$\boxed{L10}$ Last time we defined the rank of an $m \times n$ matrix A as

colrank (A) = rowrank $(A) = r$ in the standard form for equivalence

$$PAQ = D = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

Corollary 2.18 An $n \times n$ matrix is invertible iff it has rank n

proof Note that if A, B are equivalent then A invertible iff B is invertible

(because if $B = PAQ$ and A invertible (and so are P, Q) then so is B, $B^{-1} = Q^{-1}A^{-1}P^{-1}$)

By theorem 2.14 A is equivalent to

$$D = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right) \quad r = \text{rank} \quad \text{which is invertible}$$

iff $r = n$ so $D = I_n$. so

A invertible $\iff$ D invertible $\iff$ $D = I_n \iff \text{rank } D = n$

$\Updownarrow$

$\text{rank}(A)) = n$

since rank does not change under equivalence QED

Corollary 2.20 Every invertible square matrix can be transformed to the identity by col ops alone (or by row ops alone).

proof If A invertible we can write

$$I_n = \underbrace{R_s \cdots R_1}_{P} A \underbrace{C_1 \cdots C_t}_{Q}$$

for elementary row op matrices R_i and col op matrices C_j, by putting A into canonical form for equivalence. But every elementary matrix can be viewed as either a row op or a column op. So

$$A = P^{-1} I_n Q^{-1} = P^{-1} Q^{-1} = R_1^{-1} \cdots R_s^{-1} C_t^{-1} \cdots C_1^{-1}$$

can be viewed as $A = C_{t'}^{-1} \cdots C_1^{-1}$ for some larger $t' = t+s$

so $A C_1 \cdots C_{t'} = I_n$ interpreted as claimed

QED

This explains an algorithm in lin alg I for inverting a matrix A. We apply col ops $C_1, C_2 \cdots C_t$ to make $A C_1 \cdots C_t = I_n$

$\Rightarrow A^{-1} = C_1 \cdots C_t$. We can track this by applying the same col ops to

$A \mid I_n$ side by side until A is transformed to I_n and I_n is transformed to A^{-1}

$$\begin{aligned}
& A \mid I_n \\
\Rightarrow\ & AC_1 \mid I_n C_1 = C_1 \\
\Rightarrow\ & AC_1C_2 \mid C_1C_2 \\
& \vdots \\
\Rightarrow\ & I_n = AC_1 \cdots C_t \mid C_1C_2 \cdots C_t = A^{-1}
\end{aligned}$$

Example $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \Bigg| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$c_1 \leftrightarrow c_2$ $\quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \Bigg| \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$c_3 - c_1$ $\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \Bigg| \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

$c_3 - c_2$ $\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \Bigg| \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

$c_2 + \frac{1}{2} c_3$ $\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix} \Bigg| \begin{pmatrix} 0 & \frac{1}{2} & -1 \\ 1 & -1/2 & -1 \\ 0 & 1/2 & 1 \end{pmatrix}$

$c_1 + \frac{1}{2} c_3$ $\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \Bigg| \begin{pmatrix} -\frac{1}{2} & 1/2 & -1 \\ 1/2 & -1/2 & -1 \\ 1/2 & 1/2 & 1 \end{pmatrix}$

$-\frac{1}{2} c_3$ $\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Bigg| \begin{pmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{pmatrix} = A^{-1}$

Spot Quiz Is this possible (with other col ops) working over $\mathbb{F}_2 = \{0,1\}$?

☐ yes ☑ no

To see this put A into canonical form get

$$D = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \quad \text{(wk 3)}$$

which is not invertible.

or do the above to the point

$$A \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Theorem 2.21 Two $m \times n$ matrices are equivalent iff they have the same rank.

proof Suppose A, B equivalent so $B = PAQ$, P, Q invertible. By corollary 2.20 applied to P, Q, $B = R_s \cdots R_1 A C_1 \cdots C_t$

(since P, Q can be transformed to the identity by row or col ops alone, it means they can be written as product of row or col ops matrices).

