L19 How do we find the Π_i ? If α is diagonalizable with basis $v_1 \dots v_n$ of eigenvectors, let

$$P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

wrt some basis where $\alpha \leftrightarrow A = (a_{ij})$

(we regard the v_i as column vectors w.r.t. this)

Then

$$A.P = \begin{bmatrix} \lambda_1 v_1 , & \dots & \lambda_n v_n \end{bmatrix}$$

(the λ_i here are not nec. distinct)

$$= P \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\Rightarrow \quad P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

is the matrix of eigenvectors actually puts A into diagonal form (assuming α is diagonalizable)

Next, we group the eigenvectors according to the distinct eigenvalues (ie renumber the v_i)

$$P^{-1}AP = D = \begin{pmatrix} \lambda_1 I_{d_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_r I_{d_r} \end{pmatrix}, \quad \lambda_i \text{ distinct}$$

Here $V = \bigoplus_{i=1}^{n} E(\lambda_i, \alpha)$, $\quad d_i = \dim(E(\lambda_i, \alpha))$

Let $\Pi_i^D = \begin{pmatrix} 0 & & & 0 \\ & \ddots & & \\ & & I_{d_i} & \\ 0 & & & \ddots 0 \end{pmatrix}$ e.g. $\Pi_1^D = \begin{pmatrix} 1 & & & 0 \\ & \ddots 1 & \}d_1 & \\ & & 0 & \\ 0 & & & \ddots 0 \end{pmatrix}$

clearly $(\Pi_i^D)^2 = \Pi_i^D$, $\quad \Pi_i^D \Pi_j^D = 0 \quad j \neq i$, $\quad \sum_i \Pi_i^D = I_n$

and $\quad D = \sum \lambda_i \Pi_i^D$

$\Rightarrow A = PDP^{-1} = \sum_i \lambda_i \underbrace{P \Pi_i^D P^{-1}}_{}$

$\Updownarrow$.

$\Pi_i \leftrightarrow \pi_i$ as linear map.

ie $\quad \alpha = \sum_i \lambda_i \pi_i$ in the basis.

Check $\quad \Pi_i$ are projections etc.

$$(P\Pi_i^D P^{-1})^2 = P \underbrace{\Pi_i^D \cancel{P^{-1}P} \Pi_i^D}_{\Pi_i^D} P^{-1} = P\Pi_i^D P^{-1}$$

similarly for $\quad P\Pi_i^D \cancel{P^{-1}P} \Pi_j^D P^{-1} = 0 \quad i \neq j$

and $\quad \sum_i P\Pi_i^D P^{-1} = P(\underbrace{\sum_i \Pi_i^D}_{I_n})P^{-1} = I_n$

so Π_i have the required properties.

Example. $\quad A = \begin{bmatrix} -6 & 6 \\ -12 & 11 \end{bmatrix} \leftrightarrow \alpha : \mathbb{R}^2 \to \mathbb{R}^2$

wrt standard basis

we saw that $v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ eigenvectors with eigenvalues $\lambda_1 = 2$, $\lambda_2 = 3$

$\Rightarrow P = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad d_1 = d_2 = 1$

Check $\quad \underbrace{\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}}_{P} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}}_{P^{-1}} = \begin{bmatrix} -6 & 6 \\ -12 & 11 \end{bmatrix}$

$\pi_1 \leftrightarrow \Pi_1 = P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ 12 & -8 \end{bmatrix}$

$\pi_2 \leftrightarrow \Pi_2 = P \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} P^{-1} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -8 & 6 \\ -12 & 9 \end{bmatrix}$

$\alpha = 2\pi_1 + 3\pi_2 \leftrightarrow A = 2 \begin{bmatrix} 9 & -6 \\ 12 & -8 \end{bmatrix} + 3 \begin{bmatrix} -8 & 6 \\ -12 & 9 \end{bmatrix}$ ✓

Example / quiz $V =$ polynomials of degree ≤ 3 (and zero)

basis $1, y, y^2, y^3$ over $\mathbb{R}$

$$\alpha(f) = f' \leftrightarrow A = \begin{bmatrix} \alpha(1) & \alpha(y) & \alpha(y^2) & \alpha(y^3) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

as $\alpha(1) = 0$, $\alpha(y) = 1$, $\alpha(y^2) = 2y$, $\alpha(y^3) = 3y^2$

$\boxed{Q}$ Are there any eigenvectors of α? $\alpha(f) = \lambda f$ some λ

☑ yes ☐ no

$f = 1$, $\alpha(1) = 0 = 0.1$ eigenvalue 0.

