Main Examination period 2019

MTH6140, MTH6140P: Linear Algebra II

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any unauthorised notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: M. Jerrum, T. W. Müller

Question 1. [20 marks] In this question, V is a finite-dimensional vector space over a field $\mathbb{K}$.

(a) Define what it means for a list $(v_1, v_2, \ldots, v_n)$ of vectors in V to be **linearly independent** and what it means for it to be **spanning**. **[4]**

(b) Suppose $(v_1, v_2, ..., v_n)$ is a list of vectors in V. For each of the following statements, say whether the statement is equivalent to $(v_1, v_2, ..., v_n)$ being a basis for V. (No justification is required.) **[4]**

- (i) $(v_1, v_2, \ldots, v_n)$ is a maximal linearly independent list of vectors;
- (ii) $(v_1, v_2, \ldots, v_n)$ is linearly independent and spanning;
- (iii) $(v_1, v_2, \ldots, v_n)$ is a maximal spanning list of vectors;
- (iv) every vector in V is uniquely expressible as a linear combination of vectors in $(v_1, v_2, \ldots, v_n)$.

(c) Define the **dimension** of V. **[3]**

(d) Suppose U and W are subspaces of V. Define the **sum** $U + W$ of U and W. **[2]**

(e) Show that $\dim(U + W) \geq \max\{\dim(U), \dim(W)\}$. **[3]**

(f) Suppose V has dimension $n \geq 2$, and let U and W be distinct subspaces of V with $\dim(U) = \dim(W) = n - 1$. What is the dimension of the subspace $U + W$? What is the dimension of the subspace $U \cap W$? In both cases, justify your answer. You may use standard results about dimensions of subspaces. **[4]**

Question 2. [20 marks]
This question concerns not-necessarily square matrices over a field $\mathbb{K}$.

(a) Describe the three types of elementary row operations on matrices. **[3]**

(b) Which of the following statements about elementary row operations are true and which are false? (No explanations are required.) **[4]**

Elementary row operations:

(i) preserve the row rank of a matrix;

(ii) preserve the row space of a matrix;

(iii) preserve the column rank of a matrix;

(iv) preserve the column space of a matrix.

(c) Describe the **canonical form for equivalence** of matrices. **[3]**

(d) Reduce the following matrix over $\mathbb{R}$ to the canonical form for equivalence, using elementary row and column operations: **[5]**

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 0 & 3 \\ 0 & 2 & -4 & -6 \end{bmatrix}.$$

(e) Every matrix can be reduced to canonical form for equivalence using elementary row and column operations. Explain how this fact implies that the row rank of any matrix is equal to the column rank. **[3]**

(f) What is the rank of the matrix A in part (d)? **[2]**

Question 3. [20 marks] Suppose that U, V and W are finite dimensional vector spaces over a field $\mathbb{K}$, and that $\alpha : U \to V$ and $\beta : V \to W$ are linear maps.

(a) Define the **kernel** $\mathrm{Ker}(\alpha)$ and **image** $\mathrm{Im}(\alpha)$ of α. **[4]**

(b) Take a basis $(u_1, \ldots, u_k)$ for $\mathrm{Ker}(\alpha)$, and extend it to a basis $(u_1, \ldots, u_n)$ for U. Prove that the vectors $\alpha(u_{k+1}), \ldots, \alpha(u_n)$ span $\mathrm{Im}(\alpha)$. **[6]**

(c) State, without proof, a relation between $\dim(U)$, $\dim(\mathrm{Ker}(\alpha))$ and $\dim(\mathrm{Im}(\alpha))$. **[3]**

(d) Show that $\mathrm{Ker}(\alpha)$ is a subset (and hence a subspace) of $\mathrm{Ker}(\beta\alpha)$. **[4]**

(e) Suppose that $\dim(U) = 5$, $\dim(V) = 2$ and $\dim(W) = 4$. Show that the dimension of $\mathrm{Ker}(\beta\alpha)$ is at least 3. **[3]**

Question 4. [20 marks] In this question, V is a finite dimensional vector space and $\alpha : V \to V$ is a linear map.

(a) Define what it means for $v \in V$ to be an **eigenvector** of α with **eigenvalue** λ. **[2]**

(b) Define what it means for α to be **diagonalisable**. **[3]**

(c) Suppose that α is diagonalisable and that $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the distinct eigenvalues of α. Prove that

$$(\alpha - \lambda_1 I)(\alpha - \lambda_2 I) \cdots (\alpha - \lambda_r I) = 0,$$

where I is the identity map and 0 is the zero map (taking every vector to the zero vector $\mathbf{0}$). **[4]**

(d) Define the **minimal polynomial** $m_\alpha(x)$ of α. (You are not required to explain why the polynomial exists and is unique.) **[3]**

Now assume that $V = \mathbb{R}^4$, and denote by $p_\alpha(x)$ the characteristic polynomial of α.

(e) (i) Suppose $p_\alpha(x) = (x-1)(x-2)^3$. List all the possibilities for $m_\alpha(x)$.

(ii) Suppose $m_\alpha(x) = (x-2)(x-3)$. List all the possibilities for $p_\alpha(x)$.

(iii) Suppose $p_\alpha(x) = (x-1)^4$ and that α is diagonalisable. What is α?

(iv) Suppose $m_\alpha(x) = (x-2)(x^2+1)$. What is $p_\alpha(x)$? **[8]**

Question 5. [20 marks] In this question, V is a real inner product space, and $\alpha : V \to V$ is a linear map on V.

(a) Define what it means for a basis $(v_1, v_2, \ldots, v_n)$ of V to be **orthonormal**. **[3]**

(b) Let $v \in V$ be a non-zero vector, and define $U = \{u \in V : u \cdot v = 0\}$. Show that U is a subspace of V. **[5]**

(c) Define the **adjoint** α^* of α. (You are not required to prove existence and uniqueness.) What does it mean for α to be **self-adjoint**? **[4]**

From now on, assume α is self-adjoint.

(d) State a theorem (a version of the Spectral Theorem) relating the self-adjoint linear map α to a basis of V. **[3]**

(e) Suppose v is an eigenvector of α with eigenvalue λ. (We know that α has at least one such eigenvector.) Define the subspace U as in part (b). Show that $\alpha(u) \in U$ for every $u \in U$. **[5]**

End of Paper.