MTH6140 Linear Algebra II

Coursework 4 Solutions

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1. (a) Row 1 of A is the sum of rows 2 and 3, so rows 2 and 3 span the row space. Rows 2 and 3 are also independent, so form a basis. (In fact, any two rows form a basis for the row space.)

(b) We know from part (a) that the dimension of the column space is 2. So any two columns which are not multiples of each other form a basis of the column space. Columns 1 and 2 will do (as will any pair apart from 2 and 4).

(c) Following the recipe in the note, first apply the elementary operations

$$R_2 - R_1, \quad C_3 - 2C_1$$

(subtract row 1 from row 2, and cubic twice column 1 from column 3) to A, resulting in

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & -1 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & -1 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & -1 & 2 & -2 \end{bmatrix}.$$

At this point, we are effectively reduced to the situation of a 2×3 matrix. Continuing with

$$R_3 + R_2, \quad C_3 + 2C_2, \quad C_4 - 2C_2,$$

we obtain

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have reached the canonical form for equivalence.

Look again at the sequence of elementary operations used in part (c) to reduce the matrix A to canonical form, and apply the row operations in turn to a 3×3 identity matrix and the column operations in turn to a 4×4 identity matrix. Applying the row operations $R_2 - R_1$ and $R_3 + R_2$ we obtain

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} = P,$$

and applying column operations $C_3 - 2C_1$, $C_3 + 2C_2$ and $C_4 - 2C_2$ we obtain

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = Q.$$

You should verify that PAQ is indeed the canonical form computed earlier.

(d) The rank of the matrix in canonical form for equivalence is clearly 2; this matches the row- and column-ranks found in parts (a) and (b).

Remark. P and Q can also be found by multiplying elementary matrices, if desired, but the above method is simpler.

2.

- (Reflexivity.) Set $P = I_m$ and $Q = I_n$. Then $PAQ = I_m A I_n = A$, and so A is equivalent to A.
- (Symmetry.) Suppose B is equivalent to Q, i.e., there exist invertible matrices P and Q such that $B = PAQ$. Then $A = P^{-1}BQ^{-1}$ and it follows that A is equivalent to B.
- (Transitivity.) Suppose B is equivalent to A, and C is equivalent to B. Then $B = PAQ$ and $C = P'BQ'$ for some invertible matrices P, Q, P', Q'. Now note that $C = P'(PAQ)Q' = (P'P)A(QQ')$. It follows that C is equivalent to A.

3. Expanding the determinant yields

$$bc^2 + ca^2 + ab^2 - cb^2 - ba^2 - ac^2,$$

where the terms come from the permutations ι (identity permutation), $(1,2,3)$, $(1,3,2)$, $(2,3)$, $(1,3)$ and $(1,2)$. Note that the transpositions have sign -1 and the other three permutations have sign $+1$. This expression can be massaged into the required form.

4. (a) In this case $n = 5$ is odd, so $\operatorname{sign}(\pi)$ is $+1$ if π has an odd number of cycles and -1 if it has an even number. The transposition is $\tau = (1,2) = (1,2)(3)(4)(5)$ throughout; it has four cycles so always $\operatorname{sign}(\tau) = -1$. (Indeed, any transposition τ will have $\operatorname{sign}(\tau) = -1$.) Thus we need to verify in all cases that $\operatorname{sign}(\sigma\tau) = -\operatorname{sign}(\sigma)$.

i. $\sigma = (1)(2)(3,4,5)$ and $\sigma\tau = (1,2)(3,4,5)$, so $\operatorname{sign}(\sigma) = +1$ and $\operatorname{sign}(\sigma\tau) = -1$.

ii. $\sigma = (1)(2,3,4,5)$ and $\sigma\tau = (1,3,4,5,2)$, so $\operatorname{sign}(\sigma) = -1$ and $\operatorname{sign}(\sigma\tau) = +1$.

iii. $\sigma = (1,3,2,4)(5)$ and $\sigma\tau = (1,4)(2,3)(5)$, so $\mathrm{sign}(\sigma) = -1$ and $\mathrm{sign}(\sigma\tau) = +1$.

(b) I don't know a particularly attractive proof, starting with the definition of $\mathrm{sign}(\pi)$ that we are using in the module. A hands-on approach would examine what happens when we compose the transposition $\tau = (1,2)$ with an arbitrary permutation π on $\{1,\dots,n\}$. There are basically two cases. The easier to think about occurs when 1 and 2 are in different cycles of π. What happens to the number of cycles in passing from π to $\pi\tau$? After that warm-up, consider what happens if 1 and 2 are in the same cycle of π. Again, what happens to the number of cycles in passing from π to $\pi\tau$?