$\boxed{Q}$ Are there any others? No as f' lowers degree so cant have $f' = \lambda f$ except $\lambda = 0$. So α is an example of a linear map which is not diagonalizable.

5.5 Characteristic and minimal polynomials

Recall that if $\alpha : V \to V$ corresponds to A wrt a basis and we change the basis then A changes to $A' = P^{-1} A P$. The eigenvalues of α don't change and other quantities like

$\mathrm{Tr}(A) = \mathrm{Tr}(A')$

$\det(A) = \det(A')$ $\leftrightarrow$ all properties of α independent of the basis.

$P_A(x) = P_{A'}(x)$

Def 5.16 (a) We define $\det(\alpha)$ of a linear map $\alpha : V \to V$ (V f.d.) as the det of any matrix corresponding to α w.r.t a basis ("representing" α). Similarly $tr(\alpha)$ is trace of any matrix representing α.

(b) Define $P_\alpha(x)$ as the characteristic polynomial $P_A(x)$ for any A representing α.

(c) We fine $m_\alpha(x)$ the "minimal polynomial of α" as the least degree monic polynomial s.t. $m_\alpha(\alpha) = 0$ as map $V \to V$

(Here, "monic" means the coefficient of the top power of x is 1 so $x^n + a_{n-1}x^{n-1} + \dots + a_0 1$ is a monic of degree n)

We can also define $m_A(x)$ in the same way as the smallest degree monic s.t. $m_A(A) = 0$. This is independent of the choice of matrix representing α:

If $m_0 I_n + m_1 A + \cdots A^r = 0$ ($n = \dim V$)

$\Leftrightarrow$ $P^{-1}(m_0 I_n + m_1 A + \cdots A^r)P = 0$

$= m_0 I_n + m_1 A' + \cdots A'^r$ $\quad (A' = P^{-1}AP)$

(e.g. $(P^{-1}AP)^r = P^{-1}A\cancel{P}\cancel{P^{-1}}AP \cdots \cancel{P^{-1}}AP = P^{-1}A^rP$ etc)

so $m_A(A') = 0$

This is the property of $m_{A'}(x)$ and this has the same degree as $m_A(x)$.

But $m_{A'}(x)$ has ~~small~~ smallest degree s.t. $m_{A'}(A') = 0$
$\therefore \quad m_A = m_{A'}$.

So $m_\alpha(x)$ is defined, independently of the choice of basis.

We know that $p_\alpha(\alpha) = 0$ by cayley-hamilton theorem so $m_\alpha(x)$ exists.

Finally, $m_\alpha(x)$ is unique with the property stated since if $m_\alpha(x)$, $m'_\alpha(x)$ are two candidates for minimal polynomial then $(m_\alpha - m'_\alpha)(\alpha) = 0 - 0 = 0$

but $m - m'_\alpha$ is either zero or has smaller degree. Latter case is not allowed since can divide thru by the coeff of the top degree and get a monic of smaller degree that vanishes on α. (but m_α, m'_α were smallest such). So $m_\alpha = m'_\alpha$

$\boxed{L20}$

Last time we defined $m_\alpha(x)$ the minimal polynomial of a map $\alpha: V \to V$ as the lowest degree monic s.t. $m_\alpha(\alpha) = 0$

- we saw its unique
- we saw its can be computed for any matrix representation A of α as $m_A(x)$ (smallest degree monic s.t. $m_A(A) = 0$) and is independent of the choice of representative.

<u>Prop 5.17</u> For any $\alpha: V \to V$, $m_\alpha(x)$ divides $P_\alpha(x)$.

<u>Proof</u> Suppose not. We have Euclids algorithm (which works for polynomials over any field) to write
$$P_\alpha(x) = q(x)\, m_\alpha(x) + r(x)$$
where $r = 0$ <u>or</u> r has degree $<$ degree of $m_\alpha(x)$.

$\Rightarrow 0 = P_\alpha(\alpha) = \underbrace{m_\alpha(\alpha)}_{0}\, q(\alpha) + r(\alpha)$ as maps $V \to V$. $\Rightarrow r(\alpha) = 0$. If $r \neq 0$ we can normalize so that it is monic, which contradicts that m_α was minimal.

