L4 Learning Support Hour 1–2pm MB403

Glossary
l.i. = linearly independent
l.d. = " dependent
v.s. = vector space
f.d. = finite – dimensional
$:=$ means equal by definition
$\cong$ isomorphic.

Recall if $V$ is a f.d. v.s. with basis
$v_1, \dots, v_n$ then $V \cong \mathbb{K}^n = \left\{ \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, c_i \in \mathbb{K} \right\}$

check this is well-defined

$$u = \sum c_i v_i \longmapsto \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

If also $u = \sum c_i' v_i \longmapsto \begin{bmatrix} c_1' \\ \vdots \\ c_n' \end{bmatrix}$

then $\sum (c_i' - c_i) v_i = u - u = 0$

$\therefore \; c_i' - c_i = 0 \;\; \forall i$

because the $v_i$ are l.i.

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What happens if we change the basis?

Let $B = v_1, \dots v_n$ as before
$B' = v_1', \dots v_n'$ another basis.

Define $P_{B,B'}$ by $v_j' = \sum_i v_i (P_{B,B'})_{ij}$

coefficients of $v_i'$ in basis $B$

So $P_{B,B'} = [v_1']_B \, [v_2']_B \dots [v_n']_B$, where

$[\;]_B$ denotes the column vector w.r.t. the basis $B$.

this $n \times n$ matrix is called the "transition matrix" from $B$ to $B'$.

$\Rightarrow$ if $u \in V$ then $[u]_B = P_{B,B'} [u]_{B'}$

(Note this is the op. convention to Lin Alg I)

[Check: $u = \sum_i c_i v_i = \sum_j c_j' v_j'$

$= \sum_{j,i} c_j' (P_{B,B'})_{ij} v_i$

$\Rightarrow$ $\underbrace{c_i}_{[u]_B} = \sum_j (P_{B,B'})_{ij} \underbrace{c_j'}_{[u]_{B'}}$ ]

__Facts__ $P_{B,B} = I = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots 1 \end{pmatrix}$ (= $I_n$ the $n \times n$ identity matrix)

$P_{B',B} = (P_{B,B'})^{-1} \Rightarrow P_{B,B'}$ is invertible

ie $\sum_i (P_{B,B'})_{ki} (P_{B',B})_{ij} = I_{kj}$ $(= \delta_{kj})$

Similarly, if $B''$ another basis

$v_j'' = \sum_k v_k (P_{B,B''})_{kj}$

$\quad \parallel$

$\sum_i v_i' (P_{B',B''})_{ij}$

$\quad \parallel$

$\sum_{i,k} v_k (P_{B,B'})_{ki} (P_{B',B''})_{ij}$

$\Downarrow$

$$P_{B,B''} = P_{B,B'} P_{B',B''}$$

Similarly from the column vector point of view

$$P_{B,B''}[u]_{B''} = (P_{B,B'}P_{B',B''})[u]_{B''}$$
$$= P_{B,B'}(\underbrace{P_{B',B''}[u]_{B''}}_{[u]_{B'}})$$
$$= [u]_B$$

as expected

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We can also do the above with row vectors

- $u \longmapsto [c_1 \ldots c_n] = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}^t$ "transpose"

We prefer column vectors since e.g.

$$\begin{matrix} 2x+3y=5 \\ 4x+5y=9 \end{matrix} \longleftrightarrow \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

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<u>Example</u> $B = (v_1, v_2)$ basis of a 2-dimensional v.s. ✓

- $B' = (v_1', v_2')$ another basis

$$v_1' = v_1 + v_2$$
$$v_2' = 2v_1 + 3v_2$$

$$\Rightarrow \quad P_{B,B'} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$v_1'$ in basis $v_1, v_2$

