MTH 6140 Linear Algebra II

$\boxed{L0}$ **Prof Majid**

learning support hours → see Dept home page.

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4 weekly "lectures"

- Seminar = tutorial = Q&A session.

  Mondays 11-12

- 3 lectures 12 - 1 mondays

  9 - 11 Wednesday

Resources QMPlus web page.

Weekly roster
- scan of these notes.
- non-assessed cwk. — do in tutorial with solutions — at least follow
- fortnightly quizzes online worth 4 % each (x5)
- past exams, mostly with solutions
- This week : May 2019 exams

Assessments Jan exam on-campus 3 hrs

quizzes x 5

[we went thru the module web page]

[L1] This is not linear alg I

- not about algorithms
- about an abstract understanding of axioms, proofs behind them.
- logic not memory, understand proofs, ideas <u>and</u> be able to calcute.

<u>Mindset</u> <u>Secret of pure mathematics</u> Its much easier to prove something in general (eg for <u>all</u> vector spaces...) than for a specific example. [Since the only thing you have to work with is the definitions, axioms]

pure Maths is a creative subject — imagine you are in charge of inventing it.

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1. Axioms of a vector space

<u>Def 1.1</u> A <u>field</u> $(\mathbb{K}, +, \cdot)$ is an abelian group under $+$, $\mathbb{K} \setminus \{0\}$ is an abelian group under $\cdot$ (multiplication, also denoted by omission)

<u>and</u> $a(b+c) = ab + ac \quad \forall a, b, c \in \mathbb{K}$

e.g. $\mathbb{R}$ reals, $\mathbb{C}$ complex numbers, $\mathbb{Q}$ rationals

$\mathbb{F}_p = \mathbb{Z} \bmod p$ $\quad p$ prime $\quad$ ($\mathbb{F}_{p^d}$ $\quad d \geq 1$ also exist)

$\quad \parallel$

$\{0, 1, 2, \ldots p-1\}$ $\quad p$ elements

characterised by $\quad \underbrace{1+1+1+\cdots+1}_{p \text{ times}} = 0$

e.g. $\quad \mathbb{F}_2 = \{0, 1\}$ $\quad$ with $1+1=0$

$\quad \mathbb{F}_3 = \{0, 1, 2\}$ $\quad 2 = -1$ as $1+1+1=0$

$\quad 2 \cdot 2 = 4 = 1 \mod 3$

(or $2.2 \equiv 4 \equiv 1$ remaind mod 3)

Def 1.2

A vector space V over a field $\mathbb{K}$ is $(V, +, \cdot)$ with $(V, +)$ is an abelian group. and now $\cdot$ (also denoted by omission) is a "scaling action" of $\mathbb{K}$ ie $\forall a \in \mathbb{K}, v \in V$ we have $a.v \in V$ such that

$$a \cdot (v + w) = a.v + a.w \qquad \forall a \in \mathbb{K}$$

$$a \cdot (b \cdot v) = (ab) \cdot v \qquad v, w \in V, \; b \in \mathbb{K}$$

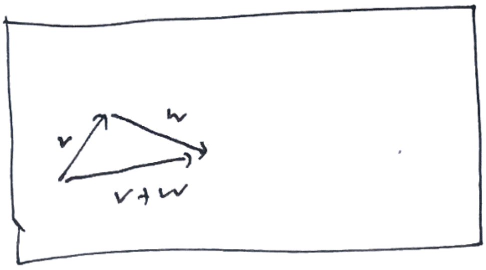
$$(a+b) \cdot v = a.v + b.v$$

$$1.v = v$$

Examples $\quad$ Ex 1 $\quad V = \mathbb{K}^n$ (n-tuples), $v = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ $\quad a_i \in \mathbb{K}$

Ex 2 Vectors in $\mathbb{R}^2$ as arrows in the plane

ie directions ie up to the location.



$0 \in V$ is the arrow of zero length.

action of $\mathbb{R}$ is to magnify.

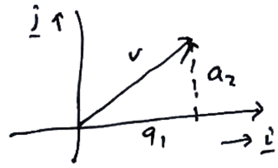
$v = \nearrow$ then $-v = \swarrow$

Choose chosen basis $\underline{i} = \longrightarrow$ vector along "x-axis"

$\underline{j} = \uparrow$ vector along "y-axis"

any $v \in V$ can be decomposed as

$$v = a_1 \underline{i} + a_2 \underline{j}$$



$$V \leftrightarrow \mathbb{R}^2$$

$$v \longleftrightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

better $V \cong \mathbb{R}^2$

"isomorphic" means

$+, \cdot$ agree on the two sides

Similarly vectors in $\mathbb{R}^n \cong \mathbb{R}^n$ as n-tuples.