$\therefore r = 0$, $P_\alpha(x) = q(x)\, m_\alpha(x)$ some poly $q(x)$ QED

<u>Theorem 5.18</u> (V is f.d.) Let $\alpha: V \to V$ be a linear map TFAE

(a) λ an eigenvalue of α

(b) λ a root of $P_\alpha(x)$

(c) λ a root of $m_\alpha(x)$

Example $A = \begin{bmatrix} -6 & 6 \\ -12 & 11 \end{bmatrix}$, $\alpha : V \longrightarrow V$, $V = \mathbb{R}^2$ with its standard basis.

$$P_A(x) = \underset{\det}{|xI_2 - A|} = \begin{vmatrix} x+6 & -6 \\ 12 & x-11 \end{vmatrix}$$

$$= (x+6)(x-11) + 72$$
$$= x^2 - 5x + 6$$
$$= (x-2)(x-3)$$

f.t.c what we found for the eigenvalues: $\lambda = 2, 3$ of A, these are the roots of $P_A(x)$ and hence $P_\alpha(x)$. $m_\alpha(x)$ has to divide $P_\alpha(x)$

So options are $m_\alpha(x) = \begin{cases} (x-2)(x-3) = P_\alpha(x) \\ x-2 & \text{X doesn't have 3 as root} \\ x-3 & \text{X doesn't have 2 as root} \end{cases}$

$\therefore \quad m_\alpha(x) = P_\alpha(x) = (x-2)(x-3)$

Example $\alpha = \frac{d}{dy}$, $V = \mathbb{R}[y]_3$ (polys degree ≤ 3)

$$\longleftrightarrow \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{in basis } 1, y, y^2, y^3$$

$$P_A(x) = \det(xI_4 - A) = \begin{vmatrix} x & -1 & 0 & 0 \\ 0 & x & -2 & 0 \\ 0 & 0 & x & -3 \\ 0 & 0 & 0 & x \end{vmatrix} = x^4 = P_\alpha(x)$$

Spot quiz What is $m_\alpha(x)$ or $m_A(x)$?

☑ x^4 ☐ x^3 ☐ x^2 ☐ x ☐ other

we know it divides $P_A = x^4$ and can't be $m_A = 1$ as vanishes on α, can't be x as $m_A(x) = 0 \Rightarrow A = 0$, but $A^2 = \begin{bmatrix} 0&0&2&0 \\ 0&0&0&6 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \neq 0$, $A^3 = \begin{bmatrix} 0&0&0&6 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \neq 0$

so similarly $m_A(x)$ cant be x^2, x^3 $\therefore m_\alpha = x^4$.

If we do this over $\overline{\mathbb{F}_3}$, define $\frac{d}{dy} y^m = m y^{m-1}$

on $\mathbb{F}_3[x]_3$ but now $A \neq 0$, $A^2 \neq 0$ but $A^3 = 0$

so $m_\alpha(x) = x^3$ in this case.

<u>Example</u> (variant of previous) $V = \mathbb{R}[y]_3$

$\beta(f) = y f' \longleftrightarrow B = [\beta(1), \beta(y), \beta(y^2), \beta(y^3)]$

$= \begin{bmatrix} 0 & & & 0 \\ & 1 & & \\ & & 2 & \\ 0 & & & 3 \end{bmatrix}$ eg $\beta(y^3) = 3y^3$

so $B = \sum_{i=1}^{4} \lambda_i \Pi_i$, $\Pi_1 = \begin{pmatrix} 1 & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix}$, $\Pi_2 = \begin{pmatrix} 0 & & & 0 \\ & 1 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix}$ etc.

$\lambda = 0, 1, 2, 3$

$P_B(x) = \begin{vmatrix} x & & & 0 \\ & x-1 & & \\ & & x-2 & \\ 0 & & & x-3 \end{vmatrix} = x(x-1)(x-2)(x-3) = P_\beta(x)$

4 distinct roots corresponding to 4 distinct eigenvalues. $m_B(x) = m_\beta(x)$ has to have the same root, so $m_B(x) = P_\beta(x)$.

