MTH6140 Linear Algebra II

Coursework 8 Solutions

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1. (a) D maps each non-constant polynomial to a polynomial of degree one lower, and each constant polynomial to 0. So D has only one eigenvalue, namely 0, with corresponding eigenvector 1 (the constant polynomial 1). Naturally, any constant polynomial not equal to 0 is an equally good eigenvector.

Or, symbolically, we are looking for solutions $f(x) = ax^3 + bx^2 + cx + d$ to $f'(x) = \lambda f(x)$. Substituting for $f(x)$ we have

$$3ax^2 + 2bx + c = f'(x) = \lambda f(x) = \lambda ax^3 + \lambda bx^2 + \lambda cx + \lambda d.$$

Equating coefficients, we see that $\lambda a = 0$, $\lambda b = 3a$, $\lambda c = 2b$ and $\lambda d = c$. There are two cases. If $\lambda = 0$ then $a = b = c = 0$, leading to the eigenvector 1 identified above. If $\lambda \neq 0$, then $a = b = c = d = 0$ and there are no corresponding eigenvectors. (Recall that the zero vector is not an eigenvalue, by definition.)

(b) If $f(x) = ax^3 + bx^2 + cx + d$, then $xf'(x) = 3ax^3 + 2bx^2 + cx$. For f to be an eigenvector, $3ax^2 + 2bx^2 + cx = \lambda(ax^3 + bx^2 + cx + d)$, for some λ, and hence $a(3 - \lambda) = b(2 - \lambda) = c(1 - \lambda) = \lambda d = 0$. So there are four possible eigenvalues, namely $\lambda = 0$, $\lambda = 1$, $\lambda = 2$ and $\lambda = 3$, with corresponding eigenvectors 1, x, x^2 and x^3 (or some scaling of these).

(c) The eigenvectors $(1, x, x^2, x^3)$ of $\widehat{D}$ form a basis for V_4, so $\widehat{D}$ is diagonalisable; in fact the matrix relative to the basis $(1, x, x^2, x^3)$ is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

However, D has only one eigenvector and so is not diagonalisable.

2. Since α is a projection, $\alpha^2 = \alpha$. Since α is invertible, we may act on both sides by α^{-1} to obtain $\alpha^{-1}\alpha^2 = \alpha^{-1}\alpha$, i.e., $\alpha = I$. So α must be the identity map.

3. (a) Following the recipe described and justified in lectures, we may take the columns of P to be the three eigenvectors v_1 v_2 and v_3 of A viewed as column vectors. Also, Q is the inverse of P. Thus.

$$P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -2 & -1 & 2 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 3 & 1 & 1 \\ -2 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix}.$$

It can be checked that the product QAP does indeed equal D or, equivalently, $A = PDQ$.

(b) We know that $A = PDP^{-1}$, and hence $A^t = (PDP^{-1})^t = PD^tP^{-1}$. Note that

$$(\tfrac{1}{3}D)^t = \begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}^t \xrightarrow[t\to\infty]{} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so $(\frac{1}{3}A)^t = P(\frac{1}{3}D)^tP^{-1}$ tends to

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ -2 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 & -1 \\ 2 & 1 & 1 \\ 4 & 2 & 2 \end{bmatrix},$$

as $t \to \infty$.

4. (a) False. It may happen that the minimal polynomial is a product of distinct linear factors, but the roots of the minimal polynomial are not real. For example, the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

are $1 \pm i$, so A has no eigenvectors when viewed as a linear map on $\mathbb{R}^2$, but has two linearly independent eigenvectors when viewed as a linear map on $\mathbb{C}^2$. These eigenvectors form a basis for $\mathbb{C}^n$. The diagonal form in this case is

$$\begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}.$$

Alternatively, the characteristic polynomial of A is $p_A(x) = x^2 - 2x + 2$. Over $\mathbb{R}$, the minimal polynomial $m_A(x)$ is equal to $p_A(x)$ which is not a product of linear factors. Over $\mathbb{C}$, it is also the case that $m_A(x) = p_A(X)$, but now $m_A(x)$ factors into distinct linear factors: $m_A(x) = (x - 1 + i)(x + 1 - i)$.

(b) True. If the minimal polynomial is a product of distinct linear factors over $\mathbb{R}$, then it is a product of linear factors (the same ones!) over $\mathbb{C}$. The claim then follows from Theorem 5.20. Alternatively, if $\mathbb{R}^n$ has a basis of real eigenvectors of A, then those same eigenvectors will form a basis of $\mathbb{C}^n$. (Stop to think why this is so.)

(c) True. The minimal polynomial has the eigenvalues of A as roots. If these are distinct, say $\lambda_1, \ldots, \lambda_n$, then the minimal polynomial must be $(x - \lambda_1) \cdots (x - \lambda_n)$. (It must have all these factors, and it can't have more as its degree is already n, which is the maximum possible.) So the matrix A is diagonalisable. Alternatively, we know that eigenvectors with distinct eigenvalues are linearly independent. So the eigenvectors of A form a linearly independent list of size n, i.e., a basis, of $\mathbb{R}^n$.

5. (a) The matrix A must satisfy $A - I = O$, so the only possibility is that $A = I$.

(b) We want the matrix A to have 1 as its only eigenvalue, but we want to avoid the situation that arose in part (a). The simplest solution is to add an extra non-zero entry to the identity matrix to yield, say, $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. This matrix satisfies $(A - I)^2 = O$ but not $A - I = O$.

(c) Since the minimal polynomial has degree 2, it must equal the characteristic polynomial. Noting that $m_\alpha(x) = (x-1)^2 + 1$, we can choose $A = \left[\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right]$, which has characteristic polynomial $x^2 - 2x + 2$.