<u>Ex 3</u> S any set, $V = \mathbb{R}^S := \{$all maps $f : S \to \mathbb{R}\}$

f sends $s \in S$ to $f(s) \in \mathbb{R}$

with $(f+g)(s) = f(s) + g(s)$

$$(a \cdot f)(s) = a(f(s)) = a f(s) \quad \forall a \in \mathbb{R}$$

$\uparrow$ product in $\mathbb{R}$

a vector space over $\mathbb{R}$.

($:=$ means equal by definition)

$\boxed{\text{Check it Yourself}}$

e.g. $\mathbb{R}^{\mathbb{R}} \supsetneq C(\mathbb{R}) \supsetneq C^{\infty}(\mathbb{R}) \supsetneq \mathbb{R}[x]$

all continuous functions — smooth functions — polynomials in x

differentiable as much as you like

If f, g are differentiable then $(f+g)' = f' + g'$ etc.

Ex 4 Polynomials in x, $V = \mathbb{K}[x]$ over a field $\mathbb{K}$. Elements are

$$f(x) = f_0 + f_1 x + f_2 x^2 + \dots + f_n x^n \qquad f_i \in \mathbb{K}$$

(some finite top degree, not fixed)

$\mathbb{K}[x]_n$ = Polynomials of degree $\leq n$ for a fixed n.

$$\cong \mathbb{K}^{n+1} \qquad f(x) \longleftrightarrow \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix}$$

$$\mathbb{K}[x] = \mathbb{K}^{\mathbb{N}_0} \qquad \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$$

is a vector space by ~~ex~~ Ex 3.

Definition 1.7 A list of vectors $(v_1, \dots, v_n)$ in a vector space V (so $v_i \in V$) is said to be linearly independent (l.i.) if (and only if)

$$c_1 v_1 + \dots + c_n v_n = 0 \implies c_1 = c_2 = \dots = c_n = 0$$

where $c_i \in \mathbb{K}$.

[i.e. $\nexists$ (does not exist) a non-zero linear combination adding to zero. i.e. we cannot write one of the vectors as a linear combination of the others]

<u>Def 1.9</u> A list of vectors $(v_1, \ldots, v_n)$ in V (so $v_i \in V$) is <u>spanning</u> if (f)

$\forall v \in V$, $v = c_1 v_1 + \ldots + c_n v_n$ for some $c_i \in \mathbb{K}$

A list $(v_1 \ldots v_n)$ is a <u>basis</u> of V if it is both linearly independent <u>and</u> spanning.

<u>Remark</u> If a list $(v_1, \ldots v_n)$ of $v_i \in V$ is not l.i. we say it is linearly dependent (l.d.) ie does exist $c_i \in \mathbb{K}$ s.t. $c_1 v_1 + \ldots c_n v_n = 0$ (not all $c_i = 0$)

<u>Spot Quiz</u> What is the dimension of the vector space $V = \{0\}$?

☑ 0 ☐ 1 ☑ doesn't exist ☐ depends on $\mathbb{K}$ ☐ other

($V = \phi$ empty set is not a vector space as need $0 \in V$)

not l.i. since $\lambda . 0 = 0 \not\Rightarrow \lambda = 0$

0^0 ☐ 0 ☐ 1 ☐ undefined

L3 $1\frac{1}{2}$ hrs

$\frac{1}{2}$ Tutorial = Q&A = "office" (here) [Also actual hour available later]

Correction: Final exam (January) will be on-line.

Example Let $V = \mathbb{R}[x]_2$ = polynomials degree ≤ 2, coefficients in $\mathbb{R}$

Consider (x, x^2-1, x^2+1) of elements of V

is this l.i.? suppose $c_1 x + c_2(x^2-1) + c_3(x^2+1) = 0$

$\Leftrightarrow ((c_3 - c_2)1 + c_1 x + (c_2+c_3)x^2 = 0$

$\Leftrightarrow \underbrace{c_3 = c_2, \quad c_3 = -c_2}_{2c_3 = 0 \Rightarrow c_3 = c_2 = 0}, \quad c_1 = 0$ so $\boxed{\text{yes}}$

is this set spanning? Suppose $v = f_0 1 + f_1 x + f_2 x^2 \in V$

Can I write $v = c_1 x + c_2(x^2-1) + c_3(x^2+1)$ for some $c_i \in \mathbb{R}$?

