MTH6140 Linear Algebra II

Coursework 4

1. Consider the following 3×4 matrix over $\mathbb{R}$:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & -1 & 2 & -2 \end{bmatrix}.$$

(a) Find a maximal independent set of rows in A. (These rows will form a basis for the row space.)

(b) Find a maximal independent set of columns in A.

(c) Use the procedure in the notes to reduce A to the canonical form for equivalence. Hence find invertible matrices P and Q such that PAQ is in the canonical form for equivalence. Verify that PAQ has the correct form by explicitly multiplying the matrices P, A and Q.

(d) From part (c), what is the rank of A? Verify that your answer is consistent with (a) and (b).

2. Recall that $m \times n$ matrices A and B are equivalent if there exist invertible matrices P and Q ($m \times m$ and $n \times n$ respectively) such that $B = PAQ$. Verify that equivalence is an equivalence relation on $m \times n$ matrices.

3. Show, by direct calculation using the Leibniz (sum-over-permutations) formula, that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b).$$

4. In the notes and lectures, we use the fact that if π is any permutation and τ is a transposition then $\operatorname{sign}(\pi\tau) = -\operatorname{sign}(\pi)$. (Recall that in computing the composition $\pi\tau$ we apply τ first and then π.)

(a) For each of the pairs of permutations π and τ on $\{1,2,3,4,5\}$ below, compute $\operatorname{sign}(\pi)$, $\operatorname{sign}(\tau)$ and $\operatorname{sign}(\pi\tau)$ and check that the claimed identity holds:

i. $\pi = (3,4,5)$ and $\tau = (1,2)$,

ii. $\pi = (2,3,4,5)$ and $\tau = (1,2)$, and

iii. $\pi = (1,3,2,4)$ and $\tau = (1,2)$.

Note that the singleton cycles have been suppressed so, for example, $(3,4,5) = (1)(2)(3,4,5)$.

(b) *Harder.* Prove that $\operatorname{sign}(\pi\tau) = -\operatorname{sign}(\pi)$ holds for any permutation π and transposition τ on $\{1,\dots,n\}$.

5. (a) Express

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + 2 \times \begin{vmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{vmatrix}$$

as a single 3×3 determinant.

(b) Express

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + 2 \times \begin{vmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{vmatrix} + 2 \times \begin{vmatrix} a & b & c \\ d' & e' & f' \\ g & h & i \end{vmatrix} + 4 \times \begin{vmatrix} a' & b' & c' \\ d' & e' & f' \\ g & h & i \end{vmatrix}$$

as a single 3×3 determinant.

6. The *permanent* of an $n \times n$ matrix $A = (a_{ij})$ over $\mathbb{K}$ is defined by the formula

$$\operatorname{per}(A) = \sum_{\pi \in S_n} a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)}.$$

(The only change from the determinant is that the factor $\operatorname{sign}(\pi)$ is missing.)

(a) Assume that $\mathbb{K} = \mathbb{R}$. Which of the three properties (D1)–(D3) hold for the permanent of a matrix? (Note that the determinant is the only function satisfying (D1)–(D3), so at least one of the properties must fail for the permanent.) In each case, either demonstrate that the property holds or provide a counterexample. By all means use the corresponding section from the notes as a model, but do present your answer in full.

(b) *Harder.* Repeat this exercise with $\mathbb{K} = \mathbb{F}_2$. How do you square your answer with the parenthetical remark in part (a)?