MTH6140 Linear Algebra II

Coursework 10 Solutions

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1. (a) Suppose $w, w' \in V$ and $c \in \mathbb{K}$. We need to show that $\alpha^*(w + w') = \alpha^*(w) + \alpha^*(w')$ and $\alpha^*(cw) = c\alpha^*(w)$. For all $v \in V$ we have

$$\begin{aligned} v \cdot \alpha^*(w + w') &= \alpha(v) \cdot (w + w') \\ &= \alpha(v) \cdot w + \alpha(v) \cdot w' \\ &= v \cdot \alpha^*(w) + v \cdot \alpha^*(w') \\ &= v \cdot (\alpha^*(w) + \alpha^*(w')), \end{aligned}$$

where the first and third equalities are from the definition of adjoint, the second and fourth use linearity of inner product. Also

$$v \cdot \alpha^*(cw) = \alpha(v) \cdot (cw) = c(\alpha(v) \cdot w) = c(v \cdot \alpha^*(w)) = v \cdot (c\alpha^*(w)).$$

Again, the first and third equalities are from the definition of adjoint, the second and fourth use linearity of inner product.

Note that if $v \cdot w = v \cdot w'$ for all $v \in V$ then $w = w'$. This is because the representation of a linear functional $\varphi : V \to \mathbb{R}$ in the form $\varphi(v) = v \cdot w$, for some fixed $w \in V$, is unique.

(b) For all $v \in V$ we have

$$v \cdot \alpha^{**}(w) = v \cdot (\alpha^*)^*(w) = \alpha^*(v) \cdot w = v \cdot \alpha(w),$$

and the claim follows, as before.

2. A matrix representing a self-adjoint linear map is symmetric, so that the missing entries in rows 1, 2 and 3 of matrix A are respectively -1, -3 and 2.

The columns of a matrix representing an orthogonal linear map form an orthonormal basis. Letting $B = (b_{ij})$ and considering the first column we find that $b_{11}^2 + b_{21}^2 + b_{31}^2 + b_{41}^2 = 1$, i.e., $b_{11}^2 = 1 - (\frac{5}{6})^2 - (\frac{1}{6})^2 - (\frac{1}{6})^2 = \frac{1}{4}$. It follows that $b_{11} = \pm\frac{1}{2}$. Similar reasoning applied to columns 2, 3 and 4 yields $b_{23} = \pm\frac{1}{6}$, $b_{32} = \pm\frac{1}{6}$ and $b_{44} = \pm\frac{5}{6}$.

We just need to determine the signs. Since column 2 is orthogonal to column 1, we know that $b_{11}b_{12} + b_{21}b_{22} + b_{31}b_{32} + b_{41}b_{42} = 0$. By trial and error, this constraint forces the sign of b_{11} to be negative. Applying similar reasoning to the remaining columns yields

$$b_{11} = -\tfrac{1}{2}, \quad b_{23} = \tfrac{1}{6}, \quad b_{32} = \tfrac{1}{6}, \quad b_{44} = -\tfrac{5}{6}.$$

3. (a) Cancellation does not hold for direct sums, so the final step is not valid. In more detail, it is possible for a vector space V to have subspaces U, U' and W such that $U \neq U'$ and $V = U \oplus W = U' \oplus W$. In $\mathbb{R}^2$ an example might be $U = \langle [\begin{smallmatrix}0\\1\end{smallmatrix}] \rangle$, $U' = \langle [\begin{smallmatrix}1\\1\end{smallmatrix}] \rangle$ and $W = \langle [\begin{smallmatrix}1\\0\end{smallmatrix}] \rangle$.

(b) Let u be an arbitrary vector in U. According to the definition of orthogonal complement every vector $u' \in U^\perp$ is orthogonal to u, i.e., $u \cdot u' = 0$. We deduce, again from the definition of orthogonal complement, that $u \in (U^\perp)^\perp$. Since $u \in U$ was arbitrary, we must have $U \subseteq (U^\perp)^\perp$.

It's not so easy to show inclusion in the other direction directly, so we take a more oblique approach. By Proposition 7.3, we know that $\dim(U) + \dim(U^\perp) = n$, where $n = \dim(V)$. By the same token, $\dim(U^\perp) + \dim((U^\perp)^\perp) = n$. Putting these two identities together yields $\dim(U) = \dim((U^\perp)^\perp)$.

Take a basis $\mathcal{B}$ for U. Since $U \subseteq (U^\perp)^\perp$ we can extend $\mathcal{B}$ to a basis $\mathcal{B}'$ of $(U^\perp)^\perp$. But $\dim(U) = \dim((U^\perp)^\perp)$, so in fact $\mathcal{B} = \mathcal{B}'$. It follows that $U = (U^\perp)^\perp$.

4. (a) For $u = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}^\top \in \mathbb{F}_2^4$ to be a member of $U^\perp$ we require

$$u \cdot \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^\top = u \cdot \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^\top = 0.$$

These conditions are satisfied precisely when $a_1 = a_2$ and $a_3 = a_4$. Therefore, $U^\perp$ is spanned by $\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^\top$ and $\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^\top$. In other words, $U = U^\perp$!

