MTH6140 Linear Algebra II

Coursework 6

1. This question asks you to verify certain steps of the proof of the Cayley-Hamilton Theorem in the context of a particular small example. Consider the 3×3 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

over $\mathbb{R}$.

(a) Compute $\mathrm{Adj}(xI - A)$ and write it in the form $x^2 B_2 + x B_1 + B_0$.

(b) Verify that $B_1 - AB_2$ and $-AB_0$ are both multiples of the identity matrix, as we discovered in the proof of the Cayley-Hamilton theorem. Hence deduce the coefficients of x^2 and x^0 (i.e., the constant term) in the characteristic polynomial of A.

2. This question involves chasing sequences of equalities based on definitions related to linear maps.

(a) Let V and W be vector spaces, and $\alpha, \beta : V \to W$ be linear maps. Verify the claim in the notes that $\alpha + \beta : V \to W$ is a linear map.

(b) Suppose that $\mathcal{B}$ and $\mathcal{B}'$ are bases of V and W, respectively. Suppose further that α and β are represented by matrices A and B, with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$. Show that the linear map $\alpha+\beta$ is represented by the matrix $A+B$. In other words, show that $[(\alpha+\beta)(v)]_{\mathcal{B}'} = (A+B)[v]_{\mathcal{B}}$, for all $v \in V$.

(c) Let U, V and W be vector spaces, and $\alpha : U \to V$ and $\beta : V \to W$ be linear maps. Carefully verify the claim in the notes that $\beta\alpha : U \to W$ is a linear map.

3. Suppose V and W are vector spaces of dimensions 3 and 4, respectively, over $\mathbb{R}$. Fix bases for V and W. A linear map $\alpha : V \to W$, is represented by the following matrix A relative to the chosen bases:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ -2 & -1 & 1 \end{bmatrix}.$$

(a) Compute bases for $\mathrm{Ker}(\alpha)$ and $\mathrm{Im}(\alpha)$ and hence find the dimensions of these two subspaces. (Note that $\mathrm{Im}(\alpha)$ is just the column space of A.)

(b) Verify that your answer to (a) is consistent with the Rank-nullity Theorem.

4. Let U, V and W be vector spaces, and $\alpha : U \to V$ and $\beta : V \to W$ be linear maps.

(a) Show that $\mathrm{Ker}(\alpha)$ is a subspace of $\mathrm{Ker}(\beta\alpha)$.

(b) Show that $\mathrm{Im}(\beta\alpha)$ is a subspace of $\mathrm{Im}(\beta)$.

(c) Suppose that $\dim(U) = 5$, $\dim(V) = 2$ and $\dim(W) = 4$. Show that the dimension of $\mathrm{Im}(\beta\alpha)$ is at most 2, and the dimension of $\mathrm{Ker}(\beta\alpha)$ is at least 3.

5. *Harder.* Recall that the rank $\varrho(\alpha)$ of a linear map α is the dimension of $\mathrm{Im}(\alpha)$ and the nullity $\nu(\alpha)$ is the dimension of $\mathrm{Ker}(\alpha)$. Let U, V and W be vector spaces, and $\alpha : U \to V$ and $\beta : V \to W$ be linear maps.

(a) Prove that $\nu(\beta\alpha) \leq \nu(\alpha) + \nu(\beta)$.

(b) Prove that $\varrho(\beta\alpha) \leq \min\{\varrho(\alpha), \varrho(\beta)\}$.

(c) Discover, state and prove some lower bounds (and maybe further upper bounds) on $\nu(\beta\alpha)$ and $\varrho(\beta\alpha)$ in terms of $\nu(\alpha)$, $\nu(\beta)$, $\varrho(\alpha)$ and $\varrho(\beta)$.

6. Let D be the map on the set of real polynomials that takes each polynomial $f(x)$ to its derivative $f'(x)$.

(a) Prove that D is a linear map.

(b) Let V_n be the vector space of polynomials of degree at most $n - 1$. Consider D as a map from V_n to itself. Find its image, its kernel, its rank, and its nullity. Check that the rank-nullity theorem is satisfied.

(c) Now consider the case when $n = 4$. Write down the matrix representing D with respect to the basis $(1, x, x^2, x^3)$ of V_4.