Quiz 4 Solutions

1. What is a Type I error for a hypothesis test?

   Rejecting the null hypothesis, when the null hypothesis is true.

2. What is a Type II error for a hypothesis test?

   Failing to reject the null hypothesis, when the null hypothesis is false.

3. Suppose you have a hypothesis test which never rejects the null hypothesis, no matter the value of the test statistic. Name a potential problem with such a test.

   Using this test would lead to making an excessive number of Type II errors.

4. The smaller the observed p-value, the more evidence you have against the null hypothesis.

   True.

5. Consider a random sample of size n from a normal distribution with unknown mean μ and unknown variance σ^2. Suppose the sample mean is $\bar{X}$ and the sample variance is S^2. We would like to test $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$.

   What test statistic should you should use to test H_0 versus H_1?

   The test statistic you would use is

$$T = \frac{\bar{X} - \mu_0}{S\sqrt{n}} \sim t_{n-1}$$

6. Now suppose that $n = 16$, the observed sample mean $\bar{x}$ is 8.9, the observed sample variance s^2 is 25 and $\mu_0 = 10.5$.

   What is the (appropriate) observed test statistic for testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$? Give answer to three decimal places.

   The observed value of the test statistic is

$$t_{obs} = \frac{8.9 - 10.5}{5/4} = -1.280$$

7. What is the p-value of the (appropriate) observed test statistic for testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ you just computed? Give answer to three decimal places.

   The p-value is given by $2 \times P(T < t_{obs}) = 2 \times P(T < -1.28) = 0.220$ *where* $T \sim t_{15}$. *You can use the R command* `pt(-1.28, 15)` *to compute* $P(T < -1.28)$.

8. The p-value indicates that there is strong evidence against H_0.

   False. The p-value indicates there is no evidence against H_0.

9. Suppose we wanted to test $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ at the 5% significance level i.e. $\alpha = 0.05$. Given our observed data, would we reject H_0?

   We can simply note that the p-value is above 0.05 *and therefore we would not reject* H_0 *at the 5% level of significance.*

10. Suppose we next want to compute a 95% confidence interval for μ.

    What is the lower endpoint of this 95% confidence interval? Give answer to three decimal places.

    Note that $\alpha = 0.05$. *The lower endpoint of the 95% confidence interval for* μ *is*

    $$\bar{x} - t_{n-1}(1 - \alpha/2)\frac{s}{\sqrt{n}} = 8.9 - 2.131 \times (5/4) = 6.236$$

    You may evaluate $t_{n-1}(1 - \alpha/2) = t_{15}(0.975) = 2.131$ *using R command* `qt(15, 0.975)`.

11. What is the upper endpoint of this 95% confidence interval? Give answer to three decimal places.

    Similar to above, the upper endpoint of the 95% confidence interval is

    $$\bar{x} + t_{n-1}(1 - \alpha/2)\frac{s}{\sqrt{n}} = 8.9 + 2.131 \times (5/4) = 11.564$$

12. Suppose we now want to test $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 \neq \sigma_0^2$.

    Which of these test statistics should we use?

    The test statistic we should use is $W = \frac{(n-1)S^2}{\sigma_0^2}$.

13. Let $\sigma_0^2 = 36$. What is the (appropriate) observed test statistic? Give answer to three decimal places.

    The value of the observed test statistic is $w_{obs} = 15 \times 25/36 = 10.417$.

14. What is the p-value of the (appropriate) observed test statistic for testing $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 \neq \sigma_0^2$ you just computed? Give answer to three decimal places.

    The p-value of the observed test statistic is $2 \times \min(P(W < w_{obs}), P(W > w_{obs})) = \min(2 \times P(W < w_{obs}), 2 \times P(W > w_{obs})) = \min(0.415, 0.793) = 0.415$.

    We can use the R command `pchisq(10.417, 15)` *to evaluate* $P(W < w_{obs})$ *and* `1 - pchisq(10.417, 15)` *to evaluate* $P(W > w_{obs})$.

15. The p-value indicates that there is strong evidence against H_0.

False. The p-value is large and does not indicate that there is evidence against H_0.

16. Suppose we wanted to test $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 \neq \sigma_0^2$ at the 5% significance level i.e. $\alpha = 0.05$. Given our observed data, would we reject H_0?

