MTH6140 Linear Algebra II

Coursework 3 Solutions

1. Suppose that V is a vector space over the field $\mathbf{k}$, and let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be vectors in V. Show that $\mathbf{v}_1, \dots, \mathbf{v}_n$ is a basis of V if, and only if, $V = \langle \mathbf{v}_1 \rangle \oplus \cdots \oplus \langle \mathbf{v}_n \rangle$.

Solution. If $B = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ is a basis of V, then each vector $\mathbf{v} \in V$ may be *uniquely* written in the form

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n \tag{1}$$

with suitable scalars $\alpha_1, \dots, \alpha_n$. Since $\alpha_i \mathbf{v}_i \in \langle \mathbf{v}_i \rangle$ for $i = 1, \dots, n$, it follows that each $\mathbf{v} \in V$ can be uniquely written in the form $\mathbf{v} = \mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_n$ with $\mathbf{u}_i \in \langle \mathbf{v}_i \rangle$, which implies that V is the direct sum of the subspaces $\langle \mathbf{v}_i \rangle$ by definition of the direct sum.

Conversely, if $V = \langle \mathbf{v}_1 \rangle \oplus \cdots \oplus \langle \mathbf{v}_n \rangle$, then each vector $\mathbf{v} \in V$ has a unique representation in the form (1), which shows that B is spanning and linearly independent, thus a basis of V.

2. If A is a (3×4)-matrix over the field of real numbers, write down elementary matrices for the following operations:

(i) add twice column 2 to column 4,

(ii) multiply column 3 by 5,

(iii) interchange rows 1 and 3,

(iv) subtract row 1 from row 2.

In each case write down the inverse operation as an elementary matrix.

Solution. (i) The required elementary matrices are, respectively,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) The required elementary matrices are, respectively,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(iii) The required elementary matrices are, respectively

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(iv) The required elementary matrices are, respectively,

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. What is the rank of the matrix $D = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

(i) regarded as a matrix over $\mathbb{R}$?

(ii) regarded as a matrix over the field $\mathbb{Z}/2\mathbb{Z}$ with two elements?

Solution. The first two rows of D are clearly linearly independent, so the decisive question is: can the third row vector be written as a linear combination of rows 1 and 2? Setting

$$(1, 1, 0) = \alpha(0, 1, 1) + \beta(1, 0, 1) = (\beta, \alpha, \alpha + \beta),$$

we compute that, equivalently, we must have $\alpha = \beta = 1$ and $\alpha + \beta = 0$.

(i) Over the field $\mathbb{R}$ of real numbers, $1 + 1 = 2 \neq 0$, so that

$$(1, 1, 0) \notin \big\langle (0, 1, 1),\ (1, 0, 1) \big\rangle,$$

and, consequently, $\mathrm{rk}_{\mathbb{R}}(D) = 3$.

(ii) If D is regarded as a matrix over $\mathbb{Z}/2\mathbb{Z}$, then $\alpha + \beta = 1 + 1 = 0$, and therefore

$$(1, 1, 0) = (0, 1, 1) + (1, 0, 1),$$

which shows that

$$(1, 1, 0) \in \big\langle (0, 1, 1),\ (1, 0, 1) \big\rangle.$$

Hence, $\mathrm{rk}_{\mathbb{Z}/2\mathbb{Z}}(D) = 2$.

4. (i) Call two $(m \times n)$-marices A and B over a field $\mathbf{k}$ *equivalent*, if there exist invertible matrices P and Q, such that $B = PAQ$. Verify that equivalence is an equivalence relation on the set of all $(m \times n)$-matrices over $\mathbf{k}$.

(ii) Call two $(n \times n)$-matrices A and B over a field $\mathbf{k}$ *similar*, if there exists an invertible matrix P such that $B = P^{-1}AP$. Show that similarity is an equivalence relation on the set of $(n \times n)$-matrices over $\mathbf{k}$.

(iii) What relationship does there exist between equivalence and similarity?

Solution. (i) First, let A be an $(m \times n)$-matrix. Then $A = I_n A I_n$, which shows that A is equivalent to itself (reflexivity of equivalence).

Next, suppose that two $(m \times n)$-matrices A and B over $\mathbf{k}$ are equivalent; that is, $B = PAQ$ for some invertible matrices P and Q. Then we have $A = P^{-1}BQ^{-1}$, showing that B is equivalent to A, so that the relation equivalence is symmetric.

Third, suppose that A, B, C are $(m \times n)$-matrices over the field $\mathbf{k}$, that A and B are equivalent, and that B and C are equivalent. Then there exist invertible matrices P_1, P_2, Q_1, Q_2, such that $B = P_1AQ_1$ and $C = P_2BQ_2$. Substituting for B, it follows that $C = (P_2P_1)A(Q_1Q_2)$. Since the product of invertible matrices is again invertible, we conclude that A is equivalent to C, whence transitivity of the relation equivalence.

(ii) Let A be an $(n \times n)$-matrix. Then $A = I_nAI_N = I_n^{-1}AI_n$, showing that A is similar to itself (reflexivity of the relation similar).

Next, let A and B be $(n \times n)$-matrices, and suppose that A is similar to B. Then there exists an invertible matrix P, such that $B = P^{-1}AP$. Thus, $A = (P^{-1})^{-1}BP^{-1}$, which tells us that B is also similar to A, whence symmetry of similarity.

Finally, suppose that A, B, C are $(n \times n)$-matrices, that A is similar to B, and that B is similar to C. Then there exist invertible matrices P and Q, such that $B = P^{-1}AP$ and $C = Q^{-1}BQ$. It follows that

$$C = Q^{-1}(P^{-1}AP)Q = (PQ)^{-1}A(PQ),$$

implying that A is similar to C, proving that the relation similar is transitive, as required.

(iii) If A and B are $(n \times n)$-matrices for some positive integer n, and if A and B are similar, then A and B are equivalent.

5. Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$.

Solution. By Laplace expansion along the first row, we have

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} b & c \\ b^2 & c^2 \end{vmatrix} - \begin{vmatrix} a & c \\ a^2 & c^2 \end{vmatrix} + \begin{vmatrix} a & b \\ a^2 & b^2 \end{vmatrix} = (bc^2 - b^2c) - (ac^2 - a^2c) + (ab^2 - a^2b)$$

$$= bc^2 - b^2c - ac^2 + a^2c + ab^2 - a^2b = (b-a)(c-a)(c-b).$$