$\therefore$ B obtained from A by row/col ops

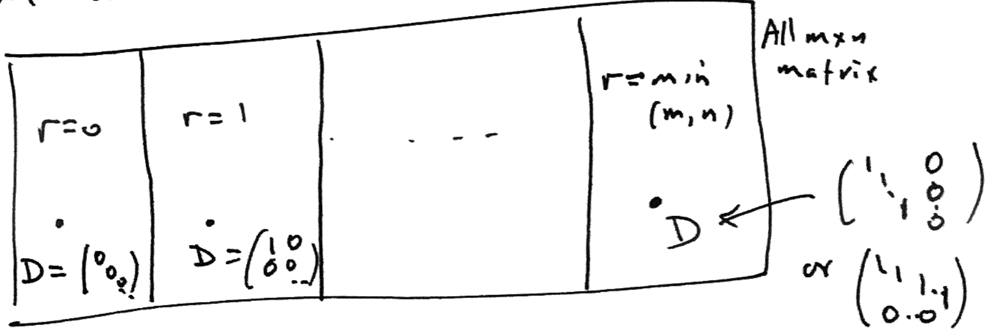
$\therefore$ $\operatorname{rank}(B) = \operatorname{rank}(A)$.

Conversely, suppose A, B have the same rank so $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = D = P_1 A Q_1 = P_2 B Q_2$ some invertible P_i, Q_i

$\Rightarrow$ $B = P_2^{-1} P_1 A Q_1 Q_2^{-1}$

$\therefore$ B equivalent to A.

Recall that an equivalence relation partitions the set on which its defined into equivalence classes



partitioned into $1 + \min(m,n)$ equivalence classes according to the rank and with representatives D.

Chapter 3 Determinants

Definition A *permutation* of set $\{1 \ldots n\}$ is a bijection of the set to itself. The set of these is called S_n, the "permutations group" or "symmetric group"

size $n!$

Example A *m-cycle* is a special type of permutation denoted $(i_1, i_2 \ldots, i_m)$ $\quad i_k \in \{1, \ldots n\}$ distinct

stands for $\quad i_1 \to i_2 \to i_3 \to i_4 \ldots \to i_{m-1} \to i_m$

eg a 2-cycle (i, j) $\quad i \neq j \quad i \rightleftarrows j$

is called a "transposition"

Fact ① – every element of S_n can be written as a product of disjoint cycles

② – every element of S_n can be written as a product of transpositions

L11

Definition for $\pi \in S_n$, define

$$\operatorname{sign}(\pi) = \begin{cases} (-1)^{n-k} & \text{if } \pi \text{ is a product of } k \text{ disjoint cycles} \\ (-1)^{d} & \text{if } \pi \text{ is a product of } d \text{ transpositions.} \end{cases}$$

Facts – these are equal

– $\operatorname{sign}(\pi\pi') = \operatorname{sign}(\pi)\operatorname{sign}(\pi')$

Examples (1) $\pi = id$ (identity map)

$$= \underbrace{(1)(2)\cdots(n)}_{k=n}$$

(1) is a 1-cycle

$$\Rightarrow \operatorname{sign}(\pi) = (-1)^{n-n} = 1$$

(2) $\pi = (12)$ (i.e. swap $1 \leftrightarrow 2$ transposition)

$$= (12)\underbrace{(3)(4)\cdots(n)}_{n-2} \qquad k = n-1$$

$$\operatorname{sign}((12)) = (-1)^{n-(n-1)} = (-1)^1 = -1$$

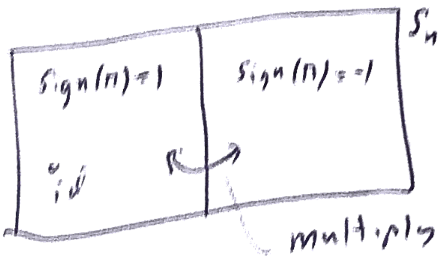
Same for any transposition τ, $\operatorname{sign}(\tau) = -1$

(3) Deduce from (1) if $\pi \in S_n$ then

$$\left.\begin{aligned} \operatorname{sign}(\pi\pi^{-1}) &= \operatorname{sign}(id) = 1 \\ \| \\ \operatorname{sign}(\pi)\operatorname{sign}(\pi^{-1}) \end{aligned}\right\} \Rightarrow \operatorname{sign}(\pi^{-1}) = \operatorname{sign}(\pi)$$

Deduce from (2) if $\pi \in S_n$, τ a transposition, then $\operatorname{sign}(\pi\tau) = \operatorname{sign}(\pi)\operatorname{sign}(\tau) = -\operatorname{sign}(\pi)$

