☑ 0 ☐ $(-1)^n$ ☐ $n!$ ☐ n^2 ☐ depends on the other rows

$r_2 \to r_2 - r_1$ gives

$$\begin{bmatrix} 1 & 2 & \cdots & n \\ n & n & \cdots & n \\ 2n+1 & \cdots & & 3n \\ & \cdots & & \end{bmatrix} \xrightarrow{r_3 - 2r_2} \begin{bmatrix} 1 & 2 & \cdots & n \\ n & n & \cdots & n \\ 1 & 2 & \cdots & n \\ \vdots & & & \end{bmatrix} = 0$$

by (D2).

$\boxed{L13}$ We've been studying a function $D : M_n(K) \to K$ defined by (D1)–(D3) showed unique such function and it $= \det(A)$.

We have a parallel theory with columns in place of rows: let $D' : M_n(K) \to K$ be characterized by

(D1') D' is linear in the ith column ($i = 1 \dots n$)

(D2') $D' = 0$ if two columns the same

(D3') $D'(I_n) = 1$

Cor 3.8 There is a unique D' obeying (D1') – (D3') and its equal to $\det(A)$.

proof By symmetry rows $\leftrightarrow$ cols, $\exists$ unique D' and its equal to

$$D'(A) = \sum_{\pi \in S_n} \text{sign}(\pi)\, a_{\pi(1)1} \cdots a_{\pi(n)n}$$

$$= \sum_{\pi \in S_n} \text{sign}(\pi)\, \underbrace{a_{\pi(1),\pi^{-1}(\pi(1))} \cdots a_{\pi(n),\pi^{-1}(\pi(n))}}_{\text{rearrange in order}}$$

$$= \sum_{\pi \in S_n} \text{sign}(\pi)\, a_{1\pi^{-1}(1)} \cdots a_{n\pi^{-1}(n)}$$

$$= \sum_{\pi^{-1} \in S_n} \operatorname{sign}(\pi^{-1}) \, a_{1\pi^{-1}(1)} \cdots a_{n\pi^{-1}(n)}$$

as $\operatorname{sign}(\pi\pi^{-1}) = \operatorname{sign}(id) = 1$

$\Rightarrow \operatorname{sign}(\pi^{-1}) = \operatorname{sign}(\pi)$

$= \det(A)$

(change π^{-1} to π)

QED

Corollary 3.9 $\det(A^t) = \det(A)$ (A^t = transpose of A)

proof $D'(A) = \sum_{\pi \in S_n} \operatorname{sign}(\pi) \, a_{\pi(1)1} \cdots a_{\pi(n)n}$

$= \det(A^t)$ $\therefore$ $= \det(A)$ QED

Example A parallelotope in $\mathbb{R}^n$ is a subset

$$P = P(v_1, \ldots, v_n) = \{ a_1 v_1 + \cdots + a_n v_n \mid 0 \le a_i \le 1 \}$$

eg $n = 2$ $P = P(u, v) = \{ a_1 u + a_2 v \mid 0 \le a_i \le 1 \}$

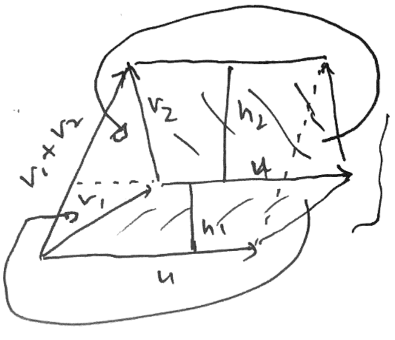
Signed Area (P) = height $h \times |u|$
if u rotates anticlockwise onto v (else its negative)

(similarly there is a signed Vol (P) in $\mathbb{R}^n$)

We'll check that this as a function of $\begin{bmatrix} u \\ v \end{bmatrix} \in M_2(\mathbb{R})$ obeys (D1)-(D3). Focus on (D1) = row linearity

$$\text{Signed Area}\left(P\begin{pmatrix} u \\ v_1 + v_2 \end{pmatrix}\right) = \underbrace{\text{signed Area}\left(P\begin{pmatrix} u \\ v_1 \end{pmatrix}\right)}_{h_1 |u|} + \underbrace{\text{signed Area}\left(P\begin{pmatrix} u \\ v_2 \end{pmatrix}\right)}_{h_2 |u|}$$

due to



move these parts to fill in at left get the LHS as $(h_1 + h_2)|u|$ ✓

Similarly (D2)–(D3) can be seen geometrically

$\therefore$ $\text{SignedArea}(P\binom{u}{v}) = \det\left(\begin{bmatrix} u \\ v \end{bmatrix}\right)$