<u>proof of</u> Theorem 5.18 <u>(a)$\Rightarrow$(c)</u>: Let λ be an eigenvalue of α with eigenvector v, $\alpha(v) = \lambda v$, $v \neq 0$

$\Rightarrow \alpha^k(v) = \lambda^k v \Rightarrow f(\alpha)v = f(\lambda)v$, for any polynomial f. Choose $f = m_\alpha$ so

$0 = m_\alpha(\alpha) = m_\alpha(\lambda)v$, but $v \neq 0 \Rightarrow m_\alpha(\lambda) = 0$

(92)

(b) $\Rightarrow$ (a): Suppose $p_\alpha(\lambda)=0$ ie $\det(\lambda I-\alpha)=0$

so $\lambda I-\alpha$ is not invertible

ie. $\lambda I-\alpha$ not of full rank, $\dim(\operatorname{Im}(\lambda I-\alpha)) < \dim V$

$\Leftrightarrow$ by rank + nullity formula,

$\dim(\ker(\lambda I-\alpha))>0$,

ie $\nu(\lambda I-\alpha)>0$ ie $\exists\, v\neq 0,\ v\in V$ s.t.

$(\lambda I-\alpha)v=0$ ie $\alpha v=\lambda v$ ie an eigenvector.

(c) $\Rightarrow$ (b): Suppose λ a root of m_α so $x-\lambda$ divides $m_\alpha(x)$. But $m_\alpha(x)$ divides $p_\alpha(x)$

so $x-\lambda$ divides $p_\alpha(x)$ [ie $m_\alpha=(x-\lambda)q(x)$, so $p_\alpha=m_\alpha\cdot q'(x)=(x-\lambda)qq'$]

$\therefore$ λ a root of p_α.

QED

$$\boxed{\begin{array}{l}\textbf{Theorem 5.20}\ \text{If } \alpha: V\to V \text{ a linear map } (V \text{ f.d.}) \text{ then } \alpha \text{ is diagonalizable iff}\\ m_\alpha \text{ is a product of } \underline{\text{distinct}} \text{ linear factors}\end{array}}$$

e.g. $A=\begin{bmatrix}-6 & 6\\ -12 & 11\end{bmatrix}$ we saw $m_\alpha(x)=(x-2)(x-3)$

linear factors, distinct.

so α diagonalizable as we saw.

By contrast $A \longleftrightarrow \alpha(f)=f'$ over $\mathbb{R}$, $m_\alpha(x)=x^4$ is a product of linear factors $(x-0)$ but repeated so not diagonalizable. Similarly over $\mathbb{F}_3$, $\mathbb{F}_2$.

This theorem tells us us that before trying to find a basis of eigenvectors to diagonalize α, check its m_α to see if it is possible, and find the eigenvalues by

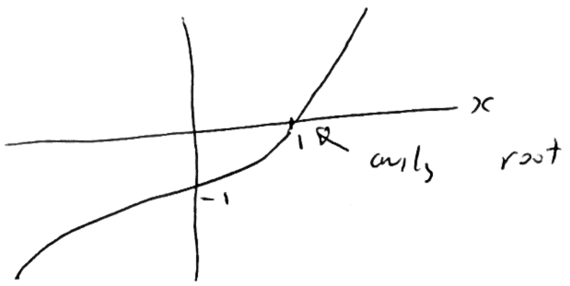
Theorem 5.18

$\boxed{L21}$ Example $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ over $\mathbb{R}$

$$P_A(x) = \begin{vmatrix} x & -1 & 0 \\ 0 & x & -1 \\ -1 & 0 & x \end{vmatrix} = x^3 - 1 = (x-1)(x^2+x+1)$$

this does not factorise further over $\mathbb{R}$ as x^2+x+1 has no roots over $\mathbb{R}$ (from the formula for roots of quadratic).

Or sketch $P_A(x) = x^3 - 1$



$m_A(x)$ divides $P_A(x)$ so the options are

$x-1$ or $(x-1)(x^2+x+1)$

(or x^2+x+1 X as m_A has the same roots as P_A)

X as $m_A(A) = 0$ would need $A - I_3 = 0$

$A = I_3$ not true.

$\therefore$ $m_A(x) = P_A(x) = (x-1)(x^2+x+1)$

not linear factor

$\therefore$ m_A not a product of distinct linear factors

$\therefore$ A not diagonalisable over $\mathbb{R}$.

