MTH6140 Linear Algebra II

Coursework 1

1. (a) Let $\mathbf{k}$ be a field. Show that, for $a, b \in \mathbf{k}$, $ab = 0 \iff a = 0$ or $b = 0$.

(b) Let V be a vector space over the field $\mathbf{k}$. Prove that, for $\alpha \in \mathbf{k}$ and $\mathbf{v} \in V$, $\alpha\mathbf{v} = \mathbf{0} \iff \alpha = 0$ or $\mathbf{v} = \mathbf{0}$.

2. Let V_2 be the vector space consisting of all polynomials of degree ≤ 2 in one variable x over the field $\mathbb{R}$. For each of the following lists of polynomials say, with justification, whether the list is *linearly independent*, and whether it is *spanning*. What is the dimension of the vector space V_2?

(i) $x - 1$, $x^2 - x$, $x^2 - 1$;

(ii) $x - 1$, $x^2 - x$;

(iii) 1, $x - 1$, $x^2 - x$, x^2;

(iv) 1, $x - 1$, $x^2 - x$.

3. Suppose that $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $B' = (\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3)$ are two bases of the 3-dimensional vector space V over $\mathbb{R}$, and that B and B' are related as follows:

$$\mathbf{v}'_1 = \mathbf{v}_1 + 2\mathbf{v}_2, \quad \mathbf{v}'_2 = 3\mathbf{v}_1 + \mathbf{v}_3, \quad \text{and} \quad \mathbf{v}'_3 = 2\mathbf{v}_1 + \mathbf{v}_2.$$

(i) Write down the transition matrix $P_{B,B'}$.

(ii) If $[u]_{B'} = [1, 2, 3]^t$, what is $[u]_B$?

(iii) From Part (i), compute the transition matrix $P_{B',B}$.

(iv) If $[w]_B = [1, 2, 3]^t$, what is $[w]_{B'}$?

4. Let A, B be subspaces of the vector space $V = \mathbb{R}^3$ given by

$$A := \{(x, y, z) : x, y, z \in \mathbb{R} \text{ and } x + y + z = 0\}$$

and

$$B := \{(x, x, z) : x, z \in \mathbb{R}\}.$$

Show that $V = A + B$. Is this sum of subspaces of V direct?