we'd need $c_1 = f_1$

$c_3 - c_2 = f_0$, $c_2 + c_3 = f_2$

$c_2 = (f_2 - f_0)/2$, $c_3 = (f_0 + f_2)/2$

so $\boxed{\text{yes}}$

What about over $\mathbb{F}_2 = \{0,1\}$?

since $+1 = -1$, the list is (x, x^2+1, x^2+1)

$1.(x^2+1) + 1.(x^2+1) = \underbrace{(1+1)}_{0}(x^2+1) = 0$

so $\boxed{\text{NO}}$

Also doesn't span e.g. x^2 can't be written as $c_1 x + c_2(x^2+1) + c_3(x^2+1)$

so $\boxed{\text{NO}}$

<u>Definition / Example of v.s.</u> Given list $(v_1, \dots, v_n)$ $v_i \in V$ a v.s., we define

$$\mathrm{Span}(v_1, \dots, v_n) := \langle v_1, \dots, v_n \rangle$$
$$:= \{ c_1 v_1 + \dots + c_n v_n \mid c_i \in \mathbb{K} \} \subseteq V$$

Check $W := \langle v_1, \dots, v_n \rangle$ is a vector space:

If $v = \sum_i c_i v_i$, $w = \sum_i d_i v_i \in W$

$v + w := \sum_i (c_i + d_i) v_i \in W$

$(= v + w \text{ inside } V)$

$a.v := \sum_i (a c_i) v_i$ $(= av \text{ in } V)$
$\in W$

ie inherits the v.s. structure from V

$0 \in \langle v_1 \dots v_n \rangle$ as $\sum_i 0 v_i = 0$ is the $0 \in V$

<u>Definition 1.16</u> If $(v_1, \dots v_n)$ is a basis of V (ie. l.i. and spanning) then we say that V is n-dimensional. A v.s. V is called finite-dimensional (f.d.) if $\exists$ a basis $(v_1 \dots v_n)$ for some finite n. (else it's called ∞-dimensional)

(We'll have to prove that this concept is well-defined!)

<u>Example</u> $\mathbb{R}[x]$ has no basis $(v_1, \dots, v_n)$ for any n but it does have a basis

$$1, x, x^2, \dots \quad \text{(all powers)}$$

e.g. l.i as $\underbrace{c_0 1 + c_1 x + \dots c_m x^m}_{\text{any } m, \text{ any } c_i \in \mathbb{R}} = 0 \Rightarrow c_i = 0$

any any $f \in \mathbb{R}[x]$ is $f = f_0 1 + f_1 x + \dots f_m x^m$ some m

so spans

Other examples like $C^\infty(\mathbb{R})$ have no basis at all by our definition.

<u>Lemma 1.12 (Excision lemma)</u> suppose $(v_1, \dots, v_m)$ is l.d. then $\exists\, i$ such that

$$v_i \in \langle v_1, \dots v_{i-1} \rangle \quad (:= \{c_1 v_1 + \dots + c_{i-1} v_{i-1} \mid c_i \in \mathbb{K}\})$$

and

$$\langle v_1, \dots v_{i-1}, v_{i+1}, \dots v_m \rangle = \langle v_1, \dots v_m \rangle$$

leave out v_i

<u>proof</u> We've know $\exists\, c_i$ not all zero s.t.

$$c_1 v_1 + \dots + c_m v_m = 0 \quad \text{(as } (v_1, \dots v_m) \text{ l.d.)}$$

Let i be the largest s.t. $c_i \neq 0$

(so all $c_j = 0$ $j > i$)

so $$c_i v_i = - \sum_{j=1}^{i-1} c_j v_j \quad \text{since}$$

and $c_i \neq 0$ so $$v_i = - \sum_{j=1}^{i-1} \left(\frac{c_j}{c_i}\right) v_j \quad (*)$$

$\in \langle v_1 \dots v_{i-1} \rangle$

If $v \in \langle v_1, \dots v_m \rangle$

ie $v = d_1 v_1 + \dots + d_i v_i + \dots + d_m v_m$

replace v_i using $(*)$ so $v \in \langle v_1 \dots v_{i-1}, v_{i+1}, \dots v_m \rangle$

so $\langle v_1 \dots v_m \rangle \subseteq \langle v_1 \dots v_{i-1}, v_{i+1} \dots v_m \rangle$

but clearly $\langle v_1 \dots v_{i-1}, v_{i+1} \dots v_m \rangle \subseteq \langle v_1 \dots v_m \rangle$

(since $v = \sum_{j \neq i} c_j v_j = \sum_j c_j v_j$ with $c_i = 0$)

$\therefore$ $\langle v_1, \dots v_m \rangle = \langle v_1 \dots v_{i-1}, v_{i+1} \dots v_m \rangle$

QED
(quod erat demonstrandum)

understanding the proof ① forces you to learn the definitions ② forces you to be logical.

