MTH5129 Probability & Statistics II

Coursework 5

1. Let X have the probability density function

$$f_X(x) = \begin{cases} 4x^3, & \text{for } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

   Find the probability density function of $Y = 2X - 1$ using the transformation of random variables method.

2. Suppose that X_1 and X_2 have joint probability density function

$$f(x_1, x_2) = \begin{cases} 8x_1x_2, & 0 < x_1 < x_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

   What is the probability density function of $Y_1 = X_1/X_2$?
   Hint: Define an additional variable, e.g. $Y_2 = X_2$.

3. Suppose X has a normal distribution, $N(\mu, \sigma^2)$, find the moment generating function of X and then deduce its mean and variance.

4. Let X be a $Bin(n, p)$ random variable.
   (a) Find the moment generating function of X.
   (b) Find the expectation of X.
   (c) Find the variance of X.

5. Suppose $Y_1, \ldots, Y_n$ are independent, normally distributed with mean $E[Y_i] = \mu_i$ and variance $\mathrm{Var[Y_i]} = \sigma_{\mathrm{i}}^2$. Prove that the sum

$$U = \sum_{i=1}^{n} \left(\frac{Y_i - \mu_i}{\sigma_i} \right)^2 \qquad \text{has a } \chi^2(n) \text{ distribution.}$$

6. Suppose $X_1, X_2, \ldots, X_n$ are independent random variables with cumulant generating functions $K_{X_i}(t),\ i = 1, \ldots, n$. Let $Y = \sum_{i=1}^{n} X_i$ and find the cumulant generating function $K_Y(t)$ in terms of the $K_{X_i}(t),\ i = 1, \ldots, n$.

7. Suppose that X is a non-negative random variable with mean μ. Prove that the median is at most 2μ. (The median is a value m with $P(X \geq m) \geq 1/2$ and $P(X \leq m) \geq 1/2$: i.e. it is the "middle value".)

8. Suppose that I toss a fair coin 100 times. Prove that the probability I get more than or equal to 60 heads or less than or equal to 40 heads is at most 1/4.