(b) Proposition 7.3 leads us to expect $U \oplus U^\perp = \mathbb{F}_2^4$. What has gone wrong? The answer is hinted at in the question. The suggested "inner product" is not valid, as it fails to be positive definite: there are non-zero vectors v, for example $v = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$, such that $v \cdot v = 0$. This causes the Gram-Schmidt procedure to fail (where?), which in turn invalidates Proposition 7.3 when applied to vector spaces over the field $\mathbb{F}_2$. Proposition 7.3 holds for real vector spaces and, with a suitable definition of inner product, for complex vector spaces too.

5. (a) From the definition of the inner product on $V^{\mathbb{C}}$,

$$\begin{aligned} v \cdot w &= (v' + iv'') \cdot (w' + iw'') \\ &= (v' \cdot w') - i(v' \cdot w'') + i(v'' \cdot w') + v'' \cdot w'' \\ &= (w' \cdot v') - i(w'' \cdot v') + i(w' \cdot v'') + w'' \cdot v'' \\ &= (w' \cdot v' + w'' \cdot v'') + i(w' \cdot v'' - w'' \cdot v') \end{aligned}$$

and

$$\begin{aligned} w \cdot v &= (w' + iw'') \cdot (v' + iv'') \\ &= (w' \cdot v') - i(w' \cdot v'') + i(w'' \cdot v') + w'' \cdot v'' \\ &= (w' \cdot v' + w'' \cdot v'') - i(w' \cdot v'' - w'' \cdot v'). \end{aligned}$$

So $w \cdot v = \overline{v \cdot w}$.

(b) That $\alpha^{\mathbb{C}}$ is self-adjoint follows from the sequence of equalities:

$$\begin{aligned} v \cdot \alpha^{\mathbb{C}}(w) &= (v' + iv'') \cdot \alpha^{\mathbb{C}}(w' + iw'') \\ &= (v' + iv'') \cdot (\alpha(w') + i\alpha(w'')) \\ &= v' \cdot \alpha(w') - i(v' \cdot \alpha(w'')) + i(v'' \cdot \alpha(w')) + v'' \cdot \alpha(w'') \\ &= \alpha(v') \cdot w' - i(\alpha(v') \cdot w'') + i(\alpha(v'') \cdot w') + \alpha(v'') \cdot w'' \\ &= (\alpha(v') + i\alpha(v'')) \cdot (w' + iw'') \\ &= \alpha^{\mathbb{C}}(v) \cdot w. \end{aligned}$$

6. The characteristic polynomial of A is

$$p_A(x) = \det(xI - A) = \begin{vmatrix} x-2 & 0 & 0 \\ 0 & x-3 & 1 \\ 0 & 1 & x-3 \end{vmatrix} = (x-2)^2(x-4).$$

So the eigenvalues are 2 (with multiplicity 2) and 4.

Letting $u = \begin{bmatrix} a & b & c \end{bmatrix}$, the solutions to $(A - 4I)u = 0$ satisfy $a = 0$ and $b + c = 0$. So we may take

$$v_1 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.$$

as the first of our orthonormal set of eigenvectors.

The solutions to $(A - 2I)u = 0$ satisfy $b - c = 0$. So we have a 2-dimensional eigenspace with a possible orthonormal basis

$$v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/2 \\ 1/2 \end{bmatrix} \quad \text{and} \quad v_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/2 \\ 1/2 \end{bmatrix}.$$

(There is flexibility in the choice of v_2 and v_3: we just need two orthonormal vectors whose second and third coordinates are equal, and the above is a natural choice.)

The required matrix P has v_1, v_2 and v_3 as its columns:

$$P = \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/2 & 1/2 \\ -1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix}$$

You may verify that $P^{-1}AP = P^{\top}AP = D$, where

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

7. (a) Since α is self-adjoint it may be written as $\alpha = \lambda_1\pi_1 + \cdots + \lambda_r\pi_r$, where the π_i are (orthogonal) projections satisfying $\pi_i\pi_j = 0$ for all $i \neq j$. Define the linear map β by $\beta = (\lambda_1)^{1/3}\pi_1 + \cdots + (\lambda_r)^{1/3}\pi_r$. Note that the cube roots of all the eigenvalues exist and are unique. (This doesn't work for square roots, which may not be real!) Now note that $\beta^3 = \alpha$, using $\pi_i^2 = \pi_i$ and and $\pi_i\pi_j = 0$.

Alternatively, choose a basis relative to which the matrix representing α is diagonal. Now show that you can take the cube root of the diagonal matrix.

(b) In general, the cube root of α is not unique, since the projections π_i are not unique. for example, both I_3 and the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

from Assignment 5 are representations of cube roots of the identity map.

(c) If β is required to be self-adjoint then the solution is unique. However, I don't know any simple demonstration of this fact.