MTH5129 Probability & Statistics II

Coursework 4

1. Suppose X and Y are jointly continuous random variables with joint density function

$$f_{X,Y}(x,y) = ce^{-|x|-|y|}, \quad \forall\ (x,y) \in \mathbb{R}^2$$

Find whether X and Y are independent.

(c is a constant chosen to make $f_{X,Y}$ a density function).

Solution: We use the Theorem which states that

X and Y are independent if and only if we can 'factorize' $f_{X,Y}$ into two bits: one depending only on x and one depending only on y.

In this case $f_{X,Y}(x,y) = ce^{-|x|}e^{-|y|}$ which is obviously composed by two parts as above. Explicitly let $g(x) = ce^{-|x|}$ and $h(y) = e^{-|y|}$. We see that $f_{X,Y}(x,y) = g(x)h(y)$ for all x and y. Hence X and Y are independent.

2. Suppose that X and Y are two random variables.

   a) Prove that if X and Y are independent then

$$E(X^kY^m) = E(X^k)\,E(Y^m),$$

   b) Deduce that if X and Y are independent then $\text{Corr}(X,Y) = 0$.

Solution:

a) We prove this for any X and Y continuous random variables (*Exercise.* Prove this for discrete random variables). Given that $E(X) = \mu_1$ and $E(Y) = \mu_2$ are constants, we have:

$$\begin{aligned}
E\left[X^kY^m\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^k y^m f_{X,Y}(x,y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^k y^m f_X(x) f_Y(y)dxdy \qquad \text{(by independence)} \\
&= \int_{-\infty}^{\infty} y^m f_Y(y)\left(\int_{-\infty}^{\infty} x^k f_X(x)dx\right)dy \\
&= E(X^k)\int_{-\infty}^{\infty} y^m f_Y(y)dy \\
&= E(X^k)\,E(Y^m)
\end{aligned}$$

b) Evaluate the above formula for $k = 1$ and $m = 1$ to get

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0.$$

3. Suppose we choose independently X and Y to be two $Uniform(0, 1)$ random variables. Use their **convolution** to find the probability density function of their sum $Z = X + Y$.

Solution: We have

$$f_X(x) = f_Y(x) = \begin{cases} 1, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise .} \end{cases}$$

The probability density function of their sum is given by

$$\begin{aligned} f_Z(z) = (f_X * f_Y)(z) &= \int_{-\infty}^{+\infty} f_X(z - y) f_Y(y) dy \\ &= \int_0^1 f_X(z - y) dy \end{aligned}$$

Note that $0 < z - y < 1$ if and only if $z - 1 < y < z$, hence:

For $z < 0$ and $z > 2$, we have $f_Z(z) = 0$.

For $z \in (0, 1)$, we have

$$f_Z(z) = \int_0^z dy = z.$$

For $z \in (1, 2)$, we have

$$f_Z(z) = \int_{z-1}^1 dy = 2 - z.$$

Overall, we have

$$f_Z(z) = \begin{cases} z, & \text{if } 0 < z < 1, \\ 2 - z, & \text{if } 1 < z < 2, \\ 0 & \text{otherwise .} \end{cases}$$

4. Let X have the probability density function

$$f_X(x) = \begin{cases} 4x^3, & \text{for } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability density function of $Y = 2X - 1$ using the cumulative distribution function method.

Solution: Using the cumulative distribution function method, we have

$$F_X(x) = \int_0^x 4t^3 dt = x^4$$

while given that $y = 2x - 1$ we have $-1 < y < 1$ corresponding to the initial $0 < x < 1$. Then, for all $-1 < y < 1$ we have

$$F_Y(y) = P(Y \le y) = P(2X - 1 \le y) = P\Big(X \le \frac{y+1}{2}\Big) = F_X\Big(\frac{y+1}{2}\Big),$$

thus

$$F_Y(y) = \begin{cases} 0, & \text{if } y \le -1 \\ \left(\frac{y+1}{2}\right)^4, & \text{if } -1 < y < -1 \\ 1, & \text{if } y \ge 1 \end{cases}$$

This finally implies that

$$f_Y(y) = \begin{cases} 2\left(\frac{y+1}{2}\right)^3 = \frac{1}{4}(y+1)^3, & \text{for } -1 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

5. Suppose X and Y are two independent $Uniform(0,1)$ random variables. Use the **cumulative distribution function method** to find the probability density function of their sum $U = X + Y$.

   Solution: Given that $X \in (0,1)$ and $Y \in (0,1)$, we have $U \in (0,2)$.

   Then for $u \in (0,2)$, the equality $U = u$ implies that $X = u - Y$. Hence, since we want $0 \le U \le u$, we observe that we need $0 \le X \le u - Y$, which is possible if and only if $Y \le u$.

   However, given that $0 < X < 1$ and $0 < Y < 1$, we need to split into two cases:

   For $u \in (0,1)$, we have $0 \le X \le u - Y$ for any $0 \le Y \le u$, thus

$$\begin{aligned} F_U(u) = P(U \le u) &= P(X + Y \le u) \\ &= P\left(X \le u - Y\right) \\ &= \int_0^u \int_0^{u-y} f_{X,Y}(x,y)\, dxdy \\ &= \int_0^u \int_0^{u-y} dxdy \\ &= \int_0^u (u-y)\ dy \\ &= \left[u\, y - \frac{1}{2}y^2\right]_{y=0}^u = u^2 - \frac{1}{2}u^2 = \frac{1}{2}u^2\,. \end{aligned}$$

For $u \in (1,2)$, we either have $0 \leq X \leq u - Y$ for any $u - 1 < Y < 1$, or $0 \leq X \leq 1$ for any $0 \leq Y \leq u - 1$, thus

$$\begin{aligned} F_U(u) = P(U \leq u) &= P(X + Y \leq u) \\ &= P\left(X \leq u - Y\right) \\ &= \int_{u-1}^{1} \int_{0}^{u-y} f_{X,Y}(x,y)\, dxdy + \int_{0}^{u-1} \int_{0}^{1} f_{X,Y}(x,y)\, dxdy \\ &= \int_{u-1}^{1} \int_{0}^{u-y} dxdy + \int_{0}^{u-1} \int_{0}^{1} dxdy \\ &= \int_{u-1}^{1} (u-y)\; dy + \int_{0}^{u-1} dy \\ &= \left[u\, y - \frac{1}{2} y^2\right]_{y=u-1}^{1} + u - 1 = 2u - \frac{1}{2}u^2 - 1\,. \end{aligned}$$

So, we have

$$f_U(u) = \frac{dF(u)}{du} = \begin{cases} u, & \text{if } 0 < u < 1, \\ 2 - u, & \text{if } 1 < u < 2, \\ 0 & \text{otherwise }. \end{cases}$$