$v_2'$ in basis $v_1, v_2$

suppose $[u]_{B'} = \begin{bmatrix} a \\ b \end{bmatrix}$ $a, b \in \mathbb{K}$

$$\Rightarrow [u]_B = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+2b \\ a+3b \end{bmatrix}$$

check $u = av_1' + bv_2'$
$$= a(v_1+v_2) + b(2v_1+3v_2)$$
$$= (a+2b)v_1 + (a+3b)v_2$$
as expected ✓

$$P_{B',B} = (P_{B,B'})^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$\left(\det \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = 1\right)$

now suppose $B'' = (v_1'', v_2'')$, $v_1'' = 3v_1' - 2v_2'$
$v_2'' = -2v_1' + v_2'$

$$P_{B',B''} = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$$

(check $\det = -1 \neq 0$ in $\mathbb{K}$ so $B''$ is a basis)

$$\Rightarrow P_{B,B'} P_{B',B''} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -3 & 1 \end{bmatrix} = P_{B,B''}$$

check $v_1'' = 3(v_1 + v_2) - 2(2v_1 + 3v_2)$
$$= -v_1 - 3v_2$$
$$v_2'' = -2(v_1+v_2) + (2v_1+3v_2)$$
$$= v_2$$
✓

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## Section 1.4 (subspaces and direct sums)

Let $V$ be a v.s. (over a field $\mathbb{K}$)

A subset $U \subseteq V$ is called a subspace if $U$ is itself a v.s. with respect to the $+$ and scaling by $\mathbb{K}$ inherited from $V$

**Lemma 1.24** Let $U$ be a non-empty subset of a v.s. $V$ over $\mathbb{K}$. Then TFAE ("the following are equivalent")

(a) $U$ is a subspace of $V$

(b) $u+u' \in U$, $cu \in U$ $\forall\, u, u' \in U$, $c \in \mathbb{K}$

(i.e. $U$ is closed under $+$ and scaling by $\mathbb{K}$ in $V$)

**proof** (a) $\Rightarrow$ (b) follows from the definition of a subspace

(b) $\Rightarrow$ (a) Suppose (b). Then for all $u \in U$

set $-u := (-1)u \in U$

Since $U \neq \emptyset$ (the empty set) $\exists\, u \in U$

$\therefore$ $0 = u - u = u + (-u) \in U$ working in $V$

If $u, v \in U$

$\Rightarrow -v \in U$ and $u - v = u + (-v) \in U$

$\therefore$ $(+, 0)$ is an abelian group structure on $U$

Now check the various axioms hold, e.g.

$c(u+u') = cu + cu'$ etc – they all hold in $U$ since the hold in $V$. Q.E.D.

**Constructions for subspaces**

(a) If $V$ is a v.s. over $\mathbb{K}$ and $v_1, \dots, v_n \in V$ then $U := \langle v_1, \dots, v_n \rangle := \{ c_1 v_1 + \dots + c_n v_n \mid c_i \in \mathbb{K} \}$ is a subspace of $V$ (spanned by $v_1 \dots v_n$)

(b) let $U, W$ be subspaces of $V$. Then

(i) $U \cap W$ ~~is a su~~ (as sets)

(ii) $U + W := \{u + w \mid u \in U, w \in W\}$

are subspaces of $V$.

$\boxed{L5}$ Proof (a). $U$ is non-empty as $\underline{0} = 0v_1 + \dots + 0v_n$

$\underline{0}$: 0 vector; $0 \in \mathbb{K}$

Closed under $+$:

If $u = \sum_i c_i v_i$, $u' = \sum_j c_j' v_j$

$\Rightarrow u + u' = \sum_i c_i v_i + \sum_i c_i' v_i = \sum_i (c_i + c_i') v_i \in U$

Similarly check $cu \in U \ \forall c \in \mathbb{K}$

and then use lemma 1.24.

(b) (i) $\underline{0} \in U$, $\underline{0} \in W$ so $\underline{0} \in U \cap W$

hence $U \cap W$ is not empty

If $u, u' \in U \cap W \Rightarrow \begin{cases} u, u' \in U \text{ so } u + u' \in U \\ u, u' \in W \text{ so } u + u' \in W \end{cases}$

$\therefore u + u' \in U \cap W$. For all $c \in \mathbb{K}$

$\left.\begin{array}{l} cu \in U \text{ as } U \text{ a subspace} \\ cu \in W \text{ as } W \text{ a subspace} \end{array}\right\} \Rightarrow cu \in U \cap W$

(ii) $\underline{0} = \underline{0} + \underline{0} \in U + W$ as $\underline{0} \in U, W$

