$\boxed{\text{L28}}$

- Quiz 5 will go live thursday (due next week)
- cwk 10 should be attempted after wed lectures.
- Next week tutorial: cwk 10 (before then continue cwk 9).

Quiz 4 feedback: weak on modular arithmetic

e.g. Q 6: $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ find $m_A(x)$ and if A is diagonalizable.

soln

$$P_A = \begin{vmatrix} x-1 & -2 & -3 \\ -3 & x-1 & -2 \\ -2 & -1 & x-3 \end{vmatrix} = (x-1)\big((x-1)(x-3)-2\big)$$

$$+2\,(-3(x-3)-4)$$

$$-3\,(3+2(x-1))$$

$$\underbrace{-6x \qquad -6x-3}_{-2(x-1)}$$

$$= (x-1)\big(\underbrace{(x-1)(x-3)}_{x^2-1-4x}-4\big)$$

$$= x^2+x+4$$

$$= (x+3)^2$$

$$(x^2+6x+9)$$

root

$$\lambda = \frac{-1 \pm \sqrt{1-4.4}}{2} = -1/2 = 2 \quad (\text{as } 2.2 = 4 = -1 \bmod 5)$$

$\therefore$

$(x-1)(x-2)^2$

$= (x+4)(x+3)^2$

check $(A+4I)(A+3I) \neq 0$

$\therefore = m_A(x)$ not product of distinct linear factors $\therefore$ not diagonalizable

Chapter 7 Inner product spaces

This just means a v.s. over $\mathbb{R}$ equipped with a positive definite quadratic form / symmetric bilinear form. We'll denote the latter as

$b(v,w) = v \cdot w$ "dot product"

7.1 Inner product spaces and orthonormal bases

Def 7.1 An inner product on a real v.s. V is a function $V \times V \to \mathbb{R}$ sending $v, w \in V$ to $v \cdot w$ "dot product" s.t.

1) the inner product is symmetric $v \cdot w = w \cdot v$ $\forall v, w \in V$

2) " " " bilinear

$$(v+v')\cdot w = v\cdot w + v'\cdot w \qquad \forall v, v', w \in V$$
$$(av)\cdot w = a(v\cdot w) \qquad a \in \mathbb{R}$$

(this is linearity in the first variable but implies linearity in the 2nd by 1)).

3) The inner product (ie the associated quadratic form) is positive-definite

ie $v \cdot v \geq 0$ $\forall v \in V$, $v \cdot v = 0$ iff $v = 0$.

Given an inner product space $(V, \cdot)$ (ie a v.s. V equipped with an inner product) we define the "length" of $v \in V$ as

$$|v| := \sqrt{v \cdot v}.$$

Similarly for any $v, w \in V \setminus \{0\}$ we define the "angle" θ between them by

$$\cos\theta = \frac{v \cdot w}{|v|\,|w|}$$

For this to be well-defined we need

$$-|v||w| \leq v \cdot w \leq |v||w|$$

so that $\cos\theta \in [-1, 1]$

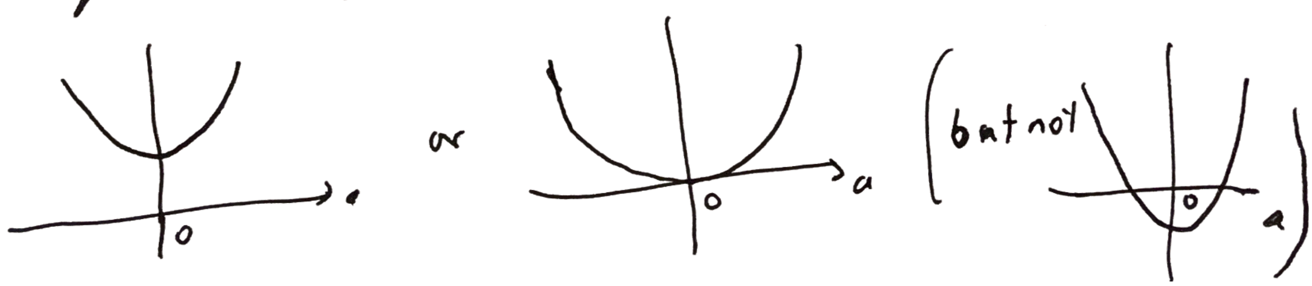
<u>Theorem 7.2</u> (Cauchy-Schwarz inequality) if $v, w \in V$ an inner product space then

