MTH6140 Linear Algebra II

Coursework 7 Solutions

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1. Suppose first that $w \in \mathrm{Im}(\beta\alpha)$, that is, $w = (\beta\alpha)(v)$ for some $v \in V$. Then $w = \beta(v')$ where $v' = \alpha(v)$, and hence $w \in \mathrm{Im}(\beta)$. Now suppose $w \in \mathrm{Im}(\beta)$, that is, $w = \beta(v)$ for some $v \in V$. Let $v' = \alpha^{-1}(v)$. (Recall that α is invertible.) Then $w = \beta(v) = \beta(\alpha(v')) = (\beta\alpha)(v')$, and hence $w \in \mathrm{Im}(\beta\alpha)$.

2.
- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so this linear map is a projection.
- $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, so this linear map is a projection.
- $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so this linear map is not a projection.
- $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, so this linear map is not a projection.
- $\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}^2 = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$, so this linear map is a projection.

3. (a) Observe that $D_1^2 = D_1$, $D_2^2 = D_2$, $D_1D_2 = D_2D_1 = O$ and $D_1 + D_2 = I$. We have

$$\Pi_1^2 = (PD_1P^{-1})(PD_1P^{-1}) = PD_1D_1P^{-1} = PD_1P^{-1} = \Pi_1,$$

so Π_1 is projection, and similarly for Π_2. Also

$$\Pi_1\Pi_2 = (PD_1P^{-1})(PD_2P^{-1}) = PD_1D_2P^{-1} = POP^{-1} = O,$$

and similarly for $\Pi_2\Pi_1$, as required. Finally

$$\Pi_1 + \Pi_2 = (PD_1P^{-1}) + (PD_2P^{-1}) = P(D_1 + D_2)P^{-1} = PIP^{-1} = I.$$

4. From the definition of direct sum, every vector $v \in V$ can be uniquely written as $v = u + w$, where $u \in U$ and $w \in W$; we define π by the rule that $\pi(v) = u$. The first step is to verify that π is a linear map. Suppose $v' = u' + w'$ where $u' \in U$ and $w' \in W$, so that $\pi(v') = u'$. Then $v + v' = (u + w) + (u' + w') = (u + u') + (w + w')$, where $u + u' \in U$ and $w + w' \in W$, and hence

$\pi(v+v') = u + u' = \pi(v) + \pi(v')$. The identity $\pi(cv) = c\pi(v)$ can be shown similarly. With v as before, $cv = cu + cw$ and hence $\pi(cv) = cu = c\pi(v)$.

It is immediate from the definition of π that $\mathrm{Im}(\pi) \subseteq U$. Also, for any $u \in U$, we have that $\pi(u) = u$ and hence $U \subseteq \mathrm{Im}(\pi)$. It follows that $\mathrm{Im}(\pi) = U$. A vector v is in $\mathrm{Ker}(\pi)$ iff it has the form $\mathbf{0} + w$ for some $w \in W$, and and hence $\mathrm{Ker}(\pi) = W$.

Finally, suppose $v = u + w$ with $u \in U$ and $w \in W$. Then $\pi(v) = u$ and $\pi^2(v) = \pi(u) = u$. Thus $\pi^2 = \pi$, and π is a projection.

5. (a) We are asked to solve

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}.$$

We have $b = \lambda a$ and $-a = \lambda b$, and hence $a = -\lambda^2 a$. If $a = 0$ then $b = 0$, but the zero vector is never an eigenvector. Otherwise, $\lambda^2 = -1$, which doesn't have a solution in $\mathbb{R}$. So in this case, the linear map has no eigenvalues (and no eigenvectors).

(b) As before, we can eliminate the possibility that $a = 0$. But over $\mathbb{C}$, when $a \neq 0$, we have the solutions (eigenvalues) $\lambda = \pm i$. The corresponding eigenvectors are $\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]$. (Any non-zero scaling of these will be eigenvectors also.)

(c) Using what we have learned so far, we might guess that something like

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

will be a suitable candidate. Indeed, repeating the above calculation, we find that $\lambda^2 = 2$, which is not solvable over $\mathbb{Q}$, but yields eigenvalues $\pm\sqrt{2}$ over $\mathbb{R}$.

6. Suppose $v, v' \in E(\lambda, \alpha)$. Then $\alpha(v) = \lambda v$ and $\alpha(v') = \lambda v'$, and hence $\alpha(v + v') = \lambda(v + v')$. This shows that $v + v' \in E(\lambda, \alpha)$. Also, $\alpha(cv) = c\alpha(v) = c\lambda v = \lambda(cv)$, and hence $cv \in E(\lambda, \alpha)$. Since $E(\lambda, \alpha)$ is non-empty (contains $\mathbf{0}$), and closed under vector addition and scalar multiplication, it is a subspace of V.

7. Letting $v = \begin{bmatrix} a & b & c \end{bmatrix}^\top$, solving $Av = 2v$ yields $2a + b + c = 0$. This equation defines a subspace of dimension 2, and a possible basis for it is $\left(\begin{bmatrix} 1 & 0 & -2 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T\right)$: this is $E(2, \alpha)$. (A variety of bases are possible: any two vectors satisfying $2a + b + c$ that are not multiples of each other.)

Solving $Av = 3v$ yields $2a + c = 0$ and $a + b = 0$. These equations define a subspace of dimension 1, with basis $\begin{bmatrix} -1 & 1 & 2 \end{bmatrix}^\top$: this is $E(3, \alpha)$.