MTH6140 Linear Algebra II

Coursework 7

1. Suppose V and W are vector spaces, and $\alpha : V \to V$ and $\beta : V \to W$ are linear maps. Prove that, if α is invertible then $\mathrm{Im}(\beta\alpha) = \mathrm{Im}(\beta)$. (Thus the rank of $\beta\alpha$ is equal to the rank of β.)

2. The following matrices represent linear maps on $\mathbb{R}^2$ with respect to some basis. Which of these linear maps are projections?

$$\text{(a)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{(d)} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \text{(e)} \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}.$$

3. Suppose $1 \le r < n$, and let D_1 and D_2 be the $n \times n$ diagonal matrices

$$D_1 = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} \quad \text{and} \quad D_2 = \begin{bmatrix} O & O \\ O & I_{n-r} \end{bmatrix},$$

over an arbitrary field $\mathbb{K}$. Let P be any invertible $n \times n$ matrix. Define the matrices $\Pi_1 = PD_1P^{-1}$ and $\Pi_2 = PD_2P^{-1}$. Show that Π_1 and Π_2 are projections on $\mathbb{K}^n$, that $\Pi_1\Pi_2 = \Pi_2\Pi_1 = O$ and that $\Pi_1 + \Pi_2 = I$. (C.f. Proposition 5.4.)

4. Complete the proof of Proposition 5.3 of the (draft) notes, following the sketch provided there. (Having defined π as in the sketch, you need to show that π is a linear map, that $\mathrm{Im}(\pi) = U$ and $\mathrm{Ker}(\pi) = W$, and that $\pi^2 = \pi$.)

5. Consider the linear map on $\mathbb{K}^2$ represented by the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(a) Find the eigenvectors and eigenvalues of A when $\mathbb{K} = \mathbb{R}$.

(b) Find the eigenvectors and eigenvalues of A when $\mathbb{K} = \mathbb{C}$.

(c) Write down a 2×2 matrix over $\mathbb{K}$ that has two eigenvalues if $\mathbb{K} = \mathbb{R}$ and none if $\mathbb{K} = \mathbb{Q}$. Explain your answer.

6. Let α be a linear map on vector space V. Recall that $E(\lambda, \alpha)$ is the set of all vectors $v \in V$ such that $\alpha(v) = \lambda v$. Verify that $E(\lambda, \alpha)$ is a subspace of V.

7. Suppose the linear map α on $\mathbb{R}^3$ is represented with respect to the standard basis by the matrix

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 2 & 3 & 1 \\ 4 & 2 & 4 \end{bmatrix}.$$

What are the dimensions of the eigenspaces $E(2, \alpha)$ and $E(3, \alpha)$? Give bases for these eigenspaces.