MTH6140 Linear Algebra II

Coursework 3

1. Suppose that V is a vector space over the field $\mathbf{k}$, and let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in V. Show that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis of V if, and only if, $V = \langle \mathbf{v}_1 \rangle \oplus \cdots \oplus \langle \mathbf{v}_n \rangle$.

2. If A is a (3×4)-matrix over the field of real numbers, write down elementary matrices for the following operations:

(i) add twice column 2 to column 4,

(ii) multiply column 3 by 5,

(iii) interchange rows 1 and 3,

(iv) subtract row 1 from row 2.

In each case write down the inverse operation as an elementary matrix.

3. What is the rank of the matrix $D = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

(i) regarded as a matrix over $\mathbb{R}$?

(ii) regarded as a matrix over the field $\mathbb{Z}/2\mathbb{Z}$ with two elements?

4. (i) Call two $(m \times n)$-marices A and B over a field $\mathbf{k}$ *equivalent*, if there exist invertible matrices P and Q, such that $B = PAQ$. Verify that equivalence is an equivalence relation on the set of all $(m \times n)$-matrices over $\mathbf{k}$.

(ii) Call two $(n \times n)$-matrices A and B over a field $\mathbf{k}$ *similar*, if there exists an invertible matrix P such that $B = P^{-1}AP$. Show that similarity is an equivalence relation on the set of $(n \times n)$-matrices over $\mathbf{k}$.

(iii) What relationship does there exist between equivalence and similarity?

5. Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$.