By the rank + nullity theorem $\nu(\alpha) + \rho(\alpha) = 3$

$\therefore \rho(\alpha) = 2 = \dim(W)$ and $\mathrm{im}(\alpha) \subseteq W$

$\therefore \mathrm{im}(\alpha) = W$ in this example.

$\boxed{L16}$

We've been studying linear maps $\alpha : V \to W$ and the associated matrix A when bases are given for V, W.

Def 4.9 Let $\alpha, \beta : V \to W$ be linear maps. Their sum $\alpha + \beta : V \to W$ is a linear map defined by $(\alpha+\beta)(v) = \alpha(v) + \beta(v) \quad \forall v \in V$

Def 4.11 Let $\alpha : U \to V$, $\beta : V \to W$ be linear maps. Define their product (or "composition") $\beta\alpha : U \to W$ by $(\beta\alpha)(u) = \beta(\alpha(u))$
$\forall u \in U$

prop 4.10 / 4.12 If α, β correspond to matrices A, B then (i) $\alpha + \beta$ corresponds to $A + B$
(ii) $\beta\alpha$ " " BA

proof (i) left as an exercise

(ii) Let $\alpha : U \to V$, $\beta : V \to W$ and fix bases $\mathcal{B}$ of U, $\mathcal{B}'$ of V, $\mathcal{B}''$ of W. Then

$$[\beta\alpha(u)]_{\mathcal{B}''} = [\beta(\alpha(u))]_{\mathcal{B}''} = B.[\alpha(u)]_{\mathcal{B}'} \quad \text{(by prop 4.8)}$$
$$= B.(A.[u]_{\mathcal{B}}) \quad \text{(by prop 4.8)}$$
$$= (B.A).[u]_{\mathcal{B}}$$
by prop 4.8

so $\beta\alpha$ corresponds to BA

Q.E.D.

4.3 Change of Bases

Recall that if B is a basis of V so that $v \in V$ corresponds to $[v]_B$ column vector w.r.t. B, and if B' is another basis then

$$[v]_B = P_{B,B'} [v]_{B'} \qquad \forall v \in V$$

where $P_{B,B'}$ is invertible ("transition matrix").

$P_{B',B} = (P_{B,B'})^{-1}$ transforms $[v]_B$ back to $[v]_{B'}$

Prop 4.13 Let $\alpha : V \to W$ be a linear map with matrix A w.r.t bases B of V, C of W. Let A' be the matrix for α w.r.t. other bases B' of V, C' of W. Then $A' = PAQ$ where $P = P_{C',C}$, $Q = P_{B,B'}$

Proof $(\underbrace{PAQ}_{A'})[v]_{B'} = PA(Q[v]_{B'}) = PAP_{B,B'}[v]_{B'}$

$$= PA[v]_B = P[\alpha(v)]_C = P_{C',C}[\alpha(v)]_C$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \parallel$$

$$[\alpha(v)]_{C'}$$

so A' sends $[v]_{B'}$ to $[\alpha(v)]_{C'}$ as required. QED.

Corollary 4.14 Any two matrices that represent the same linear map relative to possibly different vector spaces are equivalent (in our P-Q sense as before)

[Conversely, if two matrices are equivalent they can be viewed as corresponding to a single map α w.r.t. suitable bases — since P, Q are invertible, they can be used as transition matrices for suitable bases.]

4.4 Canonical form revisited

Theorem 4.15 Let $\alpha : V \to W$ be a linear map, V, W f.d. and let $r = \rho(\alpha)$ be the rank of α. Then $\exists$ bases for V, W s.t. the corresponding matrix is $A = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$

proof As in the proof of Thm 4.1 let $v_{r+1} \dots v_n$ be a basis of $\ker(\alpha)$, extend to a basis of V, $v_1 \dots v_r, v_{r+1} \dots v_n$. As in the proof there, $w_1 = \alpha(v_1) \dots w_r = \alpha(v_r)$ is a basis of $\mathrm{Im}(\alpha)$ (of dimension $r = \rho(\alpha)$). Extend this to basis of W, $w_1 \dots w_r, w_{r+1} \dots w_m$

$\left(\begin{array}{l} m = \dim(W) \\ n = \dim(V) \end{array}\right)$

Now we have

$$\alpha(v_i) = \begin{cases} w_i & \text{if } 1 \le i \le r \\ 0 & \text{else (since } v_{r+1} \dots v_n \in \ker(\alpha)) \end{cases}$$

so corresponding matrix of α is $\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$ wrt these bases

Q.E.D.

