Queen Mary University of London

Main Examination period 2019

MTH6140: Linear Algebra II (Solutions)

Duration: 2 hours

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Question 1.

(a) The list $(v_1, \ldots, v_n)$ is *linearly independent* if, for all $c_1, \ldots, c_n \in \mathbb{K}$, it is the case that $c_1v_1 + \cdots c_nv_n = \mathbf{0}$ implies $c_1 = \cdots = c_n = 0$; It is *spanning* if every vector $v \in V$ may be expressed in the form $v = c_1v_1 + \cdots c_nv_n$ for some $c_1, \ldots, c_n \in \mathbb{K}$.

(b) (i) Yes, (ii) yes, (iii) no, and (iv) yes.

(c) The dimension of V is the number of vectors in any basis of V. (All bases have the same cardinality.)

(d) The sum of U and W is defined by $U + W = \{u + w : u \in U \text{ and } w \in W\}$.

(e) $U + W \supseteq \{u + \mathbf{0} : u \in U\} = U$ an so $\dim(U + W) \geq \dim(U)$. Similarly, $\dim(U + W) \geq \dim(W)$.

(f) The subspaces are not equal, so one contains a vector that is not in the other; say, $v \in W \setminus U$. Take a basis $u_1, \ldots, u_{n-1}$ of U. Since $v \notin \langle u_1, \ldots, u_{n-1} \rangle$ we see that $u_1, \ldots, u_{n-1}, v$ is a linearly independent list of vectors. Also, since $\langle u_1, \ldots, u_{n-1}, v \rangle \subseteq U + W$, we see that $\dim(U + W) \geq n$. On the other hand, $U + W \subseteq V$, from which $\dim(U + W) \leq n$.

Finally, $\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U + W) =$
$(n - 1) + (n - 1) - n = n - 2$.

Notes. Parts (a,c,d) are basic definitions. Part (b) is an easy test of bookwork/understanding. Parts (e) and (f) are unseen, and the latter probably won't be completed by many students.

Question 2.

(a) The three types of elementary row operations are: (I) add a multiple of row j to row i, for some $i \neq j$; (II) multiply row i by a non zero scalar, and (III) interchange row i and row j, for some $i \neq j$.

(b) Elementary row operations preserve the row space of a matrix and hence the row rank. They preserve the column rank of a matrix but not the column space.

(c) The $m \times n$ matrix $A = (a_{ij})$ is in canonical form if $a_{11} = a_{22} = \cdots = a_{rr} = 1$ for some $r \leq \min\{n, m\}$ and $a_{ij} = 0$ otherwise.

(d) We already have 1 in the top left corner, so proceed immediately to zero column 1 and and row 1 using type I operations:

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 0 & 3 \\ 0 & 2 & -4 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 \\ 0 & 2 & -4 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & 2 & -4 & -6 \end{bmatrix}.$$

Now repeat on the bottom right 2×3 matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & 2 & -4 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(e) Given any matrix A, reduce it to canonical form as in the previous part. Suppose the resulting matrix, D, has r non-zero entries. It is clear that the non-zero rows of D are linearly independent, so the row-rank of D is r; by a similar argument, the column-rank of D is also r. But in reducing A to D we didn't change the row and column rank (by part (b)). So the row- and column-rank of A are both r.

(f) In the example in part (d), $r = 2$, so the rank of A is two.

Notes. Parts (a–c) are bookwork; (d) is a straightforward application of a method from the course; (e) is bookwork plus an easy deduction.

Question 3.

(a) $\mathrm{Ker}(\alpha) = \{u \in U : \alpha(u) = \mathbf{0}\}$ and $\mathrm{Im}(\alpha) = \{\alpha(u) : u \in U\}$.

(b) First, all the vectors $\alpha(u_{k+1}), \ldots, \alpha(u_n)$ are clearly in $\mathrm{Im}(\alpha)$. We just need to show that any vector $v \in \mathrm{Im}(\alpha)$ can be expressed as a linear combination of $\alpha(u_{k+1}), \ldots, \alpha(u_n)$. Since $v \in \mathrm{Im}(\alpha)$ there is a vector $u \in U$ such that $v = \alpha(u)$. Also, since $u_1, \ldots, u_n$ is a basis of U, we can write u as $c_1 u_1 + \cdots + c_n u_n$ for some scalars $c_1, \ldots, c_n$. Then

$$\begin{aligned} v = \alpha(u) &= \alpha(c_1 u_1 + \cdots + c_n u_n) \\ &= c_1 \alpha(u_1) + \cdots + c_n \alpha(u_n) \\ &= c_{k+1} \alpha(u_{k+1}) + \cdots + c_n \alpha(u_n), \end{aligned}$$

where we have used the fact that $u_1, \ldots, u_k \in \mathrm{Ker}(\alpha)$.

