MTH6140 Linear Algebra II

Coursework 10

1. (a) Prove that the adjoint α^* defined by (7.1) in the Course Notes is indeed a linear map, as claimed. Note that for reasons given in just below (7.1), it is enough to show that $v \cdot \alpha^*(w + w') = v \cdot (\alpha^*(w) + \alpha^*(w'))$ and $v \cdot \alpha^*(cw) = v \cdot (c\alpha^*(w))$, for all $v \in V$.

(b) Prove that $\alpha^{**} = (\alpha^*)^*$ satisfies $\alpha^{**} = \alpha$. Again, it is enough to show that $v \cdot \alpha^{**}(w) = v \cdot \alpha(w)$, for all $v \in V$.

2. *Festive.* Linear algebra Sudoku. The following are matrices representing linear maps with respect to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^4$. However, some entries are missing.

$$A = \begin{bmatrix} 1 & -3 & * \\ * & 2 & 2 \\ -1 & * & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} * & \frac{5}{6} & \frac{1}{6} & -\frac{1}{6} \\ \frac{5}{6} & \frac{1}{2} & * & \frac{1}{6} \\ \frac{1}{6} & * & -\frac{5}{6} & -\frac{1}{2} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{2} & * \end{bmatrix}$$

Given that the first linear map is self-adjoint and the second is orthogonal, fill in the missing entries.

3. Let U be a subspace of a real vector space V.

(a) What is wrong with the following "proof" that $U = (U^\perp)^\perp$? We know from lectures that $V = U \oplus U^\perp$, and from this it follows that $V = U^\perp \oplus (U^\perp)^\perp$. Combining these we have $U \oplus U^\perp = (U^\perp)^\perp \oplus U^\perp$. Now cancel $U^\perp$ from both sides.

(b) *Harder.* Give a correct proof of $U = (U^\perp)^\perp$. One way is to show first that $U \subseteq (U^\perp)^\perp$. Then show that $\dim(U) = \dim((U^\perp)^\perp)$.

4. (a) Define an "inner product" on $\mathbb{F}_2^4$ by

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}^\top \cdot \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix}^\top = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$$

(with arithmetic over $\mathbb{F}_2$, of course). Let U be the subspace of $\mathbb{F}_2^4$ spanned by $\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^\top$ and $\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^\top$. Find a basis for the subspace $U^\perp = \{v \in \mathbb{F}_2^4 : v \cdot u = 0 \text{ for all } u \in U\}$.

(b) *Harder.* How do you square this finding with Proposition 7.3?

5. (a) Let V be an inner product space over $\mathbb{R}$. As in the notes, form the associated vector space $V^{\mathbb{C}}$ over $\mathbb{C}$. Show that the inner product on $V^{\mathbb{C}}$ defined in the notes, i.e.,

$$v \cdot w = (v' + iv'') \cdot (w' + iw'') = (v' \cdot w') - i(v' \cdot w'') + i(v'' \cdot w') + v'' \cdot w'',$$

is indeed skew symmetric and positive definite.

(b) Let $\alpha : V \to V$ be a linear map. As in the notes, extend α to a linear map $\alpha^{\mathbb{C}} : V^{\mathbb{C}} \to V^{\mathbb{C}}$ by defining

$$\alpha^{\mathbb{C}}(v) = \alpha^{\mathbb{C}}(v' + iv'') = \alpha(v') + i\alpha(v''),$$

where $v', v'' \in V$. Verify that if α is self-adjoint as a linear map on V then $\alpha^{\mathbb{C}}$ is self-adjoint as a linear map on $V^{\mathbb{C}}$. That is, show that

$$(v' + iv'') \cdot \alpha^{\mathbb{C}}(w' + iw'') = \alpha^{\mathbb{C}}(v' + iv'') \cdot (w' + iw'')$$

for all $v', v'', w', w'' \in V$.

6. The following matrix represents a self-adjoint linear map on $\mathbb{R}^3$.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$$

Find eigenvalues and a set of orthogonal eigenvectors for A. Hence determine an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

7. (a) Let α be a self-adjoint linear map on a real vector space. Prove that α has a cube root, i.e., that there exists a linear map β such that $\alpha = \beta^3$.

(b) *Harder.* Is this cube root β unique?

(c) *Hardest.* Is the cube root β unique if we require β to be self-adjoint?