So



(Also deduce if $\pi = \tau_1 \cdots \tau_d$ then $\operatorname{sign}(\pi) = (-1)^d$)

Definition (Leibniz) let $A = (a_{ij}) \in M_n(\mathbb{k})$

Then define

$$\det(A) := \sum_{\pi \in S_n} \operatorname{sign}(\pi)\, a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}$$

Example $n=3$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix} = \sum_{\pi \in S_3} \operatorname{sign}(\pi)\, a_{1\pi(1)}\, a_{2\pi(2)}\, a_{3\pi(3)}$$

$$S_3 = \{\ \mathrm{id},\ \underset{1\leftrightarrow 2}{(12)},\ \underset{2\leftrightarrow 3}{(23)},\ \underset{1\leftrightarrow 3}{(13)},\ (123)=(13)(12),\ (321)=(12)(13)\ \}$$

(123): $1 \to 2 \to 3 \to 1$; (321): $3 \to 2 \to 1 \to 3$

$$\operatorname{sign}(\pi) = 1,\quad -1,\quad -1,\quad -1,\quad (-1)^2 = 1,\quad (-1)^2 = 1$$

(or $k=1$ so $(-1)^{3-1} = (-1)^2 = 1$)

$$\therefore \det(A) = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

$$- \underbrace{a_{13}a_{22}a_{31}}_{(13)=\pi} + \underbrace{a_{12}a_{23}a_{31}}_{\text{apply } (123)=\pi} + \underbrace{a_{13}a_{21}a_{32}}_{\text{apply } (321)=\pi}$$

Compare with the more familiar (Laplace) formula

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) \quad (\text{step } 1)$$

$$- a_{12}(a_{21}a_{33} - a_{31}a_{23}) \quad (\text{step } 2)$$

$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31}) \quad (\text{step } 3)$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23}$$

$$+ a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Same answer.

Quiz is $(231) = (123) = (312)$

☑ yes

☐ no

all

$1 \to 2 \to 3 \to 1$

We'll show that $\det(A)$ is characterised by three properties. Consider $D : M_n \to \mathbb{K}$ a function such that

(D1) For every i, $D(A)$ is a linear function of row i

(D2) If A has two equal rows then $D(A) = 0$

(D3) $D(I_n) = 1$

(D1) means if A, B differ only in the ith row

$$A = \begin{bmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{bmatrix}, \quad B = \begin{bmatrix} v_1 \\ \vdots \\ cv_i \\ \vdots \\ v_n \end{bmatrix} \quad \text{some } c \in \mathbb{K}$$

then $D(B) = c\,D(A)$. And if

$$A = \text{as above}, \quad B = \begin{bmatrix} v_1 \\ \vdots \\ v_i' \\ \vdots \\ v_n \end{bmatrix}, \quad C = \begin{bmatrix} v_1 \\ \vdots \\ v_i + v_i' \\ \vdots \\ v_n \end{bmatrix}$$

(so adding in the ith row, other rows unchanged)

then $D(C) = D(A) + D(B)$.

<u>Lemma 3.3</u> det (Leibniz formula) obeys (D1)–(D3)

<u>proof</u> (D1) Suppose $B = A$ except in the ith row where $b_{ij} = c\,a_{ij}$, $j = 1 \dots n$ some fixed $c \in \mathbb{K}$,

$$\Rightarrow \det(B) = \sum_{\pi \in S_n} \operatorname{sign}(\pi)\, \underset{b_{1\pi(1)}}{\underset{\parallel}{a_{1\pi(1)}}} \cdots \underset{b_{i\pi(i)}}{\underset{\uparrow}{c\,a_{i\pi(i)}}} \cdots \underset{b_{n\pi(n)}}{\underset{\uparrow}{a_{n\pi(n)}}}$$

$$= c\det(A)$$

Suppose B, C same as A except in the i'th row $c_{ij} = a_{ij} + b_{ij}$, $j = 1 \dots n$

$$\det(C) = \sum_{\pi \in S_n} \operatorname{sign}(\pi)\, a_{1\pi(1)} \cdots \underbrace{(a_{i\pi(i)} + b_{i\pi(i)})}_{=\, c_{i\pi(i)}} \cdots a_{n\pi(n)}$$

$a_{1\pi(1)} = c_{1\pi(1)} = b_{1\pi(1)}$ $\qquad$ $a_{n\pi(n)} = c_{n\pi(n)} = b_{n\pi(n)}$

$$= \det(A) + \det(B)$$

(D2) Suppose A has i, j th row equal $i \neq j$. Let $\tau = (i\,j)$ then

$a_{i\,\pi\tau(i)} = a_{i\,\pi(j)} = a_{j\,\pi(j)}$ $\quad$ as rows i, j same.