$$(v \cdot w)^2 \leq (v \cdot v)(w \cdot w)$$

<u>Proof</u> By definition, $(v + aw) \cdot (v + aw) \geqslant 0 \quad \forall a \in \mathbb{R}$

ie $$(w \cdot w)a^2 + 2(v \cdot w)a + (v \cdot v) \geqslant 0$$

using bilinearity and symmetry. This is a quadratic function of a so:



as $w \cdot w \geqslant 0$

$\therefore$ a has no real roots or the two roots are equal. $\therefore$ the discriminant is ≤ 0

ie $$(v\cdot w)^2 - (v\cdot v)(w\cdot w) \leq 0$$

(we assumed $w \neq 0$. The $w=0$ case is automatic)

Q.E.D.

Will say that two vectors $v, w \neq 0$ are orthogonal if $v\cdot w = 0$

<u>Definition 7.3</u> A basis $v_1 \dots v_n$ of an inner product space is called orthonormal if

$$v_i \cdot v_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad \forall\, 1 \leq i, j \leq n$$

<u>Lemma 7.4</u> If $v_1 \dots v_n \in V$ obey $v_i \cdot v_j = \delta_{ij}$ then they are l.i.

<u>proof</u> Suppose $c_1 v_1 + \dots + c_n v_n = 0$. Take the dot product with v_i:
$$\begin{aligned} 0 &= v_i \cdot (c_1 v_1 + \dots + c_n v_n) \\ &= c_1 v_i \cdot v_1 + \dots + c_n v_i \cdot v_n \\ &= c_i \end{aligned}$$

true $\forall i$ $\therefore$ $c_i = 0$ $\forall i$, $\therefore$ l.i. Q.E.D.

<u>Theorem 7.5</u> Let be an inner product on a real v.s. V of dimension n. Then $\exists$ an orthonormal basis $v_1 \dots v_n$.

Comment on the proof

1) Can do this by the "Gram-Schmidt process" (in linalg I) which takes a basis $w_1 \dots w_n$ and returns an orthonormal one $v_1 \dots v_n$.

2) We already saw the proof since $\exists$ a basis putting the real quadratic form into the canonical form for congruence (Sylvester's law of inertia) with $s = n$, $t = 0$. So the ~~an~~ canonical form is I_n ie $v_i \cdot v_j = \delta_{ij}$ for this basis.

Learning support hour Wednesday 2-3 in B-11

L29 Prop 7.6 Let B be an orthonormal basis of an inner product space $(V, \cdot)$ of dimension n. Let $[v]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $[w]_B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

Then $$v \cdot w = [a_1 \dots a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^{n} a_i b_i$$

proof If $v_1 \dots v_n$ is B then
$$v \cdot w = (a_1 v_1 + \dots a_n v_n) \cdot (b_1 v_1 + \dots + b_n v_n)$$
$$= \sum_{i=1} a_i b_i$$ by 1) bilinearity of $\cdot$ 2) $v_i \cdot v_j = \delta_{ij}$ QED

Def 7.7 An inner product on $\mathbb{R}^n$ for which the standard basis is orthonormal, is called the <u>standard inner product</u> and has the form

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = [a_1 \dots a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^{n} a_i b_i$$

This is the usual dot product on $\mathbb{R}^n$ or $\mathbb{R}^3$ and we've seen that <u>any</u> inner product space V of $\dim(V) = n$ is equivalent to this standard one by choosing an orthonormal basis.