If $u + w, u' + w' \in U + W$ (so $u, u' \in U$, $w, w' \in W$)

then in $V$, $(u + w) + (u' + w') = \underbrace{(u + u')}_{\in U} + \underbrace{(w + w')}_{\in W} \in U + W$

And if $c \in \mathbb{K}$, $c(u + w) = \underset{\in U}{cu} + \underset{\in W}{cw} \in U + W$

Then use the lemma 1.24 Q.E.D

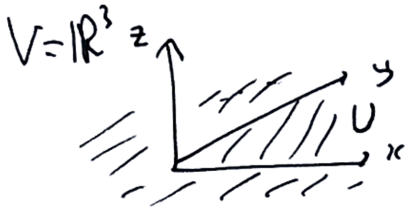
Examples (i) If $V$ a v.s. over $\mathbb{K}$ and $v \in V$ then $\langle v \rangle = \{ cv \mid c \in \mathbb{K} \} := \mathbb{K}v$ is a subspace. E.g. $\underline{0} \in V$, $\langle \underline{0} \rangle = \{ \underline{0} \} \subseteq V$ is a subspace. At the other extreme $V$ is itself a subspace of $V$. So we say $U \subseteq V$ is a proper subspace if $U$ is a subspace and $U \neq \{0\}, V$.

(ii) $\langle v_1, \ldots, v_m \rangle = \langle v_1 \rangle + \langle v_2 \rangle + \cdots + \langle v_m \rangle$

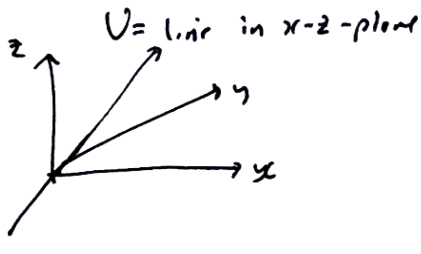
since $\langle v_i \rangle = \{ c v_i \} \subseteq V$

(iii) in $\mathbb{R}^3$, $\underline{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\underline{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

are a basis of $V = \mathbb{R}^3$. $U = \langle \underline{i}, \underline{j} \rangle =$ "x-y plane" $= \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \right\}$




$\boxed{\text{or}}$ $U = \langle \underline{i} + \underline{k} \rangle = \left\langle \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} x \\ 0 \\ x \end{bmatrix} \,\middle|\, x \in \mathbb{R} \right\}$




Spot Quiz $U = \left\{ \begin{bmatrix} x \\ 0 \\ x^2 \end{bmatrix} \,\middle|\, x \in \mathbb{R} \right\}$

is this a subspace of $V = \mathbb{R}^3$ ?

☐ yes ☑ no ☐ depends

e.g. $\begin{bmatrix} x \\ 0 \\ x^2 \end{bmatrix} + \begin{bmatrix} y \\ 0 \\ y^2 \end{bmatrix} = \begin{bmatrix} x+y \\ 0 \\ x^2+y^2 \end{bmatrix}$

so not in $U$ $\Leftarrow$ $x^2+y^2 \neq (x+y)^2$ unless $x$ or $y = 0$

[Note that in other contexts we may write $\mathbb{R}^3$ as elements $(x, y, z)$ - ordered triples.]

(iv) $V = M_n(\mathbb{R})$ (all $n \times n$ matrices entries in $\mathbb{R}$)

$U = \mathrm{Sym}_n(\mathbb{R})$ = all symmetric $n \times n$ matrices entries in $\mathbb{R}$

$= \{ (a_{ij}) \mid a_{ij} \in \mathbb{R},\ a_{ij} = a_{ji} \}$

$= \{ A \in M_n(\mathbb{R}) \mid A^t = A \}$ transpose.

is $U$ a subspace?

☑ yes ☐ no

<u>proof</u> if $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$ $\forall i, j$ then $A + B$ = matrix with entries

$(A+B)_{ij} := a_{ij} + b_{ij}$

$= a_{ji} + b_{ji}$

$= (A+B)_{ji}$

(A has entries $a_{ij}$, B " " $b_{ij}$)

so $A, B \in U \Rightarrow A + B \in U$

similarly for scaling.

what about $U = \{ A \in M_n(\mathbb{R}) \mid \mathrm{Tr}(A) = 0 \}$?

trace $\mathrm{Tr}(A) = \sum_i a_{ii}$

☑ yes ☐ no

<u>proof</u> $\mathrm{Tr}(A+B) = \sum_i (a_{ii} + b_{ii}) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$

$= 0 + 0 = 0$ if $A, B \in U$

similarly check scaling.

