MTH6140 Linear Algebra II

Coursework 6 Solutions

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1. By direct calculation from the definition of the adjugate matrix,

$$\mathrm{Adj}(xI - A) = \begin{bmatrix} x^2-1 & 0 & 0 \\ x & x^2-x & x-1 \\ 1 & x-1 & x^2-x \end{bmatrix} = x^2B_2 + xB_1 + B_0,$$

where

$$B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

Thus,

$$\begin{aligned} B_1 - AB_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -I_3. \end{aligned}$$

and

$$-AB_0 = -\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

So the coefficient of x^2 in $p_A(x)$ is -1 and the constant coefficient is 1, since $B_1 - AB_2 = c_2I$ and $-AB_0 = c_0I$. (The whole characteristic polynomial is $p_A(x) = x^3 - x^2 - x + 1$.)

2. (a) First we show linearity with respect to vector addition:

$$\begin{aligned} (\alpha + \beta)(v + v') &= \alpha(v + v') + \beta(v + v') \\ &= \alpha(v) + \alpha(v') + \beta(v) + \beta(v') \\ &= \alpha(v) + \beta(v) + \alpha(v') + \beta(v') \\ &= (\alpha + \beta)(v) + (\alpha + \beta)(v'), \end{aligned}$$

where the first and fourth equalities are from the definition of the sum of linear maps, and the second uses linearity of α and β. Then, we check linearity with respect scalar multiplication:

$$(\alpha+\beta)(cv) = \alpha(cv)+\beta(cv) = c\alpha(v)+c\beta(v) = c(\alpha(v)+\beta(v)) = c(\alpha+\beta)(v).$$

(b) Just as in the case of the product of linear maps in the notes, we just chase through the (forced) sequence of equalities:

$$\begin{aligned} [(\alpha+\beta)(v)]_{\mathcal{B}'} &= [\alpha(v)+\beta(v)]_{\mathcal{B}'} \\ &= [\alpha(v)]_{\mathcal{B}'} + [\beta(v)]_{\mathcal{B}'} \\ &= A[v]_{\mathcal{B}} + B[v]_{\mathcal{B}} \\ &= (A+B)[v]_{\mathcal{B}}. \end{aligned}$$

(c) First we show linearity with respect to vector addition:

$$\begin{aligned} (\beta\alpha)(u+u') &= \beta(\alpha(u+u')) \\ &= \beta(\alpha(u)+\alpha(u')) \\ &= \beta(\alpha(u)) + \beta(\alpha(u')) \\ &= (\beta\alpha)(u) + (\beta\alpha)(u'). \end{aligned}$$

where the first and fourth equalities are from the definition of the product of linear maps, and the second and third use linearity of α and β. Then, linearity with respect scalar multiplication goes as follows:

$$(\beta\alpha)(cv) = \beta(\alpha(cv)) = \beta(c\alpha(v)) = c(\beta(\alpha(v))) = c(\beta\alpha)(v).$$

3. Column 2 of A is the sum of columns 1 and 3. Columns 1 and 3 are clearly linearly independent, so form a basis for the column space of A, which is also $\mathrm{Im}(\alpha)$ in the given coordinate system. (Either of the other pairs of columns would also provide a basis for $\mathrm{Im}(\alpha)$.) So $\left(\begin{bmatrix}1 & 0 & 1 & -2\end{bmatrix}^\top, \begin{bmatrix}0 & 1 & -1 & 1\end{bmatrix}^\top\right)$ is a basis for $\mathrm{Im}(\alpha)$, and the dimension of $\mathrm{Im}(\alpha)$ is thus 2.

We look for vectors v such that $Av = 0$. Setting $v = \begin{bmatrix}a & b & c\end{bmatrix}^\top$ we find that $a = -b = c$. So the single vector $\begin{bmatrix}1 & -1 & 1\end{bmatrix}^\top$ is a basis for $\mathrm{Ker}(\alpha)$, which therefore has dimension 1.

Notice that $\dim \mathrm{Ker}(\alpha) + \dim \mathrm{Im}(\alpha) = 1 + 2 = 3 = \dim(V)$ as predicted by the Rank-nullity Theorem.

