MTH6140 Linear Algebra II

Coursework 1 Solutions

1. (a) Let $\mathbf{k}$ be a field. Show that, for $a, b \in \mathbf{k}$, $ab = 0 \iff a = 0$ or $b = 0$.

(b) Let V be a vector space over the field $\mathbf{k}$. Prove that, for $\alpha \in \mathbf{k}$ and $\mathbf{v} \in V$, $\alpha\mathbf{v} = \mathbf{0} \iff \alpha = 0$ or $\mathbf{v} = \mathbf{0}$.

Solution. (a) $\Leftarrow$. Without loss of generality, suppose that $a = 0$. Then

$$ab = 0 \cdot b = (0 + 0) \cdot b = 0 \cdot b + 0 \cdot b,$$

which implies $0 \cdot b = 0$ by adding $-0 \cdot b$ to both sides.

$\Rightarrow$. Suppose that $ab = 0$, and that $a \neq 0$. Then a^{-1} exists, and we have

$$0 = a^{-1}(ab) = (a^{-1}a)b = 1 \cdot b = b.$$

(b) If $\alpha = 0$, then

$$0 \cdot \mathbf{v} = (0 + 0) \cdot \mathbf{v} = 0 \cdot \mathbf{v} + 0 \cdot \mathbf{v},$$

whence $0 \cdot \mathbf{v} = \mathbf{0}$. Also, if $\mathbf{v} = \mathbf{0}$, then

$$\alpha \cdot \mathbf{v} = \alpha \cdot \mathbf{0} = \alpha \cdot (\mathbf{0} + \mathbf{0}) = \alpha \cdot \mathbf{0} + \alpha \cdot \mathbf{0},$$

thus $\alpha \cdot \mathbf{0} = \mathbf{0}$, as claimed. Conversely, suppose that $\alpha\mathbf{v} = \mathbf{0}$ and that $\alpha \neq 0$. Then α^{-1} exists, and we have

$$\mathbf{0} = \alpha^{-1}(\alpha\mathbf{v}) = (\alpha^{-1}\alpha)\mathbf{v} = 1 \cdot \mathbf{v} = \mathbf{v},$$

completing the proof.

2. Let V_2 be the vector space consisting of all polynomials of degree ≤ 2 in one variable x over the field $\mathbb{R}$. For each of the following lists of polynomials say, with justification, whether the list is *linearly independent*, and whether it is *spanning*. What is the dimension of the vector space V_2?

(i) $x - 1$, $x^2 - x$, $x^2 - 1$;

(ii) $x - 1$, $x^2 - x$;

(iii) 1, $x - 1$, $x^2 - x$, x^2;

(iv) 1, $x - 1$, $x^2 - x$.

Solution. (i) Since

$$x^2 - 1 = (x^2 - x) + (x - 1),$$

the list is *linearly dependent*. Also, since $\dim(V_2) = 3$ (the polynomials $1, x, x^2$ forming an obvious basis), this list is *not spanning*.

(ii) This list is *linearly independent*, since

$$\mathbf{0} = \alpha(x-1) + \beta(x^2 - x) = -\alpha + (\alpha - \beta)x + \beta x^2$$

implies $\alpha = \beta = 0$. Again, since $\dim(V_2) = 3$, it is *not spanning*.

(iii) Since $\dim(V_2) = 3$, this list is *linearly dependent*. More specifically, one notes, for instance, that

$$x^2 = 1 + (x-1) + (x^2 - x).$$

However, the sublist consisting of $1, x-1, x^2$ is spanning since $1 + (x-1) = x$; consequently, the original list is *spanning* as well.

(iv) This list is *linearly independent* as well as *spanning*, thus a basis of V_2. Indeed, we have $1 + (x-1) = x$, as well as

$$x^2 = 1 + (x-1) + (x^2 - x),$$

which shows that the list (iv) is *spanning*. Also, since the number of vectors in our list equals $\dim(V_2) = 3$, the list must be *linearly independent*.

3. Suppose that $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $B' = (\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3)$ are two bases of the 3-dimensional vector space V over $\mathbb{R}$, and that B and B' are related as follows:

$$\mathbf{v}'_1 = \mathbf{v}_1 + 2\mathbf{v}_2, \quad \mathbf{v}'_2 = 3\mathbf{v}_1 + \mathbf{v}_3, \quad \text{and} \quad \mathbf{v}'_3 = 2\mathbf{v}_1 + \mathbf{v}_2.$$

(i) Write down the transition matrix $P_{B,B'}$.

(ii) If $[u]_{B'} = [1,2,3]^t$, what is $[u]_B$?

(iii) From Part (i), compute the transition matrix $P_{B',B}$.

(iv) If $[w]_B = [1,2,3]^t$, what is $[w]_{B'}$?

Solution. (i) By definition of the transition matrix, we have

$$P_{B,B'} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(ii) We have

$$[u]_B = P_{B,B'}[u]_{B'} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 13 \\ 5 \\ 2 \end{bmatrix}.$$

(iii) By Part (i),

$$P_{B',B} = P_{B,B'}^{-1} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & 1 \\ \frac{2}{3} & -\frac{1}{3} & -2 \end{bmatrix}.$$

(iv) By Part (iii),

$$[w]_{B'} = P_{B',B}[w]_B = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & 1 \\ \frac{2}{3} & -\frac{1}{3} & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -6 \end{bmatrix}.$$

4. Let A, B be subspaces of the vector space $V = \mathbb{R}^3$ given by

$$A := \{(x, y, z) : x, y, z \in \mathbb{R} \text{ and } x + y + z = 0\}$$

and

$$B := \{(x, x, z) : x, z \in \mathbb{R}\}.$$

Show that $V = A + B$. Is this sum of subspaces of V direct?

Proof (i) (1,1,0) is in B but not in A and (1,0,-1) is in A but not in B, so neither subspace is a subset of the other. It follows that A+B has bigger dimension than A or B (which are each two dimensional) so at least 3, which is the dimension of V. So V=A+B.

(ii) No, since (1,1,-2) is in both A, B and is not zero. Or, the direct sum of A, B would be 4-dimensional and V is only 3-dimensional.