Corollary 4.16 Suppose $\alpha : V \to W$ is a linear map of rank r (and V, W f.d.). Then for any bases B of V, B' of W, the rank of the corresponding matrix A is also r.

proof We know $\exists$ bases s.t. the matrix has rank r (namely the canonical form as above). But by Prop 4.1 any other basis will lead to an equivalent matrix $A' = PDQ$ where $D = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$ in Thm 4.15. But rank does not change under equivalence so A' has rank r

Q.E.D.

Chapter 5 Linear maps from a vector space to itself

5.1 Projections and direct sums

Definition 5.1 A linear map $\pi : V \longrightarrow V$ is called a *projection* if $\pi^2 = \pi$

(i.e. $\pi(\pi(v)) = \pi(v)$ for all $v \in V$)

Lemma if $\pi : V \longrightarrow V$ is a projection then $v \in Im(\pi)$ iff $\pi(v) = v$

proof If $v \in Im(\pi)$, $\exists u \in V$ s.t. $v = \pi(u)$

$\Rightarrow \pi(v) = \pi(\pi(u)) = \pi^2(u) = \pi(u) = v$.

Conversely, if $\pi(v) = v$ then $\exists u$ s.t. $\pi(u) = v$ namely $u = v$, so $v \in Im(\pi)$. QED.

Prop 5.2 If $\pi : V \longrightarrow V$ is a projection then
$$V = im(\pi) \oplus ker(\pi).$$

proof Given $v \in V$, let $u = \pi(v)$, $w = v - u = v - \pi(v)$

so $u \in Im(\pi)$ and $\pi(w) = \pi(v) - \pi^2(v) = \pi(v) - \pi(v) = 0$

so $w \in ker(\pi)$. So $V = im(\pi) + ker(\pi)$

If $v \in ker(\pi) \cap im(\pi)$

$v \in Im(\pi) \Rightarrow \pi(v) = v$ by lemma

$v \in ker(\pi) \Rightarrow \pi(v) = 0 \therefore v = 0 \therefore ker\pi \cap im(\pi) = \{0\}$

QED

L17 Last time we defined a projection $\pi : V \to V$ as obeying $\pi^2 = \pi$ and showed that

$$V = \operatorname{Im}(\pi) \oplus \ker(\pi).$$

Prop 5.3 If $V = U \oplus W$ then $\exists$ a projection $\pi : V \to V$ s.t. $\operatorname{im}(\pi) = U$, $\ker(\pi) = W$

proof Recall that $V = U \oplus W \Rightarrow$ every element $v \in V$ has a unique expression as $v = u + w$, $u \in U$, $w \in W$.

We define $\pi(v) = u$ in this case.

[exercise to check that π is linear]

Clearly $\pi(\pi(v)) = \pi(u) = \pi(u + \underset{\in W}{0}) = u$ by defn of π
$= \pi(v)$ $\forall v \in V$ by defn of u

So $\pi^2 = \pi$

Also $\pi(w) = \pi(\underset{\in U}{0} + w) = 0$ by defn of π $\forall w \in W$

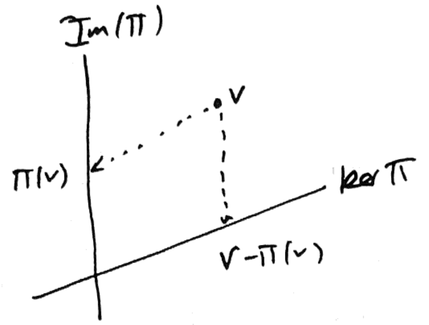
So $W \subseteq \ker \pi$. If $v \in \ker \pi$, $\pi(v) = u = 0$ $(v = u + w)$