<u>Lemma 1.13</u> $|$any l.i. list$| \leq |$any spanning list$|$

($|\ | =$ size, cardinality) in a v.s. V.

<u>proof</u> let $(v_1 \dots v_n)$ be a l.i list, let $(w_1 \dots w_m)$ be a spanning list. We want to show $n \leq m$.

Manipulate the list $(w_1, \dots w_m)$ as follows

$(w_1 \dots w_m) \xrightarrow{\text{add } v_1} (v_1, w_1 \dots w_m)$ l.d. as $(w_1, \dots w_m)$
still spanning

Spanning

$v_1 = \sum c_i w_i$

not all the $c_i = 0$

as $v_1 \neq 0$

as $(v_1 \dots v_n)$ l.i.

$\Rightarrow$ $(v_1, w_1, \dots w_{i-1}, w_{i+1}, \dots w_m)$ is a new spanning list.

lemma 1.12 (taking largest $c_i \neq 0$)

$\downarrow$

Repeat: add v_2

$(v_1, v_2, w_2, \dots w_m)$, l.d. so

$$\sum_{i=1}^{2} c_i v_i = \sum_{i=2}^{m} d_i w_i$$

can't all be zero as $\sum_{i=1}^{2} c_i v_i \neq$ as l.i.

$\swarrow$ lemma 1.12 (largest $d_i \neq 0$)

$(v_1, v_2, w_2 \dots w_{i-1}, w_{i+1}, \dots w_m)$

new spanning list.

$\downarrow$ renumber

$(v_1, v_2, w_3, \dots w_m)$ new spanning list

new spanning list

keep repeating

$\dots \longrightarrow$ Case 1 $n > m$

arrive at $(v_1, v_2, \dots, v_m)$ spanning list.

$\Rightarrow$ $v_n \in \langle v_1 \dots v_m \rangle$

$\#$ as $(v_1 \dots v_n)$ l.i.

(contradiction)

$v_n = \sum_{i=1}^{m} c_i v_i$ not allowed

$\swarrow$

case 2 $n \leq m$

ends with

$(v_1, v_2 .. v_n, w_{n+1} .. w_m)$

a spanning list with w_i permuted

Q.E.D.

<u>Theorem 1.15</u> Let V be a f.d. vector space ("finite dimensional" defined earlier).

(a) Any two bases of V have the same cardinality (called the dimension, $\dim(V)$)

(b) Any spanning list can be shortened to a basis.

(c) any l.i. list can be extended to a basis.

proof (a) Let B_1, B_2 be bases of V

$|B_1| \leq |B_2|$ by B_1 l.i., B_2 spanning } use lemma 1.13

but $|B_2| \leq |B_1|$ by B_2 l.i., B_1 spanning } use lemma 1.13

(b) let $(v_1, \dots v_m)$ be a spanning list.

if l.i. we are done.

if l.d. use lemma 1.12 to remove one v_i and renumber so $(v_1, \dots v_{m-1})$ still spanning

keep repeating until we are in the l.i. case. hence a basis.

(c) Let $(v_1 \dots v_m)$ be l.i.

if spanning, we are done

If not $\exists\, v_{m+1} \in V$ s.t. $v_{m+1} \notin \langle v_1, \dots, v_m \rangle$

then $(v_1, v_2 \dots v_m, v_{m+1})$ is l.i.

[if not ie if

$c_1 v_1 + \dots + c_{m+1} v_{m+1} = 0$ with $c_{m+1} \neq 0$

as $v_1 \dots v_m$ l.i.

$\Rightarrow v_{m+1} = -\dfrac{1}{c_{m+1}} \sum_{i=1}^{m} c_i v_i$

$\in \langle v_1, \dots v_m \rangle$]

keep repeating till we are in the spanning case, hence a basis.

Q.E.D.

<u>propisition 1.18</u> let V be a f.d. v.s. and $\mathcal{B}$ a list of vectors in V

TFAE (the following are equivalent)

(a) $\mathcal{B}$ a basis

(b) $\mathcal{B}$ a maximal l.i. list.

(c) $\mathcal{B}$ a minimal spanning list

(d) $\mathcal{B}$ a l.i. list of size $\dim(V)$

(e) $\mathcal{B}$ a spanning list of size $\dim(V)$

<u>proof</u> This follow logically from results above (see printed note).

<u>Learning support hour</u> ~~Wed 11-12 in MB 403~~

Mondays 1-2pm