The p-value is above 0.05 *hence we would not reject H_0.*

17. Finally, we will compute a 99% confidence interval for σ^2.

What is the lower endpoint of this 99% confidence interval? Give answer to three decimal places.

Note that $\alpha = 0.01$. The lower endpoint of the confidence interval is given by

$$\frac{(n-1)s^2}{\chi^2_{n-1}(1-\alpha/2)} = \frac{15 \times 25}{\chi^2_{15}(0.995)} = 11.432$$

You may use the R command `qchisq(0.995, 15)` *to compute* $\chi^2_{15}(0.995)$.

18. What is the upper endpoint of this 99% confidence interval? Give answer to three decimal places.

The upper endpoint of the confidence interval is given by

$$\frac{(n-1)s^2}{\chi^2_{n-1}(\alpha/2)} = \frac{15 \times 25}{\chi^2_{15}(0.005)} = 81.506$$

You may use the R command `qchisq(0.005, 15)` *to compute* $\chi^2_{15}(0.005)$.

19. Suppose we observe the following random sample of size $n = 117$, sampled from a discrete distribution.

Value	0	1	2	3	4	5	6
Count	32	36	21	18	6	2	2

We would like to test whether this data comes from a Poisson(λ) distribution.

The first step is to estimate the mean λ. Compute an estimate of λ using the provided data. Give answer to three decimal places.

An estimate of λ is the sample mean of the counts, equal to

$$\frac{0 \times 32 + 1 \times 36 + 2 \times 21 + 3 \times 18 + 4 \times 6 + 5 \times 2 + 6 \times 2}{117} = 1.521.$$

20. We next estimate the expected counts for each observed data value, assuming the data comes from a Poisson distribution with mean λ, where λ is the mean you just computed.

Denote by E_k with $k = 0, \ldots, 6$ the expected number of k's you expect to observe.

What is E_0? Give answer to three decimal places.

$E_0 = 117 \times e^{-\lambda} \frac{\lambda^0}{0!} = 25.564.$

21. What is E_1? Give answer to three decimal places.

    $E_1 = 117 \times e^{-\lambda}\frac{\lambda^1}{1!} = 38.882.$

22. What is E_2? Give answer to three decimal places.

    $E_2 = 117 \times e^{-\lambda}\frac{\lambda^2}{2!} = 29.570.$

23. What is E_3? Give answer to three decimal places.

    $E_3 = 117 \times e^{-\lambda}\frac{\lambda^3}{3!} = 14.992.$

24. What is E_4? Give answer to three decimal places.

    $E_4 = 117 \times e^{-\lambda}\frac{\lambda^4}{4!} = 5.701.$

25. What is E_5? Give answer to three decimal places.

    $E_5 = 117 \times e^{-\lambda}\frac{\lambda^4}{4!} = 1.734.$

26. What is E_6? Give answer to three decimal places.

    $E_6 = 117 \times (1 - P(X \leq 5)) = 0.557$ *where* $X \sim Poisson(\lambda)$. *You can compute* $P(X \leq 5)$ *in R using the command* `1 - ppois(5, 1.521))`. *Note the last bin must contain all values that are greater than, or equal to* 6.

27. Next, take the values of E_k you just computed, and calculate the observed value of the appropriate Goodness of Fit test statistic, used to test the null hypothesis that the observed data is Poisson distributed.

    Give answer to three decimal places.

    Note that since the E_k *for the last three bins are less than* 5, *we merge them into a single one to correspond to counts that are greater than or equal to* 4. *This leaves a total of* 5 *bins. Evaluating the observed value of the* X^2 *test statistic,*

    $$X^2 = \sum_{k=0}^{4} \frac{(E_i - O_i)^2}{E_i}$$

    we get $X^2 = 5.426$.

28. What is the distribution of this test statistic, assuming H_0 is true?

    Under the null hypothesis, the test statistic has a χ^2_3 *distribution. This is because after merging, we have* 5 *bins* $0, 1, 2, 3, 4+$ *and one estimated parameter* λ.

29. Compute the p-value of the observed test statistic you just computed. Give answer to three decimal places.

    The p-value is given by $P(X^2 > 5.426) = 0.143$. *This can be evaluated in R using the command* `1 - pchisq(5.426, 3)`.

30. Consider testing H_0 at the 5% significance level. Do you reject the null hypothesis?

    No. The p-value provides no evidence to reject the null hypothesis that the data comes from a Poisson distribution.