(c) $\dim(\mathrm{Ker}(\alpha)) + \dim(\mathrm{Im}(\alpha)) = \dim(U)$.

(d) Suppose $u \in \mathrm{Ker}(\alpha)$. Then $\alpha(u) = \mathbf{0}$ and hence $\beta\alpha(u) = \beta(\alpha(u)) = \beta(\mathbf{0}) = \mathbf{0}$. It follows that $u \in \mathrm{Ker}(\beta\alpha)$.

(e) Since $\mathrm{Ker}(\beta\alpha)) \supseteq \mathrm{Ker}(\alpha)$, we have

$$\begin{aligned} \dim(\mathrm{Ker}(\beta\alpha)) &\geq \dim(\mathrm{Ker}(\alpha)) \\ &= \dim(U) - \dim(\mathrm{Im}(\alpha)) \\ &\geq \dim(U) - \dim(V) = 5 - 2 = 3. \end{aligned}$$

Notes. Parts (a,c) are basic definitions/results; part (b) is bookwork; parts (d) and (e) are similar to coursework.

Question 4.

(a) A vector $v \in V$ is an eigenvector of α with eigenvalue $\lambda \in \mathbb{K}$ if $v \neq \mathbf{0}$ and $\alpha(v) = \lambda v$.

(b) α is diagonalisable if V has a basis consisting entirely of eigenectors of α.

(c) Let $\beta = (\alpha - \lambda_1 I) \cdots (\alpha - \lambda_{r-1} I)(\alpha - \lambda_r I)$. Suppose v is any eigenvector of α, and let λ_i be the corresponding eigenvector. Then

$$\begin{aligned}\beta(v) &= (\alpha - \lambda_1 I) \cdots (\alpha - \lambda_{r-1} I)(\alpha - \lambda_r I)v \\ &= (\alpha - \lambda_1 I) \cdots (\alpha - \lambda_{r-1} I)(\lambda_i - \lambda_r)v \\ &= (\lambda_i - \lambda_1) \cdots (\lambda_i - \lambda_{r-1})(\lambda_i - \lambda_r)v \\ &= 0v = \mathbf{0},\end{aligned}$$

since one of the factors in the product is zero. Since $\beta(v) = \mathbf{0}$ for any eigenvector v, and the eigenvectors of α form a basis for V, it follows that β is the zero map.

(d) The minimal polynomial of α is the monic polynomial $m_\alpha(x)$ of smallest degree such that $m_\alpha(\alpha) = 0$.

(e) (i) $(x-1)(x-2)$, $(x-1)(x-2)^2$ and $(x-1)(x-2)^3$.

(ii) $(x-2)(x-3)^3$, $(x-2)^2(x-3)^2$ and $(x-2)^3(x-3)$.

(iii) The only possibility for the minimal polynomial is $m_\alpha(x) = x - 1$. So α is the identity map.

(iv) The only possibility for the characteristic polynomial is $p_\alpha(x) = (x-2)^2(x^2+1)$.

Notes. Parts (a,b,d) are basic definitions. Part (c) comes from part of a proof in the module. Part (e) contains some not-too-hard deductions from facts about $p_\alpha(x)$ and $m_\alpha(x)$.

Question 5.

(a) $(v_1, \ldots, v_n)$ is orthonormal if $v_i \cdot v_j = \delta_{ij}$ for all i, j.

(b) It is enough to show that U is nonempty, and closed under vector addition and scalar multiplication. Suppose $u, u' \in U$ and $c \in \mathbb{R}$. Then,we have that $(u + u') \cdot v = u \cdot v + u' \cdot v = 0 + 0 = 0$, and hence $u + u' \in U$. Similarly, $(cu) \cdot v = c(u \cdot v) = c0 = 0$ and hence $cu \in U$.

(c) The adjoint $\alpha^* : V \to V$ of α is the unique linear map satisfying $v \cdot \alpha^*(w) = \alpha(v) \cdot w$ for all $v, w \in V$. The map α is self-adjoint if $\alpha^* = \alpha$.

(d) Suppose α is a self-adjoint linear map on V. Then there is an orthonormal basis for V consisting of eigenvalues of α.

(e) Suppose $u \in U$. Then, by definition, $u \cdot v = 0$. Then

$$\alpha(u) \cdot v = u \cdot \alpha(v) = u \cdot (\lambda v) = \lambda(u \cdot v) = 0.$$

It follows that $\alpha(u) \in U$.

Notes. Parts (a) and (c) are basic definitions. That the orthogonal complement of a subspace W is a subspace is bookwork; part (b) is just the special case $\dim(W) = 1$. Part (d) is bookwork. Part (e) is bookwork (provided the student recognises it as part of the proof of the Spectral Theorem).

End of Paper.