(Similarly signed Vol $(P\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix})) = \det\left(\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right)$)

3.3 Laplace's cofactor formula

Def 3.11 If A is an $n \times n$ matrix over $\mathbb{K}$,

let $K_{ij} = (-1)^{i+j} \det(A_{ij})$

A_{ij}: matrix $(n-1)\times(n-1)$ obtained by deleting ith row & jth col of A

Let $\text{Adj}(A) = K^T$ ("Adjugate")

Theorem 3.12 (a) For $1 \le i \le n$ fixed,

$$(*) \quad \det(A) = \sum_{j=1}^{n} a_{ij} K_{ij}(A)$$

(b) For $1 \le j \le n$ fixed

$$\det(A) = \sum_{i=1}^{n} a_{ij} K_{ij}(A)$$

Proof (Sketch – we'll assume RHS of $(*)$ is independent of i). Let

$$D(A) := \sum_{j=1}^{n} a_{ij} K_{ij}(A)$$

we'll show this obeys (D1)–(D3) $\therefore$ $= \det(A)$.

Let $A = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ choosing $i = 1$ say

$$D(A) = \sum_j \underbrace{(v_1)_j}_{\text{linear in } v_1} \underbrace{K_{1j}(A)}_{\text{doesn't depend on } v_1}$$

$\therefore$ (D1) holds.

If two rows of A are the same – $A = \begin{bmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_i \\ \vdots \\ v_n \end{bmatrix}$

Choose $(*)$ based on a row a different row, say row 1.

$$D(A) = \sum_j (v_1)_j \, K_{1j}(A)$$

A_{1j} still has two rows the same $\therefore \det(A_{1j}) = 0$ $\therefore K_{1j} = 0$

$\therefore D(A) = 0$, (D2) holds.

If $A = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \end{array}\right] = I_n$ $\quad D(A) = a_{11} K_{11}(A)$

as only $j=1$ contributes as $a_{1j} = 0$ if $j \neq 1$

$\det(A_{11}) = \det(I_{n-1}) = 1$ so $K_{11} = (-1)^{1+1}\det(A_{11}) = 1$

and $a_{11} = 1$ so $D(A) = 1$, (D3) holds.

(b) identical with rows & cols swapped. QED

<u>Remark</u> We could define K_{ij} using Laplace's formula for the subdeterminants of the smaller matrices A_{ij}. Then the above becomes an inductive definition, same result applies – it obeys (D1) – (D3) $\therefore = \det(A)$

<u>Theorem 3.14</u> For any $n \times n$ matrix A,

$$A \,.\, \mathrm{Adj}(A) = \mathrm{Adj}(A) \,.\, A = \det(A)\, I_n$$

matrix products

<u>Corollary</u> If A is invertible then $A^{-1} = \det(A)^{-1} \mathrm{Adj}(A)$.

(as if A invertible, $\det(A) \neq 0$).

proof (of theorem) the i-i'th entry is

$$(A.\mathrm{Adj}(A))_{ii} = \sum_{k=1}^{n} a_{ik}\,\mathrm{Adj}(A)_{ki} = \sum_{k=1}^{n} a_{ik}\,K_{ik}(A)$$

$$= \det(A) \quad \text{by Thm 3.12}$$

as expected. If $i \neq j$ then

$$(A.\mathrm{Adj}(A))_{ij} = \sum_{k=1}^{n} a_{ik}\,\mathrm{Adj}(A)_{kj} = \sum_{k=1}^{n} a_{ik}\,K_{jk}(A)$$

don't depend on jth row

$$= \sum_{k=1}^{n} a'_{ik}\,K_{jk}(A')$$

where $A' = A$ with jth row replaced by i'th row.

$$= \sum_{k=1}^{n} a'_{jk}\,K_{jk}(A')$$

as jth row same as i'th row of A'

$= \det(A')$ by Thm 3.12

$= 0$ as A' has a repeated row.

$\therefore\ A.\mathrm{Adj}(A) = \det(A)\, I_n$. Similarly for $\mathrm{Adj}(A).A$.

