MTH6140 Linear Algebra II

Coursework 2 Solutions

1. Let V be a finitely generated vector space, and let $B \subseteq V$ be a finite list of vectors. Show that B is a basis of V if, and only if, B is a maximal linearly independent list.

Solution. If $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of V then, in particular, it is linearly independent. Since B is also spanning by hypothesis, every vector $\mathbf{v} \in V$ can be written as

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n;$$

Thus, $B \cup \{\mathbf{v}\}$ is linearly dependent. Hence, B is a maximal linearly independent list of vectors.

Conversely, suppose that B is a maximal linearly independent list of vectors. Then, in particular, B is a linearly independent list. Moreover, if B is not spanning, then there exists a vector $\mathbf{v} \in V \setminus \langle B \rangle$, thus $B \cup \{\mathbf{v}\}$ would still be linearly independent, contradicting our hypothesis. Hence, B is also spanning, thus a basis of V.

2. Let U and W be subspaces of the vector space V. Show that their sum

$$U + W = \{u + w : u \in U \text{ and } w \in W\}$$

is a subspace of V containing U and W.

Solution. We first note that $U, W \subseteq V$, since $\mathbf{u} = \mathbf{u} + \mathbf{0}$ for $\mathbf{u} \in U$ and, similarly, $\mathbf{w} = \mathbf{0} + \mathbf{w}$ for $\mathbf{w} \in W$; in particular, $U + W \neq \emptyset$. Also, if $\mathbf{u}_1 + \mathbf{w}_1, \mathbf{u}_2 + \mathbf{w}_2 \in U + W$, then by commutativity and associativity of vector addition

$$(\mathbf{u}_1 + \mathbf{w}_1) + (\mathbf{u}_2 + \mathbf{w}_2) = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{w}_1 + \mathbf{w}_2) \in U + W,$$

since U and W are closed under vector addition. Moreover,

$$\alpha(\mathbf{u} + \mathbf{w}) = \alpha\mathbf{u} + \alpha\mathbf{w} \in U + W$$

for each scalar α, since U and W are closed under scalar multiplication. The fact that $U + W$ is a subspace of V now follows from the subspace criterion.

3. Let $V = \mathbb{R}^2$, and let $\mathbf{v}_1 = (\alpha, 0), \mathbf{v}_2 = (0, \beta)$. When is $(\mathbf{v}_1, \mathbf{v}_2)$ a basis of V? Please justify your answer.

Solution. If $\alpha = 0$, then $\mathbf{v}_1 = \mathbf{0}$, so that $(\mathbf{v}_1, \mathbf{v}_2)$ is not linearly independent, with a similar argument for $\mathbf{v}_2$ and β. Hence, if $(\mathbf{v}_1, \mathbf{v}_2)$ is to be a basis for V, then we must have $\alpha, \beta \neq 0$.

Conversely, suppose that $\alpha, \beta \neq 0$. Then we can write an arbitrary vector $(\gamma, \delta) \in V$ as

$$(\gamma, \delta) = \alpha^{-1}\gamma\mathbf{v}_1 + \beta^{-1}\delta\mathbf{v}_2,$$

so that $(\mathbf{v}_1, \mathbf{v}_2)$ is spanning. Since $\dim(\mathbb{R}^2) = 2$, $(\mathbf{v}_1, \mathbf{v}_2)$ is a basis of V.

4. Let X, Y, D be subspaces of the real vector space $V = \mathbb{R}^2$ given by

$$X := \{(x,0) : x \in \mathbb{R}\},$$
$$Y := \{(0,y) : y \in \mathbb{R}\},$$
$$D := \{(x,x) : x \in \mathbb{R}\}.$$

Show that $V = X \oplus Y = X \oplus D = Y \oplus D$.

Solution. We have

$$V = X + Y = X + D = Y + D,$$

since every vector $(x, y) \in V$ can be written in each of the three ways

$$(x,y) = (x,0) + (0,y) = x \cdot (1,0) + Y \cdot (0,1),$$
$$(x,y) = (x-y,0) + (y,y) = (x-y) \cdot (1,0) + y \cdot (1,1),$$
$$(x,y) = (0,y-x) + (x,x) = (y-x) \cdot (0,1) + x \cdot (1,1),$$

so that each of the lists

$$((1,0),\ (0,1)), \quad ((1,0),\ (1,1)), \quad ((0,1),\ (1,1)) \tag{1}$$

generates V; that is, each of these lists is spanning. Moreover, each of the lists in (1) is obviously also linearly independent, so that each of the lists in (1) is in fact a basis of V. Consequently, we have

$$V = \langle (1,0) \rangle \oplus \langle (0,1) \rangle = X \oplus Y,$$
$$V = \langle (1,0) \rangle \oplus \langle (1,1) \rangle = X \oplus D,$$
$$V = \langle (0,1) \rangle \oplus \langle (1,1) \rangle = Y \oplus D,$$

as claimed.