$a_{j\,\pi\tau(j)} = a_{j\,\pi(i)} = a_{i\,\pi(i)}$

$a_{k\,\pi\tau(k)} = a_{k\,\pi(k)}$, $\quad k \neq i, j$

so
$$\det(A) = \sum_{\pi \in S_n} \operatorname{sign}(\pi)\, a_{1\pi(1)} \cdots a_{n\pi(n)}$$

$$= \sum_{\substack{\pi \in S_n \\ \operatorname{sign}(\pi)=1}} a_{1\pi(1)} \cdots a_{n\pi(n)} + \underbrace{\sum_{\substack{\pi \in S_n \\ \operatorname{sign}(\pi)=+1 \\ \text{if } \pi\tau \in S_n \\ \operatorname{sign}(\pi\tau)=-1}} \underbrace{\operatorname{sign}(\pi\tau)}_{-1}\, a_{1\pi\tau(1)} \cdots a_{n\pi\tau(n)}}_{-\sum_{\substack{\pi \in S_n \\ \operatorname{sign}(\pi)=1}} a_{1\pi(1)} \cdots a_{j\pi(j)} \cdots a_{i\pi(i)} \cdots a_{n\pi(n)}}$$

$a_{i\,\pi\tau(i)} \cdots a_{j\,\pi\tau(j)} \cdots$

$$= 0$$

$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

(D3) $\det(I_n) = \sum_{\pi \in S_n} \operatorname{sign}(\pi)\, \delta_{1\pi(1)} \cdots \delta_{n\pi(n)} = 1$ as only $\pi = id$ contributes

QED

Theorem 3.4 There is only one function D obeying (D1) – (D3) (ie any such function is uniquely determined), and its given by det.

Proof We first deduce how D behaves under row operations. If R is an elementary row matrix we'll see that $D(RA) = c\,D(A)$ for some c depending on R.

Type 1 $r_i + c r_j$
(add $c \times$ row j to row i) $D(RA) = D(A)$,
(So if B given from A by this type of row op, D is unchanged). The associated constant is 1.

Type 2 $c r_i$ $D(RA) = c\,D(A)$
(scale ith row by c) associated constant is c
(so if B given from A by scaling the ith row by c, $D(B) = c\,D(A)$)

Type 3 $r_i \leftrightarrow r_j$ $D(RA) = -D(A)$
associated constant is -1
(so if B given from A by swapping two rows then $D(B) = -D(A)$)

To prove these facts:
Type 1 let A' be same as A except ith row has been set to jth row of A $(i \neq j)$

$$A = \begin{bmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_j \\ \vdots \\ v_n \end{bmatrix} \quad A' = \begin{bmatrix} v_1 \\ \vdots \\ v_j \\ \vdots \\ v_j \\ \vdots \\ v_n \end{bmatrix}, \quad B = \begin{bmatrix} v_1 \\ \vdots \\ v_i + cv_j \\ \vdots \\ v_j \\ \vdots \\ v_n \end{bmatrix}$$

$B = A$ except for the ith row where its the ith row of A + c × ith row of A'

$\therefore$ By (D1) $D(B) = D(A) + c\,D(A')$

but $D(A') = 0$ by (D2) $\therefore$ $D(B) = D(A)$.

