MTH6140 Linear Algebra II

Coursework 5

1. An $n \times n$ matrix $A = (a_{ij})$ is said to be *upper triangular* if all the entries below the main diagonal are zero; that is, if $a_{ij} = 0$ whenever $i > j$. Prove that the determinant of an upper triangular matrix is equal to the product of the diagonal entries. Experiment with one or more of the following approaches:

(a) transform A to the identity matrix using elementary row operations,

(b) use the Leibniz formula, or

(c) apply the Laplace expansion.

2. (a) Write down expressions for the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

using firstly the Leibniz formula, and then the cofactor (Laplace) expansion along the first and then second rows. Show that the two methods give the same result (up to possible rearrangement of terms).

(b) Suppose σ is any permutation of $\{2, 3, \ldots, n\}$. Extend σ to a permutation σ' of $\{1, 2, 3, \ldots, n\}$ by defining

$$\sigma'(i) = \begin{cases} 1, & \text{if } i = 1; \\ \sigma(i), & \text{otherwise.} \end{cases}$$

- Verify that $\mathrm{sign}(\sigma') = \mathrm{sign}(\sigma)$.

- Let A be an $n \times n$ matrix. Verify that the first term, $a_{1,1}K_{1,1}(A)$, in the Laplace expansion for A (along row 1) is equal to

$$\sum_{\pi \in S_n, \pi(1)=1} \operatorname{sign}(\pi) a_{1,\pi(1)} \cdots a_{n,\pi(n)},$$

i.e., the Leibniz formula restricted to permutations mapping 1 to itself.

3. Consider the following simplification of the Type 1 row operation:

Type 1′. Add row j to row i.

Show that any matrix transformation that can be done using Type 1, 2 and 3 row operations can be done using just Type 1′ and Type 2 operations. (When dealing with Type 3 operations, look to the proof of Theorem 3.4 for inspiration!)

4. Calculate the adjugate $\operatorname{Adj}(A)$ of the 3×3 matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

over $\mathbb{R}$, and verify that $A \cdot \operatorname{Adj}(A) = \det(A) I_3$.

5. (a) Calculate the characteristic polynomial $p_A(x)$ of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Verify that the matrix A satisfies the polynomial p_A, that is, $p_A(A) = O$, where O is the all-zero matrix.

(b) Show that no non-zero polynomial m_A of lower degree than p_A satisfies $m_A(A) = O$. (Let $m_A(x) = ax^2 + bx + c$. What can you deduce from $m_A(A) = O$?)

(c) *Harder.* Generalise parts (a) and (b) to $n \times n$ matrices. For $n \geq 3$ define the $n \times n$ matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1, & \text{if } 1 \leq i, j \leq n \text{ and } j = i + 1; \\ 1, & \text{if } i = n \text{ and } j = 1; \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Parts (a) and (b) correspond to the case $n = 3$. Redo both parts but with $n \geq 3$ arbitrary.

6. Suppose A is any $n \times n$ matrix over $\mathbb{K}$ and P any invertible matrix. Let $A' = P^{-1}AP$. (In this situation we say that the matrix A' is similar to A.) Show that the characteristic polynomials $p_{A'}$ and p_A of A' and A are the same.