$\therefore v = 0 + v \in W$ $\therefore v \in W$ $\therefore W = \ker \pi$

Also $\operatorname{im}(\pi) \subseteq U$ by defn of π. And if $u \in U$, $\pi(u) = \pi(u + \underset{\in W}{0}) = u$ so $u \in \operatorname{Im}(\pi)$

$\therefore U = \operatorname{im}(\pi)$ QED

Geometrically



$$v = \underset{\substack{\uparrow \\ \text{im}(\pi)}}{\pi(v)} + \underbrace{v - \pi(v)}_{\ker \pi}$$

The projection v to $\pi(v)$ subtracts something in $\ker(\pi)$ until we land in $\text{Im}(\pi)$.

Also note that if $\pi : V \longrightarrow V$ is a projection then so is $I - \pi$ (I = identity map)

$$(I-\pi)^2 = I + \pi^2 - I.\pi - \pi.I = I + \cancel{\pi} - \cancel{\pi} - \pi$$
$$= I - \pi \quad \text{so a projection.}$$

Also $\pi(I-\pi) = (I-\pi)\pi = \pi - \pi^2 = 0$

we say $\pi, I-\pi$ are "orthogonal".

If $V = U \oplus W$ & $U = \text{im}(\pi)$, $W = \ker(\pi)$ as above

then $W = \text{im}(I-\pi)$ $(= \ker \pi)$

<u>proposition 5.4</u> Suppose $\pi_1 \ldots \pi_r$ are projections on V s.t.

(a) $\pi_1 + \pi_2 + \cdots + \pi_r = I$

(b) $\pi_i \pi_j = 0 \quad \forall i \neq j$

Then $V = U_1 \oplus \cdots \oplus U_r$, $U_i := \text{Im}(\pi_i)$

(This generalizes the $r=2$ result where $\pi_1 = \pi$, $\pi_2 = I - \pi$)

Proof If $v \in V$, $v = I(v) = \Pi_1(v) + \cdots + \Pi_r(v)$

$$\begin{matrix} \Pi_1(v) & + \cdots + & \Pi_r(v) \\ \cap & & \cap \\ U_1 & \cdots & U_r \end{matrix}$$

So $v = u_1 + \cdots + u_r$

$u_i \in U_i$ $\therefore$ $V = U_1 + \cdots + U_r$

If also $v = u_1' + \cdots u_r'$, $u_i' \in U_i$

$\therefore u_i' = \Pi_i(v_i')$ some $v_i' \in V$

Then

$$\begin{aligned} u_i = \Pi_i(v) &= \Pi_i(u_1' + \cdots u_r') \\ &= \Pi_i(\Pi_1(v_1') + \cdots + \Pi_r(v_r')) \\ &= \Pi_i^2(v_i') \quad \text{as only } \Pi_i(v_i') \text{ contributes due to (b)} \\ &= \Pi_i(v_i') \\ &= u_i' \end{aligned}$$

$\therefore$ the decomposition is unique $\therefore$ $V = U_1 \oplus \cdots \oplus U_r$

QED

The converse is also true

Prop 5.5 Suppose $V = U_1 \oplus \cdots \oplus U_r$ then

$\exists \; \Pi_i$ projections $V \to V$, $i = 1 \ldots r$,

s.t. (a) $\Pi_1 + \cdots + \Pi_r = I$

(b) $\Pi_i \Pi_j = 0$ $\forall i \neq j$

and $U_i = \mathrm{Im}(\Pi_i)$

($r = 2$ case is prop 5.3)

Proof (sketch) Given $v \in V$ define $\Pi_i(v) = u_i$ in the decomposition $v = u_1 + \cdots + u_r$. Check that this gives linear maps $\Pi_i : V \to V$ and Π_i projections obeying (a), (b)

Q.E.D.