<u>Type 2</u> this is just part of (D1)

<u>Type 3</u> Swap $r_i \leftrightarrow r_j$ by a series of move

$$A = \begin{bmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_j \\ \vdots \\ v_n \end{bmatrix} \xrightarrow[r_j + r_i]{} \begin{bmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_j + v_i \\ \vdots \\ v_n \end{bmatrix} \xrightarrow{-r_i} \begin{bmatrix} v_1 \\ \vdots \\ -v_i \\ \vdots \\ v_j + v_i \\ \vdots \\ v_n \end{bmatrix} \xrightarrow{r_i + r_j} \begin{bmatrix} v_1 \\ \vdots \\ v_j \\ \vdots \\ v_j + v_i \\ \vdots \\ v_n \end{bmatrix}$$

$$\xrightarrow{r_j - r_i} \begin{bmatrix} v_1 \\ \vdots \\ v_j \\ \vdots \\ v_i \\ \vdots \\ v_n \end{bmatrix} = B$$

this type 2 move changes the sign of D

The other moves do not change D as type 1. $\therefore$ $D(B) = -D(A)$

Now suppose that A is invertible. By Cor 2.20 we can reduce it to I_n by row ops

$$R_t \cdots R_2 R_1 A = I_n$$ for elementary row op matrices R_i. Using our facts and associated constants c_i

So $1 \underset{(\text{by }(D3))}{=} D(I_n) = D(R_t \ldots R_2 R_1 A)$

$$= c_t\, D(R_{t-1} \cdots R_2 R_1 A)$$

$$= \cdots = c_t\, c_{t-1} \cdots c_2\, c_1\, D(A)$$

$\therefore\ D(A) = c_1^{-1} c_2^{-1} \ldots c_t^{-1}$ is uniquely determined

by the rules (D1) – (D3).

(And since det obeys (D1) – (D3), $D(A) = \det(A)$)

Q.E.D.

To finish the proof we have to check the case A not invertible.

<u>Corollary</u> A invertible iff $D(A) \neq 0$

(completes the proof) i.e. iff $\det(A) \neq 0$

proof If A has a row of all zeros then $D(A) = 0$. Because if r_i is zero, add r_j to r_i to get row j repeated, so $D(A) = 0$ by (D2) and since this move is type 1 so D doesn't change.

Now, if $A \in M_n(\mathbb{K})$ not invertible, it has rank $< n$. So the rows of A have a linear relation say $v_i + \sum_{j \neq i} c_j v_j = 0$ some i.

Then $r_i \longrightarrow r_i + \sum_{j \neq i} c_j r_j$ (repeated type 1 moves)

puts ith row to be 0 and doesn't change D. So $D(A) = 0$.

Q.E.D.

We can similarly prove other important facts about $\det(A)$ using only (D1) - (D3)

Theorem 3.7 If A and B are $n \times n$ matrices then $\det(AB) = \det(A)\det(B)$.

Proof If A is not invertible then neither is AB (since if $X = (AB)^{-1}$ exists we'd have $ABX = I_n = A(BX)$ etc. so we'd have that A was invertible). So in this case

$$\det(AB) = 0 = \underbrace{\det(A)}_{0}\det(B) \text{ holds.}$$

Now suppose A is invertible. By the proof of Cor 2.20, we can suppose $A = R_k \cdots R_1$ a product of elementary matrices.

So
$$\det(A) = \det(R_k R_{k-1} \cdots R_1)$$
$$= c_k \det(R_{k-1} \cdots R_1)$$
$$= \cdots = c_k c_{k-1} \cdots c_1$$

for the associated constants

(it follows that $c_i = \det(R_k)$)

Similarly
$$\det(AB) = \det(R_k R_{k-1} \cdots R_1 B)$$
$$= \underbrace{c_k \cdots c_1}\det(B)$$
$$= \det(A)\det(B) \quad \text{QED}$$

Spot quiz what is
$$\det\begin{bmatrix} 1 & 2 & \cdots & n \\ n+1 & \cdots & & 2n \\ 2n+1 & \cdots & & 3n \\ & \vdots & & \end{bmatrix} = ? \qquad n \geq 3$$

$\boxed{\checkmark}$ 0 $\square$ $(-1)^n$ $\square$ $n!$ $\square$ n^2 $\square$ depends on the other rows

$r_2 \to r_2 - r_1$

gives

$$\begin{bmatrix} 1 & 2 & \cdots & n \\ n & n & \cdots & n \\ 2n+1 & \cdots & & 3n \\ & \cdots & & \end{bmatrix} \xrightarrow{r_3 - 2r_2} \begin{bmatrix} 1 & 2 & \cdots & n \\ n & n & \cdots & n \\ 1 & 2 & \cdots & n \\ & \vdots & & \end{bmatrix} = 0$$

by (D2).