QED

In practice can combine using (D1) - (D3) with Laplace formula eg.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[c_2 - 2c_1]{} \begin{bmatrix} 1 & 0 & 3 \\ 4 & -3 & 6 \\ 7 & -6 & 9 \end{bmatrix} \xrightarrow[c_3 - 3c_1]{} \begin{bmatrix} 1 & 0 & 0 \\ 4 & -3 & -6 \\ 7 & -6 & -12 \end{bmatrix}$$

Laplace

$$\det(A) = 1 . \begin{vmatrix} -3 & -6 \\ -6 & -12 \end{vmatrix} = 0 .$$

L14 ① Reminder quiz due Thursday 11.59
② In question about is A similar to A^{-1} you can assume A is invertible.

Exam 2019
Q2 — reviewed solutions.

Section 3.4 Cayley - Hamilton theorem

Matrices can obey polynomial identities

e.g. $x^2 - 3x + 2$ could be applied to x an $n \times n$ matrix (with $x^0 = I_n$)

e.g. $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$, $A^2 - 3A + 2I_2$

$$= \begin{bmatrix} -2 & 3 \\ -6 & 7 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 6 & -9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Definition Let A be an $n \times n$ matrix. We define its characteristic polynomial $p_A(x)$ by $p_A(x) := \det(xI_n - A)$

degree n polynomial in x

e.g. $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$, $p_A(x) = \det \begin{bmatrix} x & -1 \\ 2 & x-3 \end{bmatrix} = x(x-3) + 2$

$$= x^2 - 3x + 2$$

notice $p_A(A) = A^2 - 3A + 2I_2 = 0$

Theorem 3.19 (Cayley-Hamilton theorem)

Let A be an $n \times n$ matrix over $\mathbb{K}$ with characteristic polynomial $P_A(x)$.

Then $P_A(A) = 0$ as an $n \times n$ matrix equation.

Spot Quiz Is this a proof of the Cayley-Hamilton theorem:

$$P_A(x) = \det(xI_n - A) \text{ so}$$

$$P_A(A) = \det(AI_n - A) = \det(A - A) = 0.$$

yes ☐ no ☑ maybe ☐

x is an abstract symbol which we only later write $x^0 = I_n$. That I_n has nothing to do with this I_n which is expanded in determinant. So when we replace x by A in $P_A(A)$ we are not multiplying A with that I_n.

Proof of the theorem We use our result about the adjugate applied to $xI_n - A$, i.e.

$$(xI_n - A).\mathrm{Adj}(xI_n - A) = \det(xI_n - A)I_n = P_A(x)I_n$$

$\mathrm{Adj}(xI_n - A)$ has highest power x^{n-1} in its entries. So suppose

$$\mathrm{Adj}(xI_n - A) = x^{n-1}B_{n-1} + x^{n-2}B_{n-2} + \dots + B_1 x + B_0$$

for some $n \times n$ matrices $B_0, \dots, B_{n-1}$.

Also let $\quad P_A(x) = x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$

for some $\quad c_0, c_1, \ldots c_{n-1} \in \mathbb{K}$.

So

$$x^n I_n + x^{n-1} c_{n-1} I_n + \cdots + x c_1 I_n + c_0 I_n$$

$$= P_A(x) I_n = \det(x I_n - A) I_n$$

$$= (x I_n - A) \underbrace{(x^{n-1} B_{n-1} + \cdots + x B_1 + B_0)}_{Adj\,(x I_n - A)}$$

$$= x^n B_{n-1} + x^{n-1}(-A B_{n-1} + B_{n-2}) +$$

$$+ \cdots + x(-A B_1 + B_0) - A B_0$$

Equating powers of x:

$$\Rightarrow \quad B_{n-1} = I_n$$

$$-A B_{n-1} + B_{n-2} = c_{n-1} I_n$$

$$\vdots$$

$$-A B_1 + B_0 = c_1 I_n$$

$$-A B_0 = c_0 I_n$$

Now take $A^n \times$ 1st equation, $A^{n-1} \times$ 2nd, .. etc. ... $\underbrace{A^0}_{I_n} \times$ last

and add up.

$\Rightarrow$ RHS $= A^n + c_{n-1} A^{n-1} + \cdots + c_0 I_n = P_A(A)$

LHS cancels in pairs:

$$A^n B_{n-1} + A^{n-1}(-A B_{n-1} + B_{n-2}) + \cdots$$

$$+ \cdots + A(-A B_1 + B_0) + I_n(-A B_0)$$

$= 0$

QED

Why is $P_A(x)$ important.