What about

$$U = \{ A \in M_n(\mathbb{R}) \mid \det(A) = 0 \}$$

$\square$ yes $\quad$ $\boxed{\checkmark}$ no if $n > 1$ $\quad$ $\boxed{\checkmark}$ depends if $n = 1$

$\det(A+B) \neq \det(A) + \det(B)$ unless $1\times1$ $(n=1)$

e.g. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \notin U$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in U$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in U$

(v) What about if $U, W \subseteq V$ subspaces

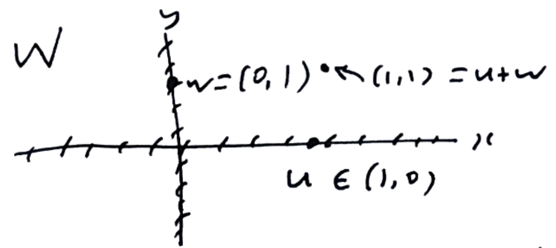
is $U \cup W$ a subspace ?

yes $\square$ $\quad$ No $\boxed{\checkmark}$ $\quad$ depends $\boxed{\checkmark}$ (in extreme cases)

e.g. $V = \mathbb{R}^2$, $U = \langle \underline{i} \rangle = x$-axis

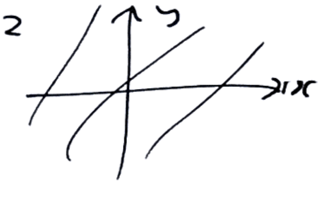
$W = \langle \underline{j} \rangle = y$-axis

$U \cup W$




$u + w = (1,1) \notin U \cup W$

$u + w \in U + W = \mathbb{R}^2$




$U \cup W \subsetneq U + W$

not a v.s. $\quad$ $\subsetneq$ means contained not equal $\quad$ $U \cup W$ extended into a v.s.

( but scaling work, $U \cup W$ closed under scaling )

[ Remark in "Schroedinger's cat" in Quantum mechanics

$u = (1,0)$ could represent "cat alive"

$w = (0,1)$ " " "cat dead"

} true state could be $u + w$ ]

(vi) Over $\mathbb{K} = \mathbb{F}_2 = \{0,1\}$, If $V$ has basis $v_1 \dots v_n$, we can identify $V \cong \underbrace{P(\{1,\dots,n\})}$

"power set" = set of subsets of $\{1,\dots,n\}$

$v \in V$
$$= \sum_i c_i v_i \longmapsto I = \{ i \in \{1,\dots,n\} \mid c_i = 1 \}$$

$$I \subseteq \{1,\dots,n\}$$

If $v, w \in V$ $\quad v \longmapsto I$

$$w = \sum_j d_j v_j, \quad w \longmapsto J = \{ j \in \{1,\dots n\}, d_j = 1 \}$$

What subset of $\{1 \dots n\}$ does $v+w$ correspond to?

$\square$ $I \cap J$ $\quad$ $\square$ $I \cup J$ $\quad$ $\boxed{\checkmark}$ $I \Delta J$ $\quad$ $\square$ $I + J$ X doesnt exist $\quad$ $\square$ other

"symmetric difference" $(I \setminus J) \cup (J \setminus I)$

$$v + w = \sum_i (c_i + d_i) v_i \longleftrightarrow \{ i \in \{1,\dots n\} \mid c_i + d_i = 1 \}$$
$$= \{ i \mid i \in I \setminus J \text{ or } i \in J \setminus I \}$$
$$= I \Delta J.$$

e.g. $U = P(\{1,2,3\}) = \langle v_1, v_2, v_3 \rangle \subseteq V = \langle v_1, v_2 \dots v_n \rangle$
if $n \geq 3$

<u>Question</u> is every subset of $V$ of this form ie is $U = P(X)$, $X \subseteq \{1,2,\dots n\}$