5.2 Linear maps on V and matrices

If $\alpha : V \longrightarrow V$ is a linear map and B a basis of V on both sides of α, say $v_1 \ldots v_n$. We define the corresponding $n \times n$ matrix $A = (a_{ij})$ by

$$\alpha(v_j) = \sum_{i=1}^{n} a_{ij} v_i$$

(this is just a special case of before with $W = V$, $w_i = v_i$ and $m = n$)

Propn If $\alpha : V \longrightarrow V$ represented by matrix A as above w.r.t. basis B, and by matrix A' w.r.t. another basis B' of V, then

$$A' = P A P^{-1} \qquad (\text{for } P = P_{B',B})$$

i.e. A' is similar to A.

Proof This is a special case of Prop 4.6 with W there now V, basis C there now B also, and $Q = P_{B,B'} = P^{-1}$, $P = P_{B',B}$ as before

Q.E.D.

Thus, the meaning of similarity-equivalence is that it corresponds to a change of basis for the same underlying linear map.

Look at May 2019 exam Q3

(a) book work

(b) [part of proof of the rank + nullity theorem]

$u_1, \dots u_k$ basis of $\ker \alpha$, extend by $u_{k+1} \dots u_n$ to basis of U. If $v \in \mathrm{Im}(\alpha) \therefore v = \alpha(u)$ some $u \in U \therefore v = \alpha(c_1 u_1 + \dots + c_k u_k + c_{k+1} u_{k+1} + \dots c_n u_n)$

(as u_i basis of U)

for some $c_i \in \mathbb{K}$

$$= c_1 \underbrace{\alpha(u_1)}_{0} + \dots + c_k \underbrace{\alpha(u_k)}_{0} + c_{k+1}\alpha(u_{k+1}) + \dots + c_n \alpha(u_n)$$

(α linear)

$$= c_{k+1}\alpha(u_{k+1}) + \dots c_n \alpha(u_n)$$ as $u_i \in \ker \alpha$ $i = 1 \dots k$

$\therefore \alpha(u_{k+1}) \dots \alpha(u_n)$ span.

(c) $\dim(U) = \underbrace{\dim \ker(\alpha)}_{\nu(\alpha)} + \underbrace{\dim \mathrm{im}(\alpha)}_{\rho(\alpha)}$

(d) $\alpha: U \to V$, $\beta: V \to W$. Show $\ker(\alpha) \subseteq \ker(\beta\alpha)$

If $u \in \ker \alpha$, $\alpha(u) = 0 \Rightarrow (\beta\alpha)(u) = \beta(\alpha(u)) = \beta(0) = 0$

$\therefore u \in \ker(\beta\alpha)$.

(e) $\dim(U) = 5$, $\dim(V) = 2$, $\dim(W) = 4$

Show $\dim(\ker(\beta\alpha)) \geq 3$.

$\dim(\ker(\beta\alpha)) \geq \dim(\ker(\alpha))$ by (d)

$= \dim(U) - \dim(\mathrm{Im}(\alpha))$

$\geq \dim(U) - \dim(V)$ $\quad$ $\mathrm{Im}(\alpha) \subseteq V$ $\therefore \dim(\mathrm{im}(\alpha)) \leq \dim(V)$

$= 5 - 2 = 3$

5.3 Eigenvalues and Eigenvectors

spot quiz: is $\underline{0}$ an eigenvector of a linear map α?

☐ yes ☑ no ☐ depends on α

Definition 5.8 Let $\alpha : V \to V$ be a linear map. A vector $v \in V$ is called an eigenvector of α with eigenvalue $\lambda \in \mathbb{K}$ if $v \neq 0$, $\alpha(v) = \lambda v$.

We define the associated eigenspace to an eigenvalue λ,

$$E(\lambda, \alpha) = \{ v \mid \alpha(v) = \lambda v \}$$

(this consists of all $\underline{0}$ and all eigenvectors with eigenvalue λ).

[Check $E(\lambda,\alpha) \subseteq V$ is a subspace.]

Note each eigenvector has a unique associated eigenvalue since if $\alpha(v) = \lambda v = \mu v \Rightarrow (\lambda - \mu) v = 0$
$\Rightarrow \lambda = \mu$ as $v \neq 0$

Example $A = \begin{bmatrix} -6 & 6 \\ -12 & 11 \end{bmatrix}$, $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\begin{bmatrix} -6 & 6 \\ -12 & 11 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

So v is an eigenvector with eigenvalue 2.