<u>Definition</u> on the space of $n\times n$ matrices, we define B is <u>similar</u> to A if $\exists$ invertible P s.t. $B = PAP^{-1}$

(another equivalence relation, stronger than the previous one with P, Q).

Observe if B is similar to A then

$$P_B(x) = \det(xI_n - \underbrace{PAP^{-1}}_{B}) = \det(P(xI_n - A)P^{-1})$$

$$= \det(P)\ \det(xI_n - A)\ \det(P^{-1})$$

$$= \underbrace{\det(P)\det(P^{-1})}_{\det(PP^{-1})=1}\ P_A(x) = P_A(x)$$

<u>Example</u> For $n=2$, $A = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$

$$P_A(x) = \det(xI_2 - A) = \det\begin{bmatrix} x-a & -b \\ -c & x-d \end{bmatrix}$$

$$= x^2 - \underbrace{(a+d)}_{Tr(A)}x + \underbrace{ad-bc}_{\det(A)}$$

Its well known that if B equivalent to A, (as above)

$$\det(B) = \det(PAP^{-1}) = \det(A)$$

$$Tr(B) = Tr(PAP^{-1}) = \sum_{i,j,k} P_{ij}A_{jk}P^{-1}_{ki}$$

$$= \sum_{i,j,k} \underbrace{P^{-1}_{ki}P_{ij}}_{\delta_{kj}} A_{jk} = \sum_j A_{jj}$$

$$= Tr(A).$$

$\boxed{\text{Challenge}}$ What do we get for $n=3$

$$P_A(x) = x^3 + \underbrace{?}\, x^2 + \underbrace{?}\, x + \underbrace{?}\, 1$$

two of these are $Tr(A)$, $\det(A)$. What is the 3rd as a function on $M_3(\mathbb{K})$.

Chapter 4 Linear maps between vector spaces

4.1 Basics

Def 4.1 Let V, W be v.s.'s over a field $\mathbb{K}$.

A map $\alpha : V \longrightarrow W$ is called a linear map

if $\quad \alpha(v_1+v_2) = \alpha(v_1) + \alpha(v_2) \quad \forall\, v, v_1, v_2 \in V$
$\quad\;\;\; \alpha(cv) = c\,\alpha(v) \quad c \in \mathbb{K}$

(or equivalently $\alpha(c_1 v_1 + c_2 v_2) = c_1 \alpha(v_1) + c_2 \alpha(v_2)$
$\forall\, v_i \in V,\ c_i \in \mathbb{K}$)

Examples (i) $V = \mathbb{K}[x]$, $\quad W = \mathbb{K}$

$\alpha(f) = f(0)$, $\quad \alpha(f_0 1 + f_1 x + \cdots\cdots) = f_0$

similarly $\alpha(f) = f(c)$ any fixed $c \in \mathbb{K}$

$\alpha(f_0 1 + f_1 x + f_2 x^2 + \cdots) = f_0 + f_1 c + f_2 c^2 + \cdots$

(ii) $V = \mathbb{K}[x]_n$ (degree $\leq n$), $\quad W = \mathbb{K}[x]_{n-1}$

$\alpha : V \longrightarrow W$, $\quad \alpha(f) = f' = \frac{df}{dx}$

$\alpha(f_0 1 + x f_1 + \cdots f_n x^n) = f_1 \cdot 1 + 2 f_2 x + \cdots + n f_n x^{n-1}$

(iii) $V = \mathbb{K}[x]_n$, $W = \mathbb{K}^{n+1}$

$$\alpha(f) = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix} \quad \text{if } f = f_0 1 + f_1 x + \dots + f_n x^n$$

here $\alpha : V \to W$ is a bijection. (we say α is an isomorphism of vector spaces $\mathbb{K}[x]_n \simeq \mathbb{K}^{n+1}$ via α.)