Remark 7.8 If V is a v.s over $\mathbb{C}$ and if we assume $\cdot$ is bilinear, then we can't assume its positive definite eg. $(i = \sqrt{-1})$

$$(iv) \cdot (iv) = i^2 v \cdot v = -v \cdot v$$

so if $v \cdot v > 0$ we can set $w = iv$, $w \cdot w < 0$. To fix this problem, keep linearity in first variable and change the symmetry axiom to $v \cdot w = \overline{w \cdot v}$ ("conjugate symmetric")

Then $(av) \cdot w = a(v \cdot w)$ as before, implies

$$v \cdot (aw) = \overline{(aw) \cdot v} = \overline{a(w \cdot v)} = \bar{a}\, \overline{w \cdot v} = \bar{a}\, v \cdot w$$

ie dot product is "conjugate linear" in the 2nd variable (ie "<u>sesquilinear</u>")

Then, for example,

$$(iv)\cdot(iv) = i(-i)v\cdot v = v\cdot v$$

and we can impose $v\cdot v \geq 0$ with equality iff $v=0$ — the defn of an inner product space over $\mathbb{C}$.

7.2 Adjoints and orthonormal linear maps

Def 7.9 Let V be a (real) inner product space and $\alpha: V \longrightarrow V$ a linear map. Then the adjoint of α is a linear map $\alpha^*: V \longrightarrow V$ defined by $v\cdot\alpha^*(w) = \alpha(v)\cdot w$ $\forall v,w \in V$.

This defines α^* implicitly by its dot product. e.g. if v_i is an orthonormal basis and $\alpha^*(v_j) = \sum_{k=1}^{n} c_k v_k$ some $c_k \in \mathbb{R}$

then $v_i \cdot \alpha^*(v_j) := \alpha(v_i)\cdot v_j$

but LHS $= c_i$ by v_i orthonormal.

so c_i are determined by α.

This is for all j so α^* is fully determined.

(The general proof without assuming V f.d. is called the Riesz representation theorem)

Exercise Using the definition of α^*, show that α^* is a linear map and

$$(\alpha^*)^* = \alpha$$

Prop 7.10 If α is represented by a matrix A w.r.t. an orthonormal basis then α^* is represented by A^t (transpose)

proof w.r.t a basis B, $[v]_B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$, $[w]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

an $A = (a_{ij})$ corresponding to α. let A^* be the matrix corresponding to α^*, $A^* = (a^*_{ij})$. The latter is characterized by

$$[v]_B \cdot (A^* [w]_B) = (A [v]_B) \cdot [w]_B .$$

By prop 7.6, this reads

$$[b_1 \dots b_n] A^* \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \left(A \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}\right)^t \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [b_1 \dots b_n] A^t \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

for all v, w $\quad \therefore \quad A^* = A^t$ QED.

Example 7.11 on $\mathbb{R}^3$, with its standard inner product, let α be clockwise rotation by $\pi/4$ looking down on the x-y plane from above. This has matrix

$$A = \begin{bmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0 \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Its adjoint is

$$A^* = A^t = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(-\frac{\pi}{4}) & \sin(-\frac{\pi}{4}) & 0 \\ -\sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so α^* is a rotation of $\frac{\pi}{4}$ anticlockwise ie α^{-1}

<u>Definition 7.12</u> let α be a linear map on an inner product space.

(a) α is <u>self-adjoint</u> if $\alpha^* = \alpha$

(b) α is <u>orthogonal</u> if α is invertible and $\alpha^* = \alpha^{-1}$

(so α in Ex 7.11 is orthogonal).

<u>Theorem 7.13</u> TFAE for a linear map α on an inner product space $(V, \cdot)$

(a) α is orthogonal

(b) α preserves the inner product
$\alpha(v) \cdot \alpha(w) = v \cdot w \quad \forall v, w \in V$.

(c) α maps any orthonormal basis of V to another orthonormal basis.

<u>Proof</u> (a)$\Rightarrow$(b): If α is orthogonal then
$\alpha(v) \cdot \alpha(w) = v \cdot (\alpha^*(\alpha(w))) = v \cdot (\alpha^{-1}\alpha(w)) = v \cdot w$

(b)$\Rightarrow$(c): If $v_1 \ldots v_n$ an orthonormal basis then

$$\alpha(v_i)\cdot\alpha(v_j) \underset{(b)}{=} v_i\cdot v_j = \delta_{ij}$$

so $\alpha(v_1) \dots \alpha(v_n)$ is an orthonormal basis.