4. (a) Suppose $u \in \mathrm{Ker}(\alpha)$. Then $\beta\alpha(u) = \beta(\alpha(u)) = \beta(0) = 0$, from which it follows that $u \in \mathrm{Ker}(\beta\alpha)$. Since $u \in \mathrm{Ker}(\alpha)$ was arbitrary, $\mathrm{Ker}(\alpha)$ is a subset of $\mathrm{Ker}(\beta\alpha)$. We know from Proposition 4.4 that $\mathrm{Ker}(\alpha)$ is a vector space, so $\mathrm{Ker}(\alpha)$ is a subspace of $\mathrm{Ker}(\beta\alpha)$.

(b) Suppose $w \in \mathrm{Im}(\beta\alpha)$. Then there exists $u \in U$ such that $w = \beta\alpha(u)$. Setting $v = \alpha(u) \in V$, we see that $w = \beta(v)$, and hence $w \in \mathrm{Im}(\beta)$. Since $w \in \mathrm{Im}(\beta\alpha)$ was arbitrary, $\mathrm{Im}(\beta\alpha)$ is a subset of $\mathrm{Im}(\beta)$. Again, by Proposition 4.4, $\mathrm{Im}(\beta\alpha)$ is a subspace of $\mathrm{Im}(\beta)$

(c) By the Rank-nullity Theorem, $\varrho(\beta) + \nu(\beta) = \dim(V) = 2$. Then, by part(b), $\varrho(\beta\alpha) \leq \varrho(\beta) \leq 2$.
Also by the Rank-nullity Theorem, $\varrho(\alpha) + \nu(\alpha) = \dim(V) = 5$. But $\varrho(\alpha) \leq \dim(V) = 2$, and so $\nu(\alpha) \geq 3$. By part (a), $\nu(\beta\alpha) \geq \nu(\alpha) \geq 3$.

5. (a) The key is to consider the restriction of the linear map α to the subspace $U' = \mathrm{Ker}(\beta\alpha)$ of U. Call this map $\alpha' : U' \to V$. Applying the Rank-Nullity Theorem to α' we deduce

$$\nu(\beta\alpha) = \dim(U') = \varrho(\alpha') + \nu(\alpha').$$

Also, since $\beta\alpha(u) = 0$ for all $u \in U'$, we have that $\mathrm{Im}(\alpha')$ is contained in $\mathrm{Ker}(\beta)$. Thus

$$\varrho(\alpha') \leq \nu(\beta).$$

Finally, since α' is a restriction of α,

$$\nu(\alpha') \leq \nu(\alpha).$$

Adding the three displayed equations gives the result.

(b) Let β' be the restriction of β to $V' = \mathrm{Im}(a) \subseteq V$. First note that $\mathrm{Im}(\beta\alpha) = \mathrm{Im}(\beta')$, so that

$$\varrho(\beta\alpha) = \varrho(\beta').$$

Also, since β' is a restriction of β, we have

$$\varrho(\beta') \leq \varrho(\beta).$$

Finally, applying the Rank-nullity Theorem to β' yields

$$\varrho(\beta') + \nu(\beta') = \dim(V') = \varrho(\alpha).$$

The first two displayed equations give $\varrho(\beta\alpha) \leq \varrho(\beta)$ and the first and the third give $\varrho(\beta\alpha) \leq \varrho(\alpha)$. Thus, $\varrho(\beta\alpha) \leq \min\{\varrho(\alpha), \varrho(\beta)\}$.

(c) You're on your own!

6. (a) We just need to show that vector addition (i.e., addition of polynomials) and scalar multiplication are preserved: $D(f + g) = f' + g' = D(f) + D(g)$ and $D(cf) = cf' = cD(f)$.

(b) Consider a general polynomial $f = a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ of degree $n-1$ and its derivative $D(f) = (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1$. It is clear that the image D contains exactly the polynomials of degree at most $n-2$. (We are using the fact that $2, 3, \ldots, n-1$ are all invertible in $\mathbb{R}$.)
So $\mathrm{Im}(D) = V_{n-1}$ and the rank of D is $n-1$. Also $D(f) = 0$ if and only if $a_{n-1} = \cdots = a_1 = 0$, i.e, if and only if f is a constant function. Thus $\mathrm{Ker}(D) = V_1$ and the nullity of D is 1. Note that

$$\varrho(D) + \nu(D) = \dim(\mathrm{Im}(D)) + \dim(\mathrm{Ker}(D)) = (n-1) + 1 = n = \dim(V_n).$$

(c) The matrix representation of D is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(Note that this matrix has rank 3, in agreement with part (b), with $n = 4$.)