L15

Def 4.3 Let $\alpha : V \to W$ be a linear map. Define

$$\text{Im}(\alpha) = \{ w \in W \mid w = \alpha(v) \text{ some } v \in V \}$$

"image(α)"

$$\text{Ker}(\alpha) = \{ v \in V \mid \alpha(v) = 0 \}$$

"kernel(α)"

Prop 4.4 $\text{Im}(\alpha) \subseteq W$, $\ker(\alpha) \subseteq V$ are subspaces.

proof Both are non-empty as $\alpha(0) = 0$

$\Rightarrow 0 \in \text{Ker}(\alpha)$, $0 \in \text{Im}(\alpha)$

If $w_1, w_2 \in \text{Im}(\alpha)$ so $w_1 = \alpha(v_1)$, $w_2 = \alpha(v_2)$ for some $v_i \in V$. Then

$$w_1 + w_2 = \alpha(v_1) + \alpha(v_2) = \alpha(v_1 + v_2)$$

by α linear

$\therefore$ $w_1 + w_2 \in \text{Im}(\alpha)$.

If $w \in \text{Im}(\alpha)$, $c \in \mathbb{K}$, $\Rightarrow$ $cw = c\alpha(v) = \alpha(cv)$
$w = \alpha(v)$ some $v \in V$ by α linear

$\therefore$ $cw \in \text{Im}(\alpha)$

$\therefore$ by lemma 1.24, $\text{Im}(\alpha)$ is a subspace.

If $v_1, v_2 \in \ker(\alpha)$ so $\alpha(v_1)=0$, $\alpha(v_2)=0$

$$\Rightarrow \quad \alpha(v_1+v_2) \underset{\substack{\uparrow \\ \text{by } \alpha \text{ linear}}}{=} \alpha(v_1)+\alpha(v_2) = 0+0=0$$

so $v_1 + v_2 \in \ker(\alpha)$

If $v \in \ker(\alpha)$, $c \in \mathbb{K} \Rightarrow \alpha(cv) = c\,\alpha(v) = c \cdot 0$
$= 0$

so $cv \in \ker(\alpha)$

$\therefore$ by lemma 1.24 $\ker(\alpha)$ is a subspace QED.

<u>Def</u> The <u>rank</u> of a linear map $\alpha : V \to W$ is denoted $\rho(\alpha)$ and defined by

$$\rho(\alpha) = \dim(\operatorname{Im}(\alpha))$$

The <u>nullity</u> of α is denoted $\nu(\alpha)$ and defined by $\nu(\alpha) = \dim(\ker(\alpha))$

(This assumes that the relevant subspaces are finite-dimensional, eg V f.d.)

<u>Theorem 4.6</u> (rank + nullity theorem)

Let $\alpha : V \to W$ be a linear map, V f.d. Then $\rho(\alpha) + \nu(\alpha) = \dim(V)$.

<u>proof</u> Let $u_1, \dots, u_q$ be a basis of $\ker(\alpha)$ ($\therefore \nu(\alpha) = q$). We can extend this to basis of V

$$u_1 \dots u_q, v_1, \dots v_s \qquad (\text{so } \dim(V) = q+s)$$

We'll show that $\alpha(v_1), \dots, \alpha(v_s)$ is a basis of $\operatorname{Im}(\alpha)$

(hence $s = \rho(\alpha)$ and so $\dim(V) = \nu(\alpha) + \rho(\alpha)$ as required). These are l.i. since, if

$$\underbrace{c_1 \alpha(v_1) + \cdots + c_s \alpha(v_s)}_{= \alpha(c_1 v_1 + \cdots c_s v_s)} = 0 \quad \text{as } \alpha \text{ linear.}$$

So $c_1 v_1 + \cdots + c_s v_s \in \ker(\alpha)$

$\therefore \quad = a_1 u_1 + \cdots + a_q u_q$ some a_i, as $\{u_i\}$ basis of $\ker(\alpha)$

$\Rightarrow \quad -a_1 u_1 + \cdots - a_q u_q + c_1 v_1 + \cdots c_s v_s = 0$

$\Rightarrow \quad c_1 = c_2 = \cdots = c_s = 0$ as $\{u_i; v_j\}$ basis of V

To see that $\alpha(v_1) \ldots \alpha(v_s)$ span, let $w = \alpha(v)$,

write $v = a_1 u_1 + \cdots a_q u_q + c_1 v_1 + \cdots + c_s v_s$ some coeffs as $\{u_i, v_j\}$ basis of V

$\therefore \quad w = \alpha(v) = a_1 \underbrace{\alpha(u_1)}_{0} + \cdots a_q \underbrace{\alpha(u_q)}_{0} + c_1 \alpha(v_1) + \cdots c_s \alpha(v_s)$

as $u_i \in \ker(\alpha)$. So $\alpha(v_1), \ldots, \alpha(v_s)$ span $\operatorname{Im}(\alpha)$. $\therefore$ a basis.