Similarly $w = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector with eigenvalue 3, e.g. over $\mathbb{R}$, and also over $\mathbb{F}_3$

$A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, $\underset{\text{eigenvalue } 2}{v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$, $\underset{\text{eigenvalue } 0}{w = \begin{bmatrix} 2 \\ 0 \end{bmatrix}}$. Also,

if we knew 2 was an eigenvalue we could solve

$\begin{bmatrix} -6 & 6 \\ -12 & 11 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$ to find v up to scale (e.g. assume x or $y = 1$, solve for the other)

We'll see that an $n\times n$ matrix has at most n distinct eigenvalues.

5.4 Diagonalizability

<u>Def</u> A linear map $\alpha : V \to V$ is <u>diagonalizable</u> if $\exists$ a basis of V consisting of eigenvectors of α.

<u>prop. 5.11</u> A linear map $\alpha : V \to V$ is diagonalizable iff $\exists$ a basis w.r.t. which the associated matrix A is diagonal.

<u>Proof</u> Suppose $v_1 \dots v_n$ basis of V of eigenvectors then $A = (a_{ij})$ has $a_{ij} = 0$ unless $i = j$ and a_{ii} the eigenvalue of v_i (by the formula that defines A) ie

$$A = \begin{pmatrix} \lambda & & 0 \\ & \ddots \mu & \\ 0 & & \ddots \end{pmatrix} \quad \begin{pmatrix}\text{eigenvalues } \lambda, \mu, \dots \\ \text{need not be} \\ \text{distinct}\end{pmatrix}$$

$\alpha(v_j) = \mu v_j$

Conversely, if A has such a diagonal form wrt some basis $\{v_i\}$ then v_i are eigenvectors because α acts by A in the basis. Q.E.D.

Given α we can pick any basis of V and compute A w.r.t. Then α is diagonalizable iff $\exists$ P invertible such that $P^{-1}AP$ is diagonal matrix. ie iff A is <u>similar</u> to a diagonal matrix.

Note every α is diagonalizable (contrast to $\alpha : V \to W$ and canonical form for equivalence)

Example $\begin{bmatrix} -6 & 6 \\ -12 & 11 \end{bmatrix}$ is diagonalizable over both $\mathbb{R}$, $\mathbb{F}_3$

as we had two eigenvectors forming a basis.

But $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable over $\mathbb{R}$

as has only one eigenvalue $\lambda = 1$

and eigenvector $\propto \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (i.e. proportional to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$)

Lemma 5.13 Let $v_1, \dots, v_r$ be eigenvectors of α with distinct eigenvalues $\lambda_1 \dots \lambda_r$. Then $v_1, \dots v_r$ are l.i.

proof Suppose they are l.d. so

$c_1 v_1 \dots + c_r v_r = 0$, not all coeffs zero.

We'll show a contradiction. Choose such a linear relation with the smallest number of $c_i \neq 0$ (there must be at least two $c_i \neq 0$ as $v_1 \dots v_r \neq 0$)

W.l.o.g. assume $c_1 \neq 0$ then

$$\begin{aligned} 0 = \alpha(0) &= \alpha(c_1 v_1 + \dots + c_r v_r) \\ &= c_1 \alpha(v_1) + \dots + c_r \alpha(v_r) \\ &= c_1 \lambda_1 v_1 + \dots + c_r \lambda_r v_r \end{aligned}$$

subtract $\lambda_1 \times$ our original relation

$$\Rightarrow \quad c_2(\lambda_2 - \lambda_1) v_2 + \dots + c_r(\lambda_r - \lambda_1) v_r = 0$$

at least one of these c_i are not zero so we have a linear relation with fewer non-zero coefficients than before. This contradicts our assumption.

Q.E.D.

This lemma $\Rightarrow$ if V has dimension n then we can't have more than n distinct eigenvalues.