(challenge) ie what do all the subspaces of $V$ over $\mathbb{F}_2$ look like

Theorem 1.25  Let $U, W$ be subspaces of a v.s. $V$ over $\mathbb{K}$. Assuming $U, W$ are f.d,

$$\dim(U \cap W) + \dim(U+W) = \dim(U) + \dim(W)$$

proof  Let $v_1, \dots v_i$ be a basis of $U \cap W$. As $U \cap W \subseteq U$ is a subspace, we can extend it to a basis of $U$

$v_1, \dots, v_i, u_1, \dots, u_j$.  (by Theorem 1.15(c))

(since $v_1 .. v_i$ are l.i. viewed in $U$ due to being l.i. in $U \cap W$ due to being a basis there)

Similarly viewed inside $W$, we extend $v_1, .. v_i$ to a basis of $W$,

$v_1, \dots, v_i, w_1, \dots, w_k$  regarding $U \cap W \subseteq W$

We'll show that

(*) $v_1, \dots, v_i, u_1, \dots, u_j, w_1, \dots w_k$ are a basis of $U+W$. If so then

$$\dim(U \cap W) + \dim(U+W)$$
$$= i + (i+j+k) = (i+j) + (i+k)$$
$$= \dim(U) + \dim(W)$$

as wanted.

We show that vectors (*) span $U+W$:

$$u+w \in \langle v_1, \dots, v_i, u_1, \dots, u_j \rangle + \langle v_1, \dots, v_i, w_1, \dots w_k \rangle$$
$$= \langle v_1, .. v_i, u_1 \dots, u_j, w_1 \dots w_k \rangle$$

Now suppose a linear relation

$$a_1 v_1 + \dots + a_i v_i + b_1 u_1 + \dots + b_j u_j + \underbrace{c_1 w_1 + \dots + c_k w_k}_{v} = 0$$

then $v := \underbrace{c_1 w_1 + \dots + c_k w_k}_{\in W} = \underbrace{- a_1 v_1 \dots - a_i v_i - b_1 u_1 \dots - b_j u_j}_{\in U}$

$\therefore \; v \in U \cap W$

$\therefore \quad v = d_1 v_1 + \dots d_i v_i \quad$ some $d_i \in \mathbb{K}$

(as $\{v_i\}$ a basis of $U \cap W$)

so

$$c_1 w_1 + \dots c_k w_k - d_1 v_1 \dots - d_i v_i = 0$$

$\Rightarrow \quad c_1 = \dots = c_k = 0 \quad$ (and also $d_1 \dots d_i = 0$)

because $v_1 \dots v_i, w_1 \dots w_k$ basis of $W$ $\therefore$ l.i.

$\Rightarrow \quad a_1 v_1 + \dots a_i v_i + b_1 u_1 + \dots b_j u_j = 0$

$\Rightarrow \quad a_1 = \dots = a_i = 0$ and $b_1 = \dots = b_j = 0$.

because $v_1 \dots v_i, u_1 \dots u_j$ a basis of $U$ $\therefore$ l.i.

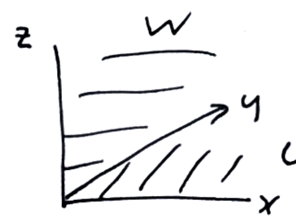
Hence $v_1 \dots v_i, u_1 \dots u_j, w_1 \dots w_k$ are l.i.

$\therefore$ a basis

QED

<u>Example</u> (a) $V = \mathbb{R}^3$, $U = \mathbb{R}^2 = \langle \underline{i}, \underline{j} \rangle$ = x-y plane

$W = \mathbb{R}^2 = \langle \underline{j}, \underline{k} \rangle$ = y-z plane




$U \cap W = \langle \underline{j} \rangle$ = y-axis

$U + W = V = \mathbb{R}^3 = \langle \underline{i}, \underline{j}, \underline{k} \rangle$

check $\quad \dim(U \cap W) + \dim(U + W) = \dim(U) + \dim(W)$

$\qquad 1 + 3 = 2 + 2$

(b) Suppose $U \subseteq W$ then $U \cap W = U$, $U + W = W$


(since $U + W \supseteq \{0\} + W = W$ and $U + W \subseteq W$)

$$\underbrace{\dim(U \cap W)}_{U} + \underbrace{\dim(U+W)}_{W} = \dim(U) + \dim(W)$$ ✓ holds automatically