QED

4.2 Representation of linear maps by matrices

Def 4.7 Let V have basis $\{v_1 \ldots v_n\} = B$

" W " " $\{w_1 \ldots w_m\} = B'$

Then every linear map $\alpha : V \to W$ defines an associated matrix $A = (a_{ij})$ ($m \times n$ matrix) by

$$\alpha(v_j) = \sum_{i=1}^{m} a_{ij} w_i \quad , \forall j = 1 \ldots n$$

Here $A = \begin{bmatrix} \alpha(v_1), \dots, \alpha(v_n) \end{bmatrix}$

col. vectors in basis $w_1 \dots w_m$

(using the basis we think of elements of W as column vector).

<u>Example</u>

$$\alpha(v_1) = w_1 + w_2$$
$$\alpha(v_2) = 2w_1 + 5w_2 \longleftrightarrow A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & -1 \end{bmatrix}$$
$$\alpha(v_3) = 3w_1 - w_2$$

$\alpha(v_1)$ $\quad$ $\alpha(v_2)$ $\quad$ $\alpha(v_3)$

<u>Prop 4.8</u> Let $\alpha : V \to W$ be a linear map and B, B' bases of V, W respectively. Then

$$[\alpha(v)]_{B'} = A \cdot [v]_B$$

col vector of $v \in V$ w.r.t. B basis of V

col vector of $\alpha(v)$ w.r.t basis B' of W

Where A is the matrix associated to α.

<u>proof</u> If $\quad v = \sum_j c_j v_j \longleftrightarrow [v]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

So $\alpha(v) = \sum_{j=1}^{n} c_j \, \alpha(v_j)$ $\quad$ by α linear

$$= \sum_{j=1}^{n} c_j \sum_{i=1}^{m} a_{ij} w_i \quad \text{by defn of } A = (a_{ij})$$

$$= \sum_{i=1}^{m} w_i \underbrace{\sum_{j=1}^{n} a_{ij} c_j}_{[\alpha(v)]_{B'}}$$

so $[\alpha(v)]_{B'}$ has entries $\sum_{j=1}^{n} a_{ij} c_j$

$$([\alpha(v)]_{B'})_i = \sum_{j=1}^{n} a_{ij} c_j = \sum_{j=1}^{n} a_{ij} ([v]_B)_j \qquad \text{QED}$$

This says that working with fixed bases, the effect of α is to apply the associated matrix A.

Example $m=2$, $n=3$ $\alpha \leftrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & -1 \end{bmatrix}$ as before

If $v = 2v_1 + 3v_2 + 4v_3 \in V \leftrightarrow [v]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

$$[\alpha(v)]_{\mathcal{B}'} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 20 \\ 13 \end{bmatrix}$$

Check $\alpha(v) = 2\alpha(v_1) + 3\alpha(v_2) + 4\alpha(v_3)$ (as α linear)

$$= 2(w_1 + w_2) + 3(2w_1 + 5w_2) + 4(3w_1 - w_2)$$

$$= 20w_1 + 13w_2 \leftrightarrow \begin{bmatrix} 20 \\ 13 \end{bmatrix}$$

We'll also illustrate the rank + nullity theorem

$$\ker(\alpha) = \{ v = av_1 + bv_2 + cv_3 \mid \alpha(v) = 0 \}$$

ie $a(w_1 + w_2) + b(2w_1 + 5w_2) + c(3w_1 - w_2) = 0$

i.e. (take coeffs of w_1, w_2)

$$a + 2b + 3c = 0$$
$$a + 5b - c = 0$$

ie $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So $\ker(\alpha)$ in matrix terms $\leftrightarrow$ column vectors which give 0 when we apply the associated matrix A. In our case 3 unknowns and 2 equations $\Rightarrow$ $\nu(\alpha) = 1$ a 1-dimensional space $\ker(\alpha)$. So

$\nu(\alpha) = 1$

By the rank + nullity theorem $\nu(\alpha) + \rho(\alpha) = 3$

$\therefore$ $\rho(\alpha) = 2 = \dim(W)$ and $\operatorname{im}(\alpha) \subseteq W$

$\therefore$ $\operatorname{im}(\alpha) = W$ in this example.