The downside of this strategy is that you may find that you have to separate out some degenerate situations — e.g., when either (1) or (2) is a cycle in π — as sub-cases. It shouldn't get too messy, though, if you go about the task systematically.

5. (a) We know from (D1) that the determinant is linear in each row and column. Applying this fact to row 1, we see that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + 2\times \begin{vmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a+2a' & b+2b' & c+2c' \\ d & e & f \\ g & h & i \end{vmatrix}.$$

(b) In the following calculation, the first equality comes from applying (D1) twice: first to the first pair of determinants and then the second pair. In each case it is linearity in row 1 that is used. The second equality comes from applying (D1) to row 2.

$$\begin{aligned} &\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + 2\times \begin{vmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{vmatrix} + 2\times \begin{vmatrix} a & b & c \\ d' & e' & f' \\ g & h & i \end{vmatrix} + 4\times \begin{vmatrix} a' & b' & c' \\ d' & e' & f' \\ g & h & i \end{vmatrix} \\ &\qquad = \begin{vmatrix} a+2a' & b+2b' & c+2c' \\ d & e & f \\ g & h & i \end{vmatrix} + 2\times \begin{vmatrix} a+2a' & b+2b' & c+2c' \\ d' & e' & f' \\ g & h & i \end{vmatrix} \\ &\qquad = \begin{vmatrix} a+2a' & b+2b' & c+2c' \\ d+2d' & e+2e' & f+2f' \\ g & h & i \end{vmatrix}. \end{aligned}$$

6. (a) Suppose $A = (a_{ij})$ is an $n \times n$ matrix and B is obtained from A by multiplying row i by a scalar c, as in equation (3.1) of the notes. Then, from the Leibniz formula for the permanent:

$$\begin{aligned}\operatorname{per}(B) &= \sum_{\pi \in S_n} a_{1,\pi(1)} \cdots a_{i-1,\pi(i-1)} \left(c a_{i,\pi(i)}\right) a_{i+1,\pi(i+1)} \cdots a_{n,\pi(n)} \\ &= c \sum_{\pi \in S_n} a_{1,\pi(1)} \cdots a_{i-1,\pi(i-1)} a_{i\pi(i)} a_{i+1,\pi(i+1)} \cdots a_{n\pi(n)} \\ &= c \operatorname{per}(A).\end{aligned}$$

Similarly, suppose A, A' and B are $n \times n$ matrices related as in equation (3.2). Then

$$\begin{aligned}\operatorname{per}(B) &= \sum_{\pi \in S_n} a_{1,\pi(1)} \cdots a_{i-1,\pi(i-1)} \left(a_{i\pi(i)} + a'_{i,\pi(i)}\right) a_{i+1,\pi(i+1)} \cdots a_{n,\pi(n)} \\ &= \sum_{\pi \in S_n} a_{1,\pi(1)} \cdots a_{i-1,\pi(i-1)} a_{i,\pi(i)} a_{i+1,\pi(i+1)} \cdots a_{n,\pi(n)} \\ &\quad + \sum_{\pi \in S_n} a_{1,\pi(1)} \cdots a_{i-1,\pi(i-1)} a'_{i\pi(i)} a_{i+1,\pi(i+1)} \cdots a_{n,\pi(n)} \\ &= \operatorname{per}(A) + \operatorname{per}(A').\end{aligned}$$

Thus (D1) holds for the perminent.

That property (D3) holds for the permanent is trivial to check.

Property (D2) fails. For example, the 2×2 matrix $J_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has permanent $1 + 1 = 2$, even though the matrix has two equal rows. (In the expansion of the determinant, the two terms cancel out to yield $1 - 1 = 0$.)

(b) This counterexample works for $\mathbb{R}$, but in $\mathbb{F}_2$ the matrix J_2 has permanent $1 + 1 = 0$, so is no longer a counterexample. In fact, if you try a few random examples, you may begin to suspect that (D2) holds for the permanent in $\mathbb{F}_2$. But if (D2) holds then the permanent must equal the determinant in $\mathbb{F}_2$. Can this be so? In fact yes, since in $\mathbb{F}_2$ we have $-1 = 1$, so the factor $\operatorname{sign}(\pi)$ no longer has an effect! So the determinant and the permanent are the same function in $\mathbb{F}_2$.