MTH6140 Linear Algebra II

Coursework 9 Solutions

---

1. (a)

$$\begin{aligned} p_A(x) &= \begin{vmatrix} x & 1 & 1 \\ -2 & x-3 & -1 \\ -4 & -2 & x-4 \end{vmatrix} \\ &= x \begin{vmatrix} x-3 & -1 \\ -2 & x-4 \end{vmatrix} - \begin{vmatrix} -2 & -1 \\ -4 & x-4 \end{vmatrix} + \begin{vmatrix} -2 & x-3 \\ -4 & -2 \end{vmatrix} \\ &= (x-2)^2(x-3). \end{aligned}$$

The minimal polynomial $m_A(x)$ divides $p_A(x)$ and has the same roots. So $m_A(x)$ is either $(x-2)^2(x-3)$ or $(x-2)(x-3)$. Computing $(A-2I)(A-3I)$ we find

$$(A-2I)(A-3I) = \begin{bmatrix} -2 & -1 & -1 \\ 2 & 1 & 1 \\ 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} -3 & -1 & -1 \\ 2 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so the minimal polynomial is $(x-2)(x-3)$. Clearly it is a product of distinct linear factors.

(b)

$$\begin{aligned} p_B(x) &= \begin{vmatrix} x-2 & 1 & 0 \\ -2 & x-3 & -1 \\ 0 & -2 & x-2 \end{vmatrix} \\ &= (x-2) \begin{vmatrix} x-3 & -1 \\ -2 & x-2 \end{vmatrix} - \begin{vmatrix} -2 & -1 \\ 0 & x-2 \end{vmatrix} \\ &= (x-2)^2(x-3). \end{aligned}$$

Again, the minimal polynomial must be either $(x-2)^2(x-3)$ or $(x-2)(x-3)$. Computing $(B-2I)(B-3I)$ we find

$$(B-2I)(B-3I) = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 0 & 0 \\ 4 & 0 & 2 \end{bmatrix},$$

so the minimal polynomial cannot be $(x-2)(x-3)$ and hence must be $(x-2)^2(x-3)$. (You can check that $(B-2I)^2(B-3I) = O$, as predicted by the Cayley-Hamilton Theorem.)

2. The minimal polynomial is composed of distinct linear factors. In $\mathbb{F}_2$ only two linear factors are possible: x and $x+1$. So the only possible minimal polynomials are x, $x+1$ and $x(x+1)$. These correspond to the zero map, the identity map, and some map α satisfying $\alpha(\alpha+I)=0$. Note that the latter is equivalent to $\alpha^2=\alpha$. All three cases are projections.

3. When a matrix in Jordan normal form is squared, the individual blocks are squared. So we just need to say what happens to an individual block. A 3×3 block is transformed as follows:

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^2 = \begin{bmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{bmatrix}.$$

More generally, we'll end up with a block that has λ^2 repeated on the diagonal, 2λ repeated immediately above the diagonal and 1 repeated immediately above that. The square of a 2×2 block will miss out the 1s.

4. (a) False. If $n=2$ and $A=B=I_2$, then

$$\mathrm{Tr}(AB)=\mathrm{Tr}(I_2)=2\neq 4=\mathrm{Tr}(I_2)\mathrm{Tr}(I_2).$$

(b) True. Let $A=(a_{ij})$ and $B=(b_{ij})$. Then

$$\mathrm{Tr}(A+B)=\sum_{i=1}^{n}(a_{ii}+b_{ii})=\sum_{i=1}^{n}a_{ii}+\sum_{i=1}^{n}b_{ii}=\mathrm{Tr}(A)+\mathrm{Tr}(B).$$

(c) False. If $n=2$ and $A=I_2$, then $\mathrm{Tr}(A^{-1})=2\neq\frac{1}{2}=\mathrm{Tr}(A)^{-1}$.

5. (a) $\mathrm{Tr}(A)=0+4-2=2$ and

$$\det(A)=\begin{vmatrix} 0 & 5 & -3 \\ 1 & -2 & 1 \\ 1 & -5 & 4 \end{vmatrix} = -5\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} - 3\begin{vmatrix} 1 & -2 \\ 1 & -5 \end{vmatrix} = -15+9=-6.$$

(b)

$$\begin{aligned} p_A(x)=\det(xI-A) &= \begin{vmatrix} x & -5 & 3 \\ -1 & x+2 & -1 \\ -1 & 5 & x-4 \end{vmatrix} \\ &= x\begin{vmatrix} x+2 & -1 \\ 5 & x-4 \end{vmatrix} + 5\begin{vmatrix} -1 & -1 \\ -1 & x-4 \end{vmatrix} + 3\begin{vmatrix} -1 & x+2 \\ -1 & 5 \end{vmatrix} \\ &= x^3-2x^2-5x+6 \\ &= (x-1)(x+2)(x-3). \end{aligned}$$

Note that the constant coefficient of $p_A(x)$ is $6=(-1)^3\det(A)$ and the coefficient of x^2 is $-2=-\mathrm{Tr}(A)$.

(c) From the factorisation of the characteristic polynomial, we see that the eigenvalues of A are 1, -2 and 3. The product of eigenvalues is -6 which is indeed equal to $\det(A)$, and the sum is 2, which is the trace of A.

6. Let $r = \dim(\mathrm{Im}(\pi))$ be the rank of π. Choose a basis $v_1, \ldots, v_r$ of $\mathrm{Im}(\pi)$ and a basis $v_{r+1}, \ldots, v_n$ of $\mathrm{Ker}(\pi)$. Since $V = \mathrm{Im}(\pi) \oplus \mathrm{Ker}(\pi)$, we know that $v_1, \ldots, v_n$ is a basis of V. Let Π be the representation of π in this basis. What does the matrix Π look like? Well, $\pi(v_i) = v_i$ if $1 \leq i \leq r$, and $\pi(v_i) = 0$ for $r < i \leq n$. So Π is a block matrix with I_r in the top left corner and zeros elsewhere. In particular, $\mathrm{Tr}(\Pi) = r$.

Although Π is a particular (and special) matrix representing π, we know that any matrix representing Π has trace r, since similar matrices have the same trace.

7. There are three properties to check.

- Symmetric. For all $p, q \in V_n$,

$$p \cdot q = \int_0^1 p(x)q(x)\,dx = \int_0^1 q(x)p(x)\,dx = q \cdot p.$$

- Bilinear. For all polynomials $p, p', q \in V_n$, and scalars $c \in \mathbb{R}$ we have

$$\begin{aligned}(p + p') \cdot q &= \int_0^1 (p(x) + p'(x))q(x)\,dx \\ &= \int_0^1 p(x)q(x)\,dx + \int_0^1 p'(x)q(x)\,dx \\ &= p \cdot q + p' \cdot q.\end{aligned}$$

  and

$$(cp) \cdot q = \int_0^1 cp(x)q(x)\,dx = c\int_0^1 p(x)q(x)\,dx = c(p \cdot q).$$

  The same clearly holds for the second argument.

- Positive definite. For all polynomials $p \in V_n$,

$$p \cdot p = \int_0^1 p(x)^2\,dx \geq 0,$$

  with equality only if p is identically zero.