Spot quiz is A above diagonalisable over $\mathbb{C}$?

☑ yes ☐ no

Either factorize $p_A(x)$ using roots of x^2+x+1

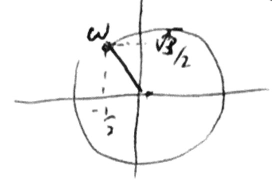
as $\dfrac{-1 \pm \sqrt{-3}}{2} = \dfrac{-1 \pm i\sqrt{3}}{2} = \begin{cases} \omega = e^{\frac{2\pi i}{3}} & (+ \text{case}) \\ \omega^2 = e^{-\frac{2\pi i}{3}} \end{cases}$

So $m_\alpha(x) = (x-1)(x-\omega)(x-\omega^2)$

three distinct linear factors $\therefore$ diagonalizable

with eigenvalues $1, \omega, \omega^2$

$\mathbb{C}$



or $e^{iy} = \cos y + i \sin y$

$y = 2\pi/3$

OR without knowing the value of ω as root of x^2+x+1, we know ω is not real $\therefore$ $\bar{\omega}$ is also root as $\omega^2+\omega+1=0 \Rightarrow \bar{\omega}^2+\bar{\omega}+1=0$

and $\bar{\omega} \neq \omega$ so $m_\alpha(x) = (x-1)(x-\omega)(x-\bar{\omega})$

$\therefore$ diagonalizable as product of distinct linear factors.

Note in this example that since A over $\mathbb{C}$ is diagonalizable, $\exists$ a basis of eigen-vectors, namely

$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix}$

$\lambda = 1$ $\qquad$ $\lambda = \omega$ $\qquad$ $\lambda = \omega^2$

hence $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$ diagonalizes A,

$$P \underbrace{\begin{bmatrix} 1 & & 0 \\ & \omega & \\ 0 & & \omega^2 \end{bmatrix}}_{\text{diagonal form}} P^{-1} = A$$

There could be other reasons that m_α is not a product of distinct linear factor.

e.g. $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_B(x) = \begin{vmatrix} x-2 & -1 & 0 \\ 0 & x-2 & 0 \\ 0 & 0 & x-1 \end{vmatrix}$

$= (x-2)^2(x-1)$

$m_B(x)$ has same roots so must contain at least $(x-2)(x-1)$ so options are

$m_B = (x-2)(x-1)$, or $(x-2)^2(x-1) = P_B$

check $(B - 2I_3)(B - I_3) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0$

$\therefore m_B \neq (x-2)(x-1)$ $\therefore$ $m_B = P_A = (x-2)^2(x-1)$

$\therefore$ m_B is not the product of distinct linear factors $\therefore$ not diagonalizable by Theorem 5.20.

Proof of Theorem 5.20 (in one direction) Assume α is diagonalizable so $\exists$ basis $v_1 \dots v_n$ ($n = \dim V$) of eigenvectors. Let $\lambda_1 \dots \lambda_r$ be the distinct eigenvalues. ($r \le n$) and consider $m(x) = (x-\lambda_1)(x-\lambda_2) \dots (x-\lambda_r)$.

We'll show this is m_α. Indeed

$$m(\alpha)v_j = (\alpha - \lambda_1 I) \dots (\alpha - \lambda_r I) v_j$$

$= 0$ as $(\alpha - \lambda_i I)v_j = 0$ if v_j eigenvect of eigenvalue λ_i

and $(\alpha - \lambda_i I)$ is among the $(\alpha - \lambda_1 I) \dots (\alpha - \lambda_r I)$

(Here each basis element v_j is associated to some λ_i in the list $\lambda_1, \dots \lambda_r$ as its eigenvalue)

Here the order of factors doesn't matter so we can bring this $(\alpha - \lambda_i I)$ to the right to act on v_j.

So $m(\alpha)v_j = 0$ for all eigenvectors $V_1 \ldots V_n$, but these are a basis so $m(\alpha) = 0$. Now by Thm 5.18 all the eigenvalues of α are roots of m_α so m_α must have degree $\geq r$ and obey $m_\alpha(\alpha) = 0$.

So m already does this, so $m_\alpha = m$.

(Proof in the other direction is in the printed notes)

Q.E.D.

Looked at Q4 May 2019 Exam