<u>Theorem 5.14</u> Suppose $\alpha : V \longrightarrow V$ is a linear map, V f.d. and $\lambda_1 \dots \lambda_r$ are all the distinct eigenvalues of α. TFAE

(a) α diagonalizable

(b) $V = E(\lambda_1, \alpha) \oplus \cdots \oplus E(\lambda_r, \alpha)$

(i.e. a direct sum of the eigenspaces)

(c) $\alpha = \lambda_1 \pi_1 + \dots + \lambda_r \pi_r$ for $\pi_i : V \longrightarrow V$ projections s.t. $\pi_1 + \dots + \pi_r = I$, $\pi_i \pi_j = 0$ $\forall i \neq j$

<u>proof</u> (a) $\Rightarrow$ (b) If α diagonalizable, $\exists$ basis of eigenvectors and every $v \in V$ is a linear combination of them, so
$$V = E(\lambda_1, \alpha) + \cdots + E(\lambda_r, \alpha)$$
ie
$$v = u_1 + \dots + u_r, \quad u_i \in E(\lambda_i, \alpha)$$
Suppose also
$$v = u_1' + \dots + u_r' \qquad u_i' \in E(\lambda_i, \alpha)$$
$$\Rightarrow \underbrace{(u_1 - u_1')}_{\in E(\lambda_1, \alpha)} + \cdots + \underbrace{(u_r - u_r')}_{\in E(\lambda_r, \alpha)} = 0$$
$\Rightarrow u_1 - u_1' = u_2 - u_2' = \cdots = u_r - u_r'$ since if not, we would have a linear relation between eigenvectors with different eigenvalues, contradicting lemma 5.13.

(b) $\Rightarrow$ (a) If B_i basis of $E(\lambda_i, \alpha)$, by prop 1.28 $B = B_1 \cup \dots \cup B_r$ is a basis of V

$\Rightarrow$ V diagonalizable as has basis of eigenvectors

(b) $\Rightarrow$ (c) If $V = E(\lambda_1, \alpha) \oplus \cdots \oplus E(\lambda_r, \alpha)$

by prop 5.5 $\exists$ projections π_i as stated and with $\operatorname{Im}(\pi_i) = E(\lambda_i, \alpha)$. Here

$\pi_i(u) = u$ iff $u \in E(\lambda_i, \alpha)$.

For any $v \in V$, $\alpha(v) = \alpha\left(\sum_i \pi_i(v)\right)$ $\left(\text{as } \sum_i \pi_i = I\right)$

$$= \sum_i \alpha(\pi_i(v))$$

$$= \sum_i \lambda_i \pi_i(v) \quad \text{as } \pi_i(v) \in E(\lambda_i, \alpha)$$

(c) $\Rightarrow$ (b) by prop 5.4 $V = \bigoplus_i U_i$ $\quad U_i = \operatorname{Im}(\pi_i)$

If $\alpha = \sum_j \lambda_j \pi_j$, $u \in U_i$ (so $\pi_i(u) = u$)

$$\Rightarrow \alpha(u) = \sum_j \lambda_j \pi_j(u) = \sum_j \lambda_j \underbrace{\pi_j \pi_i}_{0 \text{ unless } i=j}(u)$$

$$= \lambda_i \pi_i(u) = \lambda_i u$$

$\therefore u \in E(\lambda_i, \alpha)$ so $U_i \subseteq E(\lambda_i, \alpha)$

Conversely if $u \in E(\lambda_i, \alpha)$

$$\alpha(u) = \sum_j \lambda_j \pi_j(u) = \lambda_i u$$

apply $\underset{j \neq i}{\pi_j}$: $\lambda_j \pi_j(u) = \lambda_i \pi_j(u)$

(to be continued next time)

$\boxed{\text{L19}}$ $\Rightarrow \underbrace{(\lambda_j - \lambda_i)}_{\neq 0} \pi_j(u) = 0 \Rightarrow \pi_j(u) = 0,\ j \neq i$

$\therefore u = \sum_j \pi_j(u) = \pi_i(u) \quad \therefore u \in U_i$

so $E(\lambda_i, \alpha) \subseteq U_i$. QED