(c) $\Rightarrow$ (a): If (c) hold and

$(\alpha^*\alpha)(v_i) = c_1 v_1 + \dots + c_n v_n$ say.

$$\Rightarrow \quad v_j\cdot\left(\alpha^*(\alpha(v_i))\right) := \alpha(v_j)\cdot\alpha(v_i) \underset{(c)}{=} \delta_{ij}$$

$\parallel$ if v_i an orthonormal basis.

$v_j\cdot(c_1 v_1 + \dots c_n v_n) = c_j$

so $c_i = 1$, $c_j = 0$ $\forall j \neq i$

$\therefore$ $(\alpha^*\alpha)(v_i) = v_i$, True for all i, so

$\alpha^*\alpha = I$ on V $\therefore$ $\alpha^* = \alpha^{-1}$. Q.E.D.

$\boxed{L30}$ <u>Corollary 7.14</u> If α is represented by A w.r.t an orthonormal basis

then (a) α is selfadjoint iff A is symmetric

(b) α is orthogonal iff $A^tA = I$

(i.e. A is an "orthogonal matrix"), I = identity matrix.

<u>proof</u> We saw that α^* is represented by A^t. Q.E.D.

<u>Ex 7.15</u> revisit Ex 7.11 $\alpha = \frac{\pi}{4}$ rotation of $\mathbb{R}^3$ (of the x-y plane). We saw $\alpha^* \leftrightarrow A^t$ was a $\frac{-\pi}{4}$ rotation matrix, A as in Ex 7.11.

so $\alpha^*\alpha = I$ or $A^tA = I_3$, α orthogonal.

$\Leftrightarrow$ (by the theorem) α preserves dot products

$\Leftrightarrow$ α preserves lengths and angles — as expected for a rotation.

Corollary 7.16 Let $\alpha: V \to V$ be represented by A w.r.t. an orthonormal basis. Then α is orthogonal iff the columns of A form an orthonormal basis when viewed in V via the original basis.

proof Let $A = \begin{bmatrix} \bar{v}_1 & \cdots & \bar{v}_n \end{bmatrix}$, $\bar{v}_i := [v_i]_B$

column vectors w.r.t. original basis B.

$$A^t = \begin{bmatrix} \bar{v}_1^t \\ \vdots \\ \bar{v}_n^t \end{bmatrix} \Rightarrow A^T A = \begin{bmatrix} \bar{v}_1^t \bar{v}_1 & \cdots & \bar{v}_1^t \bar{v}_n \\ \vdots & & \vdots \\ \bar{v}_n^t \bar{v}_1 & \cdots & \bar{v}_n^t \bar{v}_n \end{bmatrix}$$

but $\bar{v}_i^t \bar{v}_j = v_i \cdot v_j$ by prop 7-6. for the the corresponding $v_i \in V$.

$\therefore$ $A^T A = I_n$ iff $v_i \cdot v_j = \delta_{ij}$ ie v_i an orthonormal basis.

Q.E.D.

Ex In Ex 7.11 and Ex 7.15 about the $\frac{\pi}{4}$ rotation, can check that the column of A are indeed an orthonormal basis:

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -1/\sqrt{2} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition 7.18 Two $n \times n$ matrices A, A' are called <u>orthogonally similar</u> if $\exists$ an orthogonal matrix P (i.e. $P^{-1} = P^t$) s.t. $A' = P^{-1}AP$ $(= P^tAP)$

From Theorem 7.13 we know that orthogonal matrices P take an orthonormal basis to another. Hence A, A' are orthogonally similar iff they represent the same linear map $\alpha : V \to V$ with respect to possibly different orthonormal bases.

Chapter 8 The spectral theorem

We'll study ~~we~~ when a matrix over $\mathbb{R}$ is orthogonally similar to a diagonal matrix.

8.1 Orthogonal projections and decompositions

Def 8.1 In an inner product space $(V, \cdot)$ two vectors $v, w \in V$ are <u>orthogonal</u> if $v \cdot w = 0$. We say two subspaces $U, W \subseteq V$ are orthogonal if $u \cdot w = 0$ $\forall u \in U$, $w \in W$

Def 8.2 let $(V, \cdot)$ be an inner product space and $U \subseteq V$ a subspace. Its orthogonal complement is the subspace

$$U^{\perp} = \{ w \in V \mid w \cdot u = 0 \quad \forall u \in U \}$$

(i.e. the largest subspace orthogonal to U).

(Check for yourself that $U^{\perp}$ is indeed a subspace).

Prop 8.3 If V is an n-dimensional inner product space and $U \subset V$ is a subspace of dimension r then $U^{\perp}$ is a subspace of dimension $n-r$ and

$$V = U \oplus U^{\perp}.$$

proof Let $u_1, \dots u_r$ be a basis of U and extend to $u_1, \dots u_n$ a basis of V. Apply the Gram-Schmidt process to convert this to an orthonormal basis $v_1, \dots v_n$ of V. All we need to know is that it can be done inductively, ie at all stages

$$\langle v_1, \dots v_i \rangle = \langle u_1, \dots u_{i-1}, v_i \rangle \qquad i = 1 \dots n$$

So at $i = r$, we'll have $\langle v_1, \dots v_r \rangle = \langle u_1, \dots u_r \rangle$

and $v_1, \dots v_r$ will be an orthonormal basis of U. Also $v_{r+1}, \dots v_n \in U^{\perp}$. So every $v \in V$ can be written as

$$v = \underbrace{u}_{\in \langle v_1, \dots v_r \rangle} + \underbrace{w}_{\in \langle v_{r+1}, \dots v_n \rangle}, \qquad u \in U,\ w \in U^{\perp}$$

so $V = U + U^\perp$. If $u \in U \cap U^\perp$

$\Rightarrow$ $\underset{u \in U \nearrow \quad \nwarrow u \in U^\perp}{u \cdot u = 0}$ $\Rightarrow$ $u = 0$ by positive def. of dot product.

So $V = U \oplus U^\perp$ $\therefore$ $\dim(U^\perp) = \dim(V) - \dim(U) = n - r$ Q.E.D.

Def 8.4 Let $(V, \cdot)$ be an inner product space. A linear map $\pi : V \to V$ is called an orthogonal projection if

(a) π is a projection (ie $\pi^2 = \pi$)

(b) π is self-adjoint (ie $\pi^* = \pi$).

Definition 8.5 If $(V, \cdot)$ is an inner product space and $U_1 \cdots U_r \subseteq V$ subspaces, we say that a direct sum $V = U_1 \oplus \cdots \oplus U_r$ is an orthogonal decomposition of V if U_i are orthogonal to U_j $\forall i \neq j$ (ie $U_i \subseteq U_j^\perp$ $\forall i \neq j$).

proposition 8.6 Suppose $\pi_1 \cdots \pi_r$ are orthogonal projections on an inner product space $(V, \cdot)$ such that

(a) $\pi_1 + \cdots + \pi_r = I$ (identity map)

(b) $\pi_i \pi_j = 0$ $\forall i \neq j$

let $U_i = \text{Image}(\pi_i)$. Then $V = U_1 \oplus \dots \oplus U_r$ is an orthogonal decomposition.

<u>proof</u> The new part (beyond prop 5.4) is to prove that U_i, U_j are orthogonal for all $i \neq j$. Recall that if π is a projection then $v \in \text{Image}(\pi)$ iff $\pi(v) = v$.

Consider $i \neq j$, $u_i \in U_i = \text{image}(\pi_i)$
$u_j \in U_j = \text{image}(\pi_j)$

then
$$\begin{aligned} u_i \cdot u_j &= \pi_i(u_i) \cdot \pi_j(u_j) \\ &= u_i \cdot \pi_i^* \pi_j(u_j) && \text{(def of } \pi_i^*) \\ &= u_i \cdot \pi_i \pi_j(u_j) && \text{as } \pi_i^* = \pi_i \\ &= 0 && \text{as } \pi_i \pi_j = 0 \ \forall i \neq j \end{aligned}$$

so U_i is orthogonal to U_j. Q.E.D.