0$.

Thm 5.20 $\alpha : V \to V$ is diagonalizable iff $m_\alpha(x)$ is a product of distinct linear factors.

(e.g. not $(x-\lambda)^2(x-\mu)$, not $(x-\lambda)(x^2+1)$ over $\mathbb{R}$)

Over $\mathbb{C}$ every $n \times n$ matrix has a jordan form unique up to ordering of block. (is similar to such)

Chap 6 quadratic forms 3 points of view ($1+1 \neq 0$ in $\mathbb{K}$)

(1) $q = q_A(x_1, \dots x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ $A = (a_{ij})$ symmetric

(2) $q : V \to \mathbb{K}$ st. $q(cv) = c^2 q(v)$ $\forall c \in \mathbb{K}$

and $b(v,w) := \frac{1}{2}\left(q(v+w) - q(v) - q(w)\right)$
is bilinear ("polarization of q")

(3) $b: V \times V \to \mathbb{K}$ symmetric, bilinear form
$q(v) := b(v,v)$

If fix a basis $\{v_i\}$ then

$q(v = \sum_i x_i v_i) = q_A(x_1 \ldots x_n)$, $a_{ij} = b(v_i, v_j)$

<u>Thm 6.7</u> For any $q(x_1 \ldots x_n)$ in n variables
$\exists$ a linear change of variables y_i (ie a change of basis) s.t $q(x_1 \ldots x_n) = \sum_i c_i y_i^2$

over $\mathbb{R}$ can choose s.t $c_i = \begin{cases} +1 & \leftarrow \# \text{ of these} = s \\ -1 & \leftarrow \# \text{ " " } = t \\ 0 \end{cases}$

s, t are properties of q ($s+t$ = rank, $s-t$ = signature)

ie indept of choice of bases $\Rightarrow$ <u>Sylvesters law of inertia</u>

which says in matrix terms that over $\mathbb{R}$,

$\exists P$ s.t. $A' = P^t A P = \begin{bmatrix} I_s & & & 0 \\ & -I_t & & \\ & & 0 & \\ 0 & & & \ddots \; 0 \end{bmatrix}$

here $A \to A' = P^t A P$

is <u>congruence</u> corresponds to a linear change of variables. <u>Two methods</u> (1) use change of variables algorithm in proof of Thm 6.7.

(2) by inspection "complete the square"

<u>Ch 7/8</u> V over $\mathbb{R}$ is equipped with a quadratic form / symmetric bilinear "dot product" which is positive definite ie $v \cdot v \geq 0$, $v \cdot v = 0$

iff $v=0$ (ie $t=0$, $s=n$ in Sylvester's law)

— called an inner product space.

$\Rightarrow$ ① an orthonormal basis $v_i \cdot v_j = \delta_{ij}$

② concept of adjoint to $\alpha : V \to V$ is

$\alpha^* : V \to V$ defined by $w \cdot \alpha^*(v) = \alpha(w) \cdot v$
$\forall v, w$

$\alpha^* \leftrightarrow A^t$ wrt an orthonormal basis.

③ concept of an orthogonal complement

$U^\perp = \{ w \in V \mid w \cdot v = 0 \ \ \forall v \in U \}$ of

a subspace $U \subseteq V$

<u>Theorem 8.8</u> (Spectral theorem) if $\alpha^* = \alpha$

$\exists$ an orthonormal basis of eigenvectors

ie $V = \bigoplus_{i=1}^{r} E(\lambda_i, \alpha)$ λ_i distinct eigenvalues

is an orthogonal decomposition.

($\Rightarrow$ $V = \mathbb{R}^n$ with its standard inner product, any real symmetric matrix A can be diagonalized ie $A' = P^{-1}AP$ with $P^{-1} = P^t$

(ie P orthogonal) "orthogonally similar")

<u>CWK 10 Q6</u> $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ is self-adjoint on $\mathbb{R}^3$ with its standard inner product

Find eigenvalues, set of orthogonal eigenvectors and P s.t. $P^{-1}AP$ is diagonal

Soln $\quad p_A(\lambda) = \det \begin{bmatrix} \lambda-2 & 0 & 0 \\ 0 & \lambda-3 & 1 \\ 0 & 1 & \lambda-3 \end{bmatrix} = (\lambda-2)^2(\lambda-4)$

so eigenvalues are 2 (with multiplicity 2) and $\lambda=4$. $\quad$ Solve $(A-4I)v=0$

$$\Rightarrow \quad v_1 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$ after normalization w.r.t standard inner product.

Solve $(A-2I)v=0 \Rightarrow$

$$v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/2 \\ 1/2 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/2 \\ 1/2 \end{bmatrix}$$

chosen so that $v_2 \cdot v_3 = 0$, then normalize.

$P = [v_1, v_2, v_3]$. $\quad$ check $P^{-1}AP = P^{t}AP = \begin{bmatrix} 4 & & 0 \\ & 2 & \\ 0 & & 2 \end{bmatrix}$.