(c) Suppose $U, W \subseteq V$ subspaces and $U \cap W = \{0\}$. In this case we write $U+W$ as $U \oplus W$

$U \oplus W := U + W$ in case $U \cap W = \{0\}$

"direct sum"

$$\underbrace{\dim(\underbrace{U \cap W}_{\{0\}})}_{0} + \dim(U+W) = \dim(U) + \dim(W)$$

so $\dim(U \oplus W) = \dim(U) + \dim(W)$

e.g. $U =$ x-y plane $= \mathbb{R}^2$, $W =$ z-axis $= \mathbb{R}$. $U \cap W = \{0\}$ and indeed $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$

With respect to Theorem 1.25 we have $U \cap W = \{0\}$ so no vectors $v_1 \ldots v_i$. So $u_1, \ldots u_j$ is a basis of $U$, $w_1 \ldots w_k$ a basis of $W$ and $u_1, \ldots u_j, w_1 \ldots w_k$ is a basis of $U \oplus W$

$\Rightarrow$ $\boxed{\text{every element of } U \oplus W \text{ is uniquely written as } u+w \text{ for } u \in U,\ w \in W}$ (*)

since of the form $\underbrace{a_1 u_1 + \cdots a_j u_j}_{u} + \underbrace{b_1 w_1 + \cdots + b_k w_k}_{w}$

In fact direct sums do not need the v.s.'s to be f.d. This property (*) still holds.

Proof If $v = u + w = u' + w'$ $\quad u, u' \in U$, $w, w' \in W$

$\Rightarrow$ $\underset{\in U}{u - u'} = \underset{\in W}{w' - w}$ $\therefore \in U \cap W = \{0\}$

so $u - u' = 0$, $w - w' = 0$

QED

<u>Example</u> $V = C^\infty(\mathbb{R})$ (all differentiable functions as many times as you like)

$U = \{ f(x) = ax \mid a \in \mathbb{R} \} \subseteq V$

$\cong \mathbb{R}$

$f \longmapsto a \in \mathbb{R}$ Its a subspace by lemma 1.24

$ax + bx = (a+b)x$
$a(bx) = (ab)x$

$W = \{ f \in C^\infty(\mathbb{R}) \mid f'(0) = 0 \} \subseteq V$

a subspace by lemma 1.24

$U \cap W = \{ f \in C^\infty(\mathbb{R}) \mid f(x) = ax,\ f'(0) = 0 \}$

$f'(0) = a$ $\qquad$ $a = 0$

$= \{0\}$

so we have $U \oplus W$. In fact

$U \oplus W = V$ any $f \in C^\infty(\mathbb{R})$
$= u + w$

$u(x) = f'(0)x$, $\quad w(x) = f(x) - u(x)$

$\in U$ $\qquad w'(0) = 0$ so $w \in W$

check $\quad u = a v_1' + b v_2'$

$$= a(v_1 + v_2) + b(2v_1 + 3v_2)$$

$$= (a+2b)v_1 + (a+3b)v_2$$

as expected ✓

$$P_{B',B} = (P_{B,B'})^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$\left(\det \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = 1\right)$

now suppose $\quad B'' = (v_1'', v_2'')$, $\quad v_1'' = 3v_1' - 2v_2'$

$v_2'' = -2v_1' + v_2'$

$$P_{B',B''} = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$$

(check $\det = -1 \neq 0$ in $\mathbb{K}$ so $B''$ is a basis)

$$\Rightarrow P_{B,B'} P_{B',B''} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -3 & 1 \end{bmatrix} = P_{B,B''}$$

check $\quad v_1'' = 3(v_1 + v_2) - 2(2v_1 + 3v_2)$

$$= -v_1 - 3v_2$$

$$v_2'' = -2(v_1 + v_2) + (2v_1 + 3v_2)$$

$$= v_2$$

✓

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## Section 1.4 (subspaces and direct sums)

Let $V$ be a v.s. (over a field $\mathbb{K}$)

A subset $U \subseteq V$ is called a <u>subspace</u> if $U$ is itself a v.s. with respect to the $+$ and scaling